# Problem Set 2: Conservation Laws

## 1 Some Derivations (10 points)

In this problem, we'll go through a couple of derivations that we didn't have time to cover in class. **Credit will only be given if you use index notation and the Levi-Civita symbol**.

(a) In this part of the problem, we'll derive an expression for energy conservation in a medium.

(i) Show that the work done per unit time per unit volume on free charges and currents in the presence of an electric field $E^i$ and a magnetic field $B^i$ is $E_i J_f^i$, where $J_f^i$ is the free current.

> **SOLUTION:**
> The force density on the free charges and current is $f^i = \rho_f E^i + \epsilon^{ijk} J_{fj} B_k$, with $J_{fj} = \rho_f v_j$, and thus the work done per unit time per unit volume is $v_i f^i = J_f^i E_i + \epsilon^{ijk} \rho_f v_i v_j B_k = J_f^i E_i$, due to the antisymmetry of the Levi-Civita symbol.

(ii) Using the Ampere-Maxwell equation in a medium, derive the following expression:

$$\partial_t \left(u_{\text{EM}} + u_{\text{mech}}\right) + \partial_i S^i = 0 \,, \tag{1}$$

where $\partial_t u_{\text{mech}} = E_i J_f^i$, the in-medium Poynting vector is

$$S^i = (\mathbf{E} \times \mathbf{H})^i \,, \tag{2}$$

and

$$\partial_t u_{\text{EM}} = E_i \partial_t D^i + H_i \partial_t B^i \,. \tag{3}$$

> **SOLUTION:**
> The Ampere-Maxwell equation in a medium reads
>
> $$\epsilon^{ijk} \partial_j H_k = J_f^i + \partial_t D^i \,, \tag{4}$$
>
> and hence
>
> $$\partial_t u_{\text{mech}} = J_f^i E_i = \epsilon^{ijk} E_i \partial_j H_k - E_i \partial_t D^i \,. \tag{5}$$
>
> By the product rule, we know that
>
> $$\partial_j (\epsilon^{ijk} E_i H_k) = \epsilon^{ijk} [(\partial_j E_i) H_k + E_i \partial_j H_k] \,, \tag{6}$$
>
> and therefore we have
>
> $$\begin{aligned} \partial_t u_{\text{mech}} &= \partial_j (\epsilon^{ijk} E_i H_k) - \epsilon^{ijk} H_k \partial_j E_i - E_i \partial_t D^i \\ &= -\partial_j (\mathbf{E} \times \mathbf{H})^j + H_k \epsilon^{kji} \partial_j E_i - E_i \partial_t D^i \\ &= -\partial_j S^j + H_k (-\partial_t B^k) - E_i \partial_t D^i \,, \end{aligned} \tag{7}$$
>
> where in the last line I used Faraday's law. Putting everything on the LHS gives the desired expression.

(iii) Show that for a linear medium, we expect

$$u_{\text{EM}} = \frac{1}{2} (E_i D^i + B_i H^i) \,. \tag{8}$$

**SOLUTION:**
For a linear medium, we have $\mathbf{B} = \mu\mathbf{H}$ and $\mathbf{D} = \epsilon\mathbf{E}$, and so

$$\begin{aligned}\partial_t u_{\mathrm{EM}} &= \epsilon E_i \partial_t E^i + \frac{1}{\mu} H_i \partial_t H^i \\ &= \frac{1}{2}\partial_t(\epsilon E_i E^i) + \frac{1}{2}\partial_t\left(\frac{1}{\mu}H_i H^i\right) \\ &= \frac{1}{2}\partial_t\left(D_i E^i + B_i H^i\right)\,, \end{aligned} \tag{9}$$

and so we can handwavingly conclude that

$$u_{\mathrm{EM}} = \frac{1}{2}(D_i E^i + B_i H^i)\,, \tag{10}$$

although the astute among you would have noted that $\partial_t f = \partial_t g$ only implies that $f = g + h$, where $\partial_t h = 0$.

(b) Derive the expression we saw in class for momentum conservation,

$$f^i + \epsilon_0\mu_0 \partial_t S^i - \partial_j T^{ij} = 0\,, \tag{11}$$

where

$$f^i = \rho E^i + (\mathbf{J}\times\mathbf{B})^i \tag{12}$$

is the force density acting on charges,

$$S^i = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})^i \tag{13}$$

is the Poynting vector, and

$$T^{ij} = \epsilon_0\left(E^i E^j - \frac{1}{2}\delta^{ij}E^2\right) + \frac{1}{\mu_0}\left(B^i B^j - \frac{1}{2}\delta^{ij}B^2\right)\,. \tag{14}$$

**SOLUTION:**
Let's start by the taking the spatial derivative of $T^{ij}$. This gives

$$\partial_j T^{ij} = \epsilon_0\left(E^j\partial_j E^i + E^i\partial_j E^j - \frac{1}{2}\partial^i(E^2)\right) + \frac{1}{\mu_0}\left(B^j\partial_j B^i + B^i\partial_j B^j - \frac{1}{2}\partial^i(B^2)\right)\,. \tag{15}$$

Using Gauss's law for electric and magnetic fields, we have $\partial_j E^j = \rho/\epsilon_0$, and $\partial_j B^j = 0$. Furthermore, $\partial^i(E^2) = \partial^i(E_j E^j) = 2E_j\partial^i E^j$, and similarly for $B^2$. This leaves us with

$$\partial_j T^{ij} = \epsilon_0\left(E^j\partial_j E^i - E^j\partial^i E_j\right) + \rho + \frac{1}{\mu_0}\left(B^j\partial_j B^i - B^j\partial^i B_j\right)\,. \tag{16}$$

We could go on, but it's easier to set this aside and take a look at another term instead. For the term with the Poynting vector, we find

$$\begin{aligned}\epsilon_0\mu_0\partial_t S^i &= \epsilon_0\partial_t\left(\epsilon^{ijk}E_j B_k\right) \\ &= \epsilon_0\epsilon^{ijk}\left(B_k\partial_t E_j + E_j\partial_t B_k\right)\,. \end{aligned} \tag{17}$$

Using the Ampere-Maxwell law, i.e. $\mu_0\epsilon_0\partial_t E_j = \epsilon_{jik}\partial^i B^k - \mu_0 J_j$, and Faraday's law, i.e. $\partial_t B_k =$

$-\epsilon_{kij}\partial^i E^j$,

$$\epsilon_0\mu_0\partial_t S^i = \epsilon_0\epsilon^{ijk}\left[B_k\left(\frac{1}{\mu_0\epsilon_0}\epsilon_{jmn}\partial^m B^n - \frac{1}{\epsilon_0}J_j\right) + E_j(-\epsilon_{kmn}\partial^m E^n)\right], \tag{18}$$

where I have relabeled some indices to make sure I only have single free indices and contracted pairs of indices.

At this point, let's take things one term and a time. First,

$$\begin{aligned}\epsilon_0\epsilon^{ijk}B_k\left(\frac{1}{\mu_0\epsilon_0}\epsilon_{jmn}\partial^m B^n\right) &= \frac{B_k}{\mu_0}\epsilon^{ijk}\epsilon_{jmn}\partial^m B^n \\ &= -\frac{B_k}{\mu_0}\left(\delta^i_m\delta^k_n - \delta^k_m\delta^i_n\right)\partial^m B^n \\ &= -\frac{B_k}{\mu_0}\left(\partial^i B^k - \partial^k B^i\right) \\ &= \frac{1}{\mu_0}\left(B_j\partial^j B^i - B_j\partial^i B^j\right), \end{aligned} \tag{19}$$

where in the last line I've relabeled $k \to j$ so that you can see the resemblance between this and Eq. (16). Very similarly, we have

$$\begin{aligned}-\epsilon_0\epsilon^{ijk}E_j\epsilon_{kmn}\partial^m E^n &= -\epsilon_0 E_j\epsilon^{kij}\epsilon_{kmn}\partial^m E^n \\ &= \epsilon_0 E_j(\partial^j E^i - \partial^i E^j). \end{aligned} \tag{20}$$

Putting everything together, we have

$$\epsilon_0\mu_0\partial_t S^i = \epsilon_0(E^j\partial_j E^i - E^j\partial^i E_j) - \epsilon^{ijk}J_j B_k + \frac{1}{\mu_0}\left(B^j\partial_j B^i - B^j\partial^i B_j\right). \tag{21}$$

Comparing this with Eq. (16), we see that

$$\epsilon_0\mu_0\partial_t S^i = \partial_j T^{ij} - \rho - \epsilon^{ijk}J_j B_k = \partial_j T^{ij} - f^i, \tag{22}$$

proving the expression.

## 2 Verifying Coulomb's Law (10 points)

Use the Maxwell stress tensor to find the Coulomb force between two point charges, placed at $-a$ and $a$ along the $z$-axis. To simplify the calculation, let the charge magnitudes be equal, and use an integration surface consisting of the midplane between the charges and a hemispherical surface of a very large radius enclosing the right-hand charge. Consider charges of

(a) opposite polarity, and

(b) same polarity.

In both cases, obtain the full expressions for the stress tensor.

**SOLUTION:**
Consider:

$$q_1 = q \text{ at } \vec{r}_1 = a\hat{z} \quad \text{and} \quad q_2 = sq \text{ at } \vec{r}_2 = -a\hat{z}, \qquad s = \begin{cases} +1 & \text{(same polarity)} \\ -1 & \text{(opposite polarity).}\end{cases} \tag{23}$$

Since the charges are stationary, there is no magnetic field. Hence, the Maxwell stress tensor is just

$$T^{ij} = \varepsilon_0 \Big( E^i E^j - \tfrac{1}{2}\delta^{ij} E^2 \Big). \tag{24}$$

Considering the hemispherical surface mentioned, the force on the charge $q_1$ is given by

$$F^i = \oint_{\partial V} T^{ij} \cdot n_j \, \mathrm{d}A, \tag{25}$$

where $\hat{n}$ the outward normal of the closed surface $\partial V$. The electric field is given by:

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\varepsilon_0} \left( \frac{\vec{r} - \vec{r}_1}{|\vec{r} - \vec{r}_1|^3} + s \frac{\vec{r} - \vec{r}_2}{|\vec{r} - \vec{r}_2|^3} \right). \tag{26}$$

On the curved surface of the hemisphere, when its radius $R \to \infty$, the electric field on the surface would, at best, scale as $E \sim 1/R^2$ and hence $T \sim 1/R^4$. However, since the surface area scales as $R^2$, the surface integral of $T$ on the curved surface would scale as $R^2/R^4 \to 0$ as $R \to \infty$.
To compute the surface integral on the flat plane of the hemisphere, $z = 0$, it is easier to work in the cylindrical coordinates $\vec{r} = (\rho, \phi, z)$. The total electric field on the plane is:

$$\vec{E}(\rho, \phi, 0) = \frac{q}{4\pi\varepsilon_0} \left( \frac{\rho\hat{\rho} - a\hat{z}}{(\rho^2 + a^2)^{3/2}} + s \; \frac{\rho\hat{\rho} + a\hat{z}}{(\rho^2 + a^2)^{3/2}} \right) \tag{27}$$

(a) When both charges have the same polarity ($s = 1$):

$$\vec{E}(\rho, \phi, 0) = \frac{q}{4\pi\varepsilon_0} \frac{2\rho \; \hat{\rho}}{(\rho^2 + a^2)^{3/2}} = E_\rho \, \hat{\rho} \,, \tag{28}$$

At this point, we note that $\hat{\rho} = \cos\phi \, \hat{x} + \sin\phi \, \hat{y}$, and so we can work out the components:

$$T_{xx} = \varepsilon_0 \left( E_x^2 - \frac{1}{2} E^2 \right) = \varepsilon_0 E_\rho^2 \left( \cos^2\phi - \frac{1}{2} \right) = \frac{1}{2} \epsilon_0 E_\rho^2 \cos 2\phi \tag{29}$$

$$T_{xy} = \varepsilon_0 E_x E_y = \varepsilon_0 E_\rho^2 (\cos\phi \sin\phi) = \frac{1}{2} \epsilon_0 E_\rho^2 \sin 2\phi \tag{30}$$

$$T_{yx} = T_{xy} \tag{31}$$

$$T_{yy} = \varepsilon_0 \left( E_y^2 - \frac{1}{2} E^2 \right) = \varepsilon_0 E_\rho^2 \left( \sin^2\phi - \frac{1}{2} \right) = -\frac{1}{2} \varepsilon_0 E_\rho^2 \cos 2\phi \tag{32}$$

$$T_{xz} = T_{zx} = T_{yz} = T_{zy} = 0 \tag{33}$$

$$T_{zz} = \varepsilon_0 \left( E_z^2 - \frac{1}{2} E^2 \right) = -\frac{1}{2} \varepsilon_0 E_\rho^2 \tag{34}$$

Putting this altogether, we get

$$T = \frac{1}{2} \varepsilon_0 E_\rho^2 \begin{pmatrix} \cos 2\phi & \sin 2\phi & 0 \\ \sin 2\phi & -\cos 2\phi & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{35}$$

Since the normal to the flat surface is $\hat{n} = -\hat{z}$, the only non-zero component of the force is:

$$F_z = -\int_0^\infty T_{zz} (2\pi\rho \, d\rho) \tag{36}$$

$$= \int_0^\infty \frac{\varepsilon_0}{2} \left( \frac{q}{4\pi\varepsilon_0} \frac{2\rho}{(\rho^2 + a^2)^{3/2}} \right)^2 (2\pi\rho \, \mathrm{d}\rho) \tag{37}$$

$$= \frac{q^2}{4\pi\varepsilon_0} \int_0^\infty \frac{\rho^3}{(\rho^2 + a^2)^3} \mathrm{d}\rho \tag{38}$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{q^2}{(2a^2)} \tag{39}$$

(b) When the two charges have the opposite polarity ($s = -1$):

$$\vec{E}(\rho, \phi, 0) = -\frac{q}{4\pi\varepsilon_0} \frac{2a\ \hat{z}}{(\rho^2 + a^2)^{3/2}} = E_z\, \hat{z}\,, \tag{40}$$

and so there is only a $z$-component. We can easily work out the Maxwell stress tensor as

$$\begin{aligned} T_{xx} &= E_x^2 - \frac{1}{2}E^2 = -\frac{1}{2}E_z^2\,, && (41)\\ T_{yy} &= E_y^2 - \frac{1}{2}E^2 = -\frac{1}{2}E_z^2\,, && (42)\\ T_{zz} &= E_z^2 - \frac{1}{2}E^2 = \frac{1}{2}E_z^2\,, && (43) \end{aligned}$$

where $E_x = E_y = 0$, and all other entries are zero. To summarize,

$$T = \varepsilon_0 \begin{pmatrix} -\frac{1}{2}E_z^2 & 0 & 0 \\ 0 & -\frac{1}{2}E_z^2 & 0 \\ 0 & 0 & \frac{1}{2}E_z^2 \end{pmatrix}\,. \tag{44}$$

In this case, the non-zero component of the force is:

$$\begin{aligned} F_z &= -\int_0^\infty T_{zz}(2\pi\rho\, d\rho) && (45)\\ &= -\frac{q^2}{4\pi\varepsilon_0} \int_0^\infty \frac{a^2\rho}{(\rho^2 + a^2)^3} \mathrm{d}\rho && (46)\\ &= -\frac{1}{4\pi\varepsilon_0} \frac{q^2}{(2a^2)} && (47) \end{aligned}$$

# 3 Constant Fields (10 points)

Consider an object occupying some volume $V$, with surface $\partial V$. Show that a constant, static electric field $\mathbf{E}$ on the surface cannot exert a net electric force on this object, regardless of the charge and current distribution within the object. The same argument should also show that this is true for a constant, static magnetic field $\mathbf{B}$.

This is a remarkable statement that can be proven using the Maxwell stress tensor. Let's just look at the case of the magnetic field. For a static and constant magnetic field $\mathbf{B}$, the Maxwell stress tensor is

$$T^{ij} = \frac{1}{\mu_0}\left(B^i B^j - \frac{1}{2}\delta^{ij} B^2\right)\,, \tag{48}$$

and the net force acting on the object occupying some volume $V$ with surface $\partial V$ is

$$\begin{aligned} F^i &= \oint_{\partial V} \mathrm{d}A_j\, T^{ij} \\ &= \frac{1}{\mu_0} \oint_{\partial V} \mathrm{d}A_j \left(B^i B^j - \frac{1}{2}\delta^{ij} B^2\right)\,. && (49) \end{aligned}$$

Since the magnetic fields are all constant, we can pull all the fields out from the integral, leaving

$$F^i = \frac{1}{\mu_0}\left(B^i B^j - \frac{1}{2}\delta^{ij} B^2\right) \oint_{\partial V} \mathrm{d}A_j\,. \tag{50}$$

To show that the closed surface integral is zero, write

$$\mathrm{d}A_j = \mathrm{d}\mathbf{A} \cdot \mathbf{e}\,, \tag{51}$$

where $e^j = 1$, and all other components are zero. We can now use the divergence theorem to see that

$$\oint_{\partial V} \mathrm{d}A_j = \oint_{\partial V} \mathrm{d}\mathbf{A} \cdot \mathbf{e} = \int_V \mathrm{d}^3\boldsymbol{r}\,(\nabla \cdot \mathbf{e}) = 0\,, \tag{52}$$

since $\mathbf{e}$ is simply a constant! We conclude therefore that $F^i = 0$, as required.
The same argument can be used for the electric field. In fact, for the electric field, the fact that the flux is zero also shows that the charge enclosed in the volume must be zero, by Gauss's law.

# 4 Concentric Spherical Shells (10 points)

Two concentric spherical shells carry uniformly distributed charges $+Q$ (at radius $a$) and $-Q$ (at radius $b > a$). They are immersed in a uniform magnetic field $\mathbf{B} = B_0\hat{\mathbf{z}}$.

(a) Find the angular momentum of the fields (with respect to the center).

**SOLUTION:**
First, let's start with computing the electric field due to the charge on the shells. From the spherical symmetry in the system, we can identify that the electric fields will point along the radial direction, and that there is uniform electric field on all spherical surfaces, i.e. in spherical coordinates $\vec{E}(r,\theta,\phi) = \vec{E}(r)$. Using Gauss law with spherical surfaces $V$, we can see that:

$$\oint_V \vec{E} \cdot d\vec{S} = E(r) * 4\pi r^2 = \frac{Q_{\mathrm{enc}}}{\varepsilon_0} \tag{53}$$

$$\vec{E}(r) = \begin{cases} 0 & r < a, \\ \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{r^2}\,\hat{r} & a < r < b, \\ 0 & r > b. \end{cases} \tag{54}$$

The momentum density of the electromagnetic field is

$$\vec{g} = \varepsilon_0 \vec{E} \times \vec{B}, \tag{55}$$

and the field angular momentum about the origin is

$$\vec{L}_{\mathrm{field}} = \int \vec{r} \times \vec{g}\,\mathrm{d}^3\mathrm{r} = \varepsilon_0 \int \tilde{\mathrm{r}} \times (\tilde{\mathrm{E}} \times \tilde{\mathrm{B}})\,\mathrm{d}^3\mathrm{r}. \tag{56}$$

We get a non zero contribution only within the region $a < r < b$ contributes. With $\vec{B} = B_0\hat{z}$,

$$\vec{E} \times \vec{B} = \left(\frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}\right) \times (B_0\hat{z}) \tag{57}$$

$$= -\frac{QB_0}{4\pi\varepsilon_0}\frac{\sin\theta}{r^2}\,\hat{\phi}. \tag{58}$$

$$\vec{r} \times (\vec{E} \times \vec{B}) = \frac{QB_0}{4\pi\varepsilon_0}\frac{\sin\theta}{r}\,\hat{\theta}. \tag{59}$$

For the integral of the above quantity (the angular momentum density) over all space, notice that only the $z$-component adds up, since $\hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}$ and the $z$-component is the only component even around $\theta = \pi/2$. This can also be understood from a symmetry perspective: with the magnetic field considered, the spherical symmetry is broken down into an axial symmetry along the $z$-axis. Hence, we arrive at:

$$\vec{L}_{\text{field}} = \varepsilon_0 \int_a^b \mathrm{dr} \int_0^\pi \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi \; \mathrm{r}^2 \sin\theta \left( -\frac{\mathrm{QB}_0}{4\pi\varepsilon_0} \frac{\sin^2\theta}{\mathrm{r}} \hat{\mathrm{z}} \right) \tag{60}$$

$$= \frac{QB_0}{3} (a^2 - b^2) \, \hat{z} \tag{61}$$

(b) Now the magnetic field is gradually turned off. Find the torque on each sphere, and the resulting angular momentum of the system.

**SOLUTION:**
When the magnetic field is slowly turned off, there is an induced electric field $\vec{E}_{\text{ind}}$, by Faraday's law. By cylindrical symmetry, we know that that $\vec{E}_{\text{ind}}$ must be in the $\hat{\phi}$ direction. We apply Faraday's law to a circle with axis parallel to the $z$-axis, located at $z = r\cos\theta$ where $\theta$ is the polar angle, which gives

$$\oint_{\text{circle}} \vec{E} \cdot \mathrm{d}\vec{\ell} = -\frac{\mathrm{d}}{\mathrm{d}t} \int \vec{B} \cdot \mathrm{d}\vec{A} \tag{62}$$

$$2\pi (r\sin\theta) E_{\text{ind}} = -\pi (r\sin\theta)^2 \dot{B} \tag{63}$$

$$E_{\text{ind}} = -\frac{r}{2} \sin\theta \dot{B} \,, \tag{64}$$

or

$$\vec{E}_{\text{ind}} = -\frac{1}{2} \, r \sin\theta \, \dot{B} \, \hat{\phi} \,. \tag{65}$$

The force from the circulating electric field leads to the torque on the shells. The force on an elemental area on the inner sphere and the corresponding torque due to the force is given by

$$\mathrm{d}\vec{F} = \vec{E}_{\text{ind}} \sigma_a \mathrm{d}S \,, \tag{66}$$

$$\mathrm{d}\tau = \vec{r} \times \mathrm{d}\vec{F} \tag{67}$$

where $\sigma_a = \frac{Q}{4\pi a^2}$ is the charge density of the inner shell. The total torque on the inner shell is

$$\vec{\tau}_a = \oint \mathrm{d}S \; \vec{r} \times (\sigma_a \vec{E}_{\text{ind}}) \tag{68}$$

$$= \int_0^\pi \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi \; a^2 \sin\theta \left( \frac{Q}{4\pi a^2} \right) \left( -\frac{1}{2} \, a \sin\theta \, \dot{B} \right) (-\hat{\theta}) \tag{69}$$

$$= -\frac{Qa^2}{3} \, \dot{B} \, \hat{z}. \tag{70}$$

Note that we, once again, can use axial symmetry to simplify the integral. Similarly, the torque on the outer shell is:

$$\vec{\tau}_b = \frac{Qa^2}{3} \dot{B} \hat{z} \tag{71}$$

The angular momentum gained by the the shells is given by:

$$\vec{L}_a = \int \mathrm{dt}\ \vec{\tau} \tag{72}$$
$$= -\frac{Qa^2}{3}\int \mathrm{dt}\,\dot{B}\,\hat{z} \tag{73}$$
$$= \frac{B_0 Q a^2}{3}\hat{z}, \tag{74}$$
$$\vec{L}_b = -\frac{B_0 Q b^2}{3}\hat{z}. \tag{75}$$

Hence the total final angular momentum

$$\vec{L}_a + \vec{L}_b = \frac{QB_0}{3}(a^2 - b^2)\,\hat{z} \tag{76}$$

is the same as the initial angular momentum.

# 5 Magnetic Fields Do No Work (10 points)

One thing that Griffiths emphasizes a lot in Chapter 8 of this book—and that we didn't have time to cover in class—is that the magnetic field never does any work. This is evident from the Lorentz force law, which for a point charge reads

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})\,, \tag{77}$$

and so the magnetic field always acts orthogonal to the direct of motion. But every-day experience appears to contradict this statement: permanent magnets, for example, do appear to pull on magnetic materials, so what's up with that? The punchline is that magnetic fields only facilitate the conversion of kinetic energy in the form of bound currents in magnetic materials into translational kinetic energy, without doing any net work on the system. This problem is a nice example of that happening.

A massless circular disk of radius $R$ carries $n$ point charges $q$, each with mass $m$, attached at regular intervals around its rim. At time $t = 0$, the disk lies in the $x$-$y$ plane, with its center at the origin, and is rotating about the $z$-axis with angular velocity $\omega_0$, when it is released. The disk is immersed in a (time-independent) external magnetic field

$$\mathbf{B}(s, z) = k(-s\hat{\mathbf{s}} + 2z\hat{\mathbf{z}})\,, \tag{78}$$

where $s$ is the cylindrical coordinate system distance from the $z$-axis, and $k$ is a constant. Neglect gravity for this problem.

(a) Find the position of the center of the ring, $z(t)$, and its angular velocity $\omega(t)$, as functions of time.

**SOLUTION:**
The force on the charges due to the magnetic field can lead to both net force and torque on the disk. For a single charge at the rim, the velocity of the charge is $\vec{v} = R\omega(t)\,\hat{\phi} + \dot{z}(t)\,\hat{z}$ and the force on it is:

$$\vec{F} = q\,\vec{v}\times\vec{B} \tag{79}$$
$$= q(R\omega\,\hat{\phi} + \dot{z}\,\hat{z})\times k(-R\,\hat{s} + 2z\,\hat{z}) \tag{80}$$
$$= kqR\,(R\omega\,\hat{z} + 2z\omega\,\hat{s} - \dot{z}\,\hat{\phi}) \tag{81}$$

From axial symmetry, the sum of the forces on the disk along $\hat{s}$ and $\hat{\phi}$ get cancelled and hence, the center of mass of the disk is governed by :

$$(nm)\ddot{z} = n(kqR^2\omega) \tag{82}$$

$$\ddot{z} = \frac{kqR^2}{m}\omega \tag{83}$$

Only the tangential force contributed to the net torque on the disk. The rotational equation of motion, thus gives:

$$I\dot{\omega} = \tau_{\text{net}} \tag{84}$$

$$(nmR^2)\dot{\omega} = -n(R)(kqR\dot{z}) \tag{85}$$

$$\dot{\omega} = -\frac{kq}{m}\dot{z} \tag{86}$$

Taking the derivative of the above equation and using the other equation of motion to eliminate $z$ we get:

$$\ddot{\omega} = -\left(\frac{kqR}{m}\right)^2 \omega \implies \omega(t) = \omega_0 \cos(\Omega\, t) \tag{87}$$

where, $\Omega = kqR/m$. We also have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\omega + \frac{kq}{m}z\right) = 0 \implies \omega(t) = \omega_0 + \frac{kq}{m}z(t) \tag{88}$$

This gives

$$z(t) = \frac{m\omega_0}{kq}(1 - \cos(\Omega t)) \tag{89}$$

(b) Describe the motion, and check that the total kinetic energy, translational plus rotational, is constant, confirming that the magnetic force does no work.

**SOLUTION:**
The center of mass oscillates about $z = 0$ and the angular velocity of the ring is simple harmonic about $\omega_0$. There is a steady exchange between the rotational energy and the translational kinetic energy of the disk.
The translational energy of the disk is:

$$E_T = \frac{1}{2}(mn)\dot{z}^2 \tag{90}$$

$$= \frac{1}{2}mn(R\omega_0 \sin(\Omega t))^2 \tag{91}$$

$$= \frac{1}{2}mnR^2\omega_0^2 \sin^2(\omega t) \tag{92}$$

The rotational energy of the disk is:

$$E_R = \frac{1}{2}I\omega^2 \tag{93}$$

$$= \frac{1}{2}mnR^2\left(\omega_0 \cos(\Omega t)\right)^2 \tag{94}$$

$$= \frac{1}{2}mnR^2\omega_0^2 \cos^2(\omega t) \tag{95}$$

The total energy is

$$E_R + E_T = \frac{1}{2} mnR^2\omega_0^2 \tag{96}$$

is constant with time and hence, no work is done by the magnetic field.