Problem Set 11: Special Relativity II

1 Cops and Robbers (10 points)

A stationary observer outside a bank sees a group of bank robbers in a getaway car, traveling at speed 3/4. Behind the getaway car is a police car, traveling toward the robbers at speed 1/2. From the car, a police officer fires a gun toward the robbers, with the speed of the bullet relative to the gun being 1/3. Let the direction of travel of the robbers be the $+x$ direction.

(a) In the frame of the stationary observer, determine the velocity of the observer at the bank, the robbers, the police car, and the bullet. Does the bullet catch up to the robbers?

The velocities of the observer, robbers, and the police in the observer's frame S are:

$$v_{\rm obs} = 0\,, \tag{1}$$

$$v_{\rm rob} = +\frac{3}{4}\,, \tag{2}$$

$$v_{\rm pol} = +\frac{1}{2}\,. \tag{3}$$

For the bullet, we use relativistic velocity addition. In the police frame S', the velocity of the bullet is $v'_{\rm bul} = +1/3$. In the frame S, it is:

$$v_{\rm bul} = \frac{v_{\rm pol} + v'_{\rm bul}}{1 + v_{\rm pol}\, v'_{\rm bul}} = \frac{\frac{1}{2} + \frac{1}{3}}{1 + \frac{1}{2}\cdot\frac{1}{3}} = \frac{\frac{5}{6}}{\frac{7}{6}} = \frac{5}{7}\,. \tag{4}$$

Since $v_{\rm bul} = 5/7 \approx 0.714 < 3/4 = 0.75 = v_{\rm rob}$, the bullet does not catch up to the robbers.

(b) Repeat the problem in the frame of the police car.

In the police frame S', all velocities need relativistic velocity addition with $-v_{\rm pol} = -1/2$:

$$v' = \frac{v - v_{\rm pol}}{1 - v\, v_{\rm pol}}\,. \tag{5}$$

Applying this to each object:

$$v'_{\rm pol} = \frac{\frac{1}{2} - \frac{1}{2}}{1 - \frac{1}{2}\cdot\frac{1}{2}} = 0\,, \tag{6}$$

$$v'_{\rm obs} = \frac{0 - \frac{1}{2}}{1 - 0} = -\frac{1}{2}\,, \tag{7}$$

$$v'_{\rm rob} = \frac{\frac{3}{4} - \frac{1}{2}}{1 - \frac{3}{4}\cdot\frac{1}{2}} = \frac{\frac{1}{4}}{\frac{5}{8}} = \frac{2}{5}\,, \tag{8}$$

$$v'_{\rm bul} = +\frac{1}{3}\,. \tag{9}$$

We see that $v'_{\rm bul} < v'_{\rm rob}$, as expected from part (a).

(c) Repeat the problem in the frame of the robbers' getaway car.

In the robber's frame, we do the same as in part (b), now with the velocity $-v_{\text{rob}} = -3/4$:

$$v'_{\text{rob}} = 0\,, \tag{10}$$
$$v'_{\text{obs}} = \frac{0 - \frac{3}{4}}{1 - 0} = -\frac{3}{4}\,, \tag{11}$$
$$v'_{\text{pol}} = \frac{\frac{1}{2} - \frac{3}{4}}{1 - \frac{1}{2}\cdot\frac{3}{4}} = \frac{-\frac{1}{4}}{\frac{5}{8}} = -\frac{2}{5}\,, \tag{12}$$
$$v'_{\text{bul}} = \frac{\frac{5}{7} - \frac{3}{4}}{1 - \frac{5}{7}\cdot\frac{3}{4}} = \frac{\frac{20-21}{28}}{\frac{28-15}{28}} = \frac{-\frac{1}{28}}{\frac{13}{28}} = -\frac{1}{13}\,. \tag{13}$$

Again, $v'_{\text{bul}} < v'_{\text{rob}}$.

(d) Repeat the problem in the frame of the bullet.

We repeat the same process, now with the velocity $-v_{\text{bul}} = -5/7$, that we found in part (a):

$$v'_{\text{bul}} = 0\,, \tag{14}$$
$$v'_{\text{obs}} = \frac{0 - \frac{5}{7}}{1 - 0} = -\frac{5}{7}\,, \tag{15}$$
$$v'_{\text{pol}} = \frac{\frac{1}{2} - \frac{5}{7}}{1 - \frac{1}{2}\cdot\frac{5}{7}} = \frac{-\frac{3}{14}}{\frac{9}{14}} = -\frac{1}{3}\,, \tag{16}$$
$$v'_{\text{rob}} = \frac{\frac{3}{4} - \frac{5}{7}}{1 - \frac{3}{4}\cdot\frac{5}{7}} = \frac{\frac{1}{28}}{\frac{13}{28}} = +\frac{1}{13}\,. \tag{17}$$

Yet again, $v'_{\text{bul}} < v'_{\text{rob}}$.

2 Pion Decay (10 points)

A neutral pion of rest mass m and relativistic momentum $p = 3m/4$ decays into two photons. Suppose one of the photons is emitted in the same direction as the original pion, and the other in the opposite direction. Find the energy of each photon.

First, we determine the energy of the pion using the energy-momentum relation:

$$E_\pi^2 = p^2 + m^2 = \frac{9m^2}{16} + m^2 = \frac{25m^2}{16}\,, \tag{18}$$
$$E_\pi = \frac{5m}{4}\,. \tag{19}$$

Let photon 1 travel in the $+x$ direction (same as the pion) with energy E_1, and photon 2 travel in the $-x$ direction with energy E_2. For photons, $|p| = E$. Conservation of energy and momentum give:

$$E_1 + E_2 = E_\pi = \frac{5m}{4}\,, \tag{20}$$
$$E_1 - E_2 = p_\pi = \frac{3m}{4}\,. \tag{21}$$

Adding and subtracting:

$$E_1 = \frac{1}{2}\left(\frac{5m}{4} + \frac{3m}{4}\right) = m\,, \tag{22}$$

$$E_2 = \frac{1}{2}\left(\frac{5m}{4} - \frac{3m}{4}\right) = \frac{m}{4}\,. \tag{23}$$

Hence the forward-moving photon has energy $E_1 = m$ and the backward-moving photon has energy $E_2 = m/4$.

3 Fixed Target vs. Colliding Beams (10 points)

Suppose I would like to design a particle accelerator that collides protons of mass m with each other, and I have the technology to accelerate protons up to an energy E. I have two design options: A) accelerate one proton up to energy E, and collide it with a proton at rest, or B) accelerate both protons to energy E, and collide them with each other head on.

(a) In Newtonian mechanics, calculate the total center-of-mass energy, i.e. the total kinetic energy of the particles in the frame where the center-of-mass is stationary, for Option A (K_A) and Option B (K_B). Evaluate the ratio K_A/K_B.

In Newtonian mechanics, for a proton with velocity v, the Kinetic energy is $K = \frac{1}{2}mv^2$.
Option A: Proton 1 has velocity v, proton 2 is at rest. In the center-of-mass (CM) frame, each proton moves at $v/2$. The total kinetic energy is:

$$K_A = 2 \cdot \frac{1}{2} m \left(\frac{v}{2}\right)^2 = \frac{1}{4} m v^2 = \frac{1}{2} K \tag{24}$$

Option B: Both protons have energy E and collide head-on with equal and opposite momenta. The CM frame is the lab frame, so:

$$K_B = 2K\,. \tag{25}$$

The ratio is

$$\frac{K_A}{K_B} = \frac{1}{4}\,. \tag{26}$$

(b) In special relativity, calculate the total center-of-mass energy, i.e. the total *energy* (not just kinetic) in the center-of-mass frame, for Option A (E_A) and Option B (E_B). Evaluate the ratio E_A/E_B.

In special relativity, the kinetic energy is $K = E - m$. The magnitude of the four-momentum is an invariant. For a single particle, this gives the mass of the particle: $p^\mu p_\mu = E^2 - \mathbf{p}^2 = m^2$. For multiple particles, the total center-of-mass energy squared is a Lorentz invariant, since the sum of momenta is zero.
Option A: Proton 1 has four-momentum $(E, \mathbf{p})$ and proton 2 has four-momentum $(m, \mathbf{0})$:

$$E_A^2 = (E+m)^2 - p^2 = E^2 + 2Em + m^2 - p^2\,. \tag{27}$$

Using $E^2 - \mathbf{p}^2 = m^2$:

$$E_A^2 = 2m^2 + 2Em = 2m(m+E)\,, \tag{28}$$

$$E_A = \sqrt{2m(m+E)}\,. \tag{29}$$

Option B: Both protons have energy E and equal and opposite momenta. The lab frame is the CM frame:

$$E_B = 2E\,. \tag{30}$$

The ratio is:

$$\frac{E_A}{E_B} = \frac{\sqrt{2m(m+E)}}{2E}\,. \tag{31}$$

For $E \gg m$, this becomes $E_A/E_B \approx \sqrt{m/(2E)} \to 0$. The fixed-target option becomes increasingly inefficient at high energies.

(c) Which strategy does the Large Hadron Collider adopt, and why?

The LHC uses Option B. As shown in part(b), the center-of-mass energy scales as $2E$ for colliding beams but only as $\sqrt{2mE}$ for fixed target at high energies. Since the purpose of the LHC is to reach the highest possible center-of-mass energies to produce heavy particles (like the Higgs boson), colliding beams is vastly more efficient.

4 Acceleration (10 points)

(a) For a particle of mass m, show that the rate of change of (relativistic) momentum $\mathrm{d}\mathbf{p}/\mathrm{d}t$ is given by

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = m\gamma[\mathbf{a} + \gamma^2\mathbf{v}(\mathbf{v}\cdot\mathbf{a})]\,, \tag{32}$$

where $\mathbf{v}$ is the ordinary three-velocity, and $\mathbf{a} = \mathrm{d}\mathbf{v}/\mathrm{d}t$.

The relativistic momentum is $\mathbf{p} = m\gamma\mathbf{v}$. Taking the time derivative:

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = m\left(\frac{\mathrm{d}\gamma}{\mathrm{d}t}\mathbf{v} + \gamma\mathbf{a}\right)\,. \tag{33}$$

Since $\gamma = (1-v^2)^{-1/2}$,

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = -\frac{1}{2}(1-v^2)^{-3/2}(-2\mathbf{v}\cdot\mathbf{a}) = \gamma^3(\mathbf{v}\cdot\mathbf{a})\,. \tag{34}$$

Substituting:

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = m\left[\gamma^3(\mathbf{v}\cdot\mathbf{a})\mathbf{v} + \gamma\mathbf{a}\right] = m\gamma\left[\mathbf{a} + \gamma^2\mathbf{v}(\mathbf{v}\cdot\mathbf{a})\right]\,. \tag{35}$$

(b) For a particle of mass m with some trajectory in spacetime, we can define a quantity called the **four-acceleration**, given by

$$\alpha^\mu = \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau}\,, \tag{36}$$

where u^μ is the four-velocity, and τ is the proper time along the trajectory. Find α^0 and α^i in terms of $\mathbf{v}$ and $\mathbf{a}$.

The four-velocity is $u^\mu = \gamma(1, \mathbf{v})$. Since $\mathrm{d}\tau = \mathrm{d}t/\gamma$, the four-acceleration is:

$$\alpha^\mu = \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = \gamma \frac{\mathrm{d}u^\mu}{\mathrm{d}t}\,. \tag{37}$$

Using what we found in part (a), we can compute the components of the four-acceleration tensor:

$$\alpha^0 = \gamma \frac{\mathrm{d}}{\mathrm{d}t}(\gamma) = \gamma \cdot \gamma^3 (\mathbf{v}\cdot\mathbf{a}) = \gamma^4 (\mathbf{v}\cdot\mathbf{a})\,, \tag{38}$$

$$\alpha^i = \gamma \frac{\mathrm{d}}{\mathrm{d}t}(\gamma v^i) = \gamma \left[\gamma^3 (\mathbf{v}\cdot\mathbf{a}) v^i + \gamma a^i\right] = \gamma^2 \left[a^i + \gamma^2 v^i (\mathbf{v}\cdot\mathbf{a})\right]\,. \tag{39}$$

(c) Express $\alpha_\mu \alpha^\mu$ in terms of $\mathbf{v}$ and $\mathbf{a}$.

For the magnitude of the four-vector, we need the magnitude of the three-vector in part(b):

$$\begin{aligned} |\boldsymbol{\alpha}|^2 &= \gamma^4 \left[a^2 + 2\gamma^2 (\mathbf{v}\cdot\mathbf{a})^2 + \gamma^4 v^2 (\mathbf{v}\cdot\mathbf{a})^2\right] &(40)\\ &= \gamma^4 \left[a^2 + \gamma^2 (\mathbf{v}\cdot\mathbf{a})^2 (2 + \gamma^2 v^2)\right]\,. &(41)\end{aligned}$$

Using $\gamma^2 v^2 = \gamma^2 - 1$, we get

$$|\boldsymbol{\alpha}|^2 = \gamma^4 \left[a^2 + \gamma^2 (1+\gamma^2)(\mathbf{v}\cdot\mathbf{a})^2\right]\,. \tag{42}$$

With this, we have

$$\begin{aligned} \alpha_\mu \alpha^\mu &= |\boldsymbol{\alpha}|^2 - (\alpha^0)^2 &(43)\\ &= \gamma^4 a^2 + \gamma^6 (1+\gamma^2)(\mathbf{v}\cdot\mathbf{a})^2 - \gamma^8 (\mathbf{v}\cdot\mathbf{a})^2 &(44)\\ &= \gamma^4 a^2 + \gamma^6 (\mathbf{v}\cdot\mathbf{a})^2 &(45)\\ &= \gamma^4 \left[a^2 + \gamma^2 (\mathbf{v}\cdot\mathbf{a})^2\right]\,. &(46)\end{aligned}$$

(d) Show that $u^\mu \alpha_\mu = 0$.

Using the expressions in part(b), we get:

$$\begin{aligned} u^\mu \alpha_\mu &= -u^0 \alpha^0 + \mathbf{u}\cdot\boldsymbol{\alpha} &(47)\\ &= -\gamma \cdot \gamma^4 (\mathbf{v}\cdot\mathbf{a}) + \gamma \mathbf{v} \cdot \gamma^2 \left[\mathbf{a} + \gamma^2 \mathbf{v} (\mathbf{v}\cdot\mathbf{a})\right] &(48)\\ &= -\gamma^5 (\mathbf{v}\cdot\mathbf{a}) + \gamma^3 (\mathbf{v}\cdot\mathbf{a}) + \gamma^5 v^2 (\mathbf{v}\cdot\mathbf{a}) &(49)\\ &= (\mathbf{v}\cdot\mathbf{a}) \left[-\gamma^5 + \gamma^3 + \gamma^5 v^2\right] &(50)\\ &= (\mathbf{v}\cdot\mathbf{a}) \left[\gamma^3 + \gamma^5 (v^2 - 1)\right] &(51)\\ &= (\mathbf{v}\cdot\mathbf{a}) \left[\gamma^3 - \gamma^5/\gamma^2\right] &(52)\\ &= (\mathbf{v}\cdot\mathbf{a}) \left[\gamma^3 - \gamma^3\right] = 0\,. &(53)\end{aligned}$$

Alternatively, this follows from differentiating $u^\mu u_\mu = -1$:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(u^\mu u_\mu) = 2 u^\mu \alpha_\mu = 0\,. \tag{54}$$

5 Moving Charged Wire (10 points)

Consider an infinite straight wire in the lab, oriented along the z-axis, carrying a charge density λ traveling in the $+z$ direction at speed v. Let the lab frame be S, and the frame in which the wire is stationary be S'.

(a) In S', determine the electromagnetic field tensor, as well as the dual tensor, at the point $(x', 0, 0)$.

In S', the wire is stationary, and hence, the current and consequently the magnetic field are zero. The charge density in S' is $\lambda' = \lambda/\gamma$, since the wire is length-contracted in S. The electric field at a distance x' from the wire is:

$$\mathbf{E}' = \frac{\lambda'}{2\pi\epsilon_0 x'}\hat{\mathbf{x}}\,, \quad \mathbf{B}' = 0\,. \tag{55}$$

The field tensor $F'^{\mu\nu}$ at $(x', 0, 0)$ is:

$$F'^{\mu\nu} = \begin{pmatrix} 0 & E'^x & 0 & 0 \\ -E'^x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{56}$$

with $E'^x = \frac{\lambda'}{2\pi\epsilon_0 x'}$. The dual tensor $G'^{\mu\nu}$ is:

$$G'^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -E'^x \\ 0 & 0 & E'^x & 0 \end{pmatrix}. \tag{57}$$

(b) Determine the electromagnetic field tensor and the dual tensor at the point corresponding point $(x, 0, 0)$ in the lab frame S.

The wire moves in the $+z$ direction at speed v, so we boost along z with velocity $-v$. The relevant Lorentz transformation is $\Lambda^\mu{}_\nu$ for the boost in the z-direction is:

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & 0 & 0 & \gamma v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma v & 0 & 0 & \gamma \end{pmatrix} \tag{58}$$

Under this boost, $x' = x$ and the field tensor transforms as:

$$F^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta F'^{\alpha\beta} \tag{59}$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & E^x & 0 & 0 \\ -E^x & 0 & 0 & -B^y \\ 0 & 0 & 0 & 0 \\ 0 & B^y & 0 & 0 \end{pmatrix}. \tag{60}$$

where,

$$E^x = \gamma E'^x\,, \quad B^y = \gamma v E'^x\,. \tag{61}$$

Since $\gamma\lambda' = \lambda$ and $x' = x$:

$$E^x = \frac{\gamma\lambda'}{2\pi\epsilon_0 x'} = \frac{\lambda}{2\pi\epsilon_0 x}\,, \tag{62}$$

$$B^y = \frac{\gamma v\lambda'}{2\pi\epsilon_0 x'} = \frac{\mu_0 \lambda v}{2\pi x}\,. \tag{63}$$

where we used $\mu_0 = 1/\epsilon_0$. The dual tensor is:

$$G^{\mu\nu} = \begin{pmatrix} 0 & 0 & B^y & 0 \\ 0 & 0 & 0 & 0 \\ -B^y & 0 & 0 & -E^x \\ 0 & 0 & E^x & 0 \end{pmatrix} . \tag{64}$$

(c) Repeat the two parts with the points $(0, y', 0)$ and corresponding point in S $(0, y, 0)$.

In S': At the point $(0, y', 0)$, the electric field points in the $\hat{\mathbf{y}}$ direction:

$$E'^y = \frac{\lambda'}{2\pi\epsilon_0 y'} , \quad \mathbf{B}' = 0 . \tag{65}$$

The field tensor is:

$$F'^{\mu\nu} = \frac{\lambda'}{2\pi\epsilon_0 y'} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{66}$$

The dual tensor is:

$$G'^{\mu\nu} = \frac{\lambda'}{2\pi\epsilon_0 y'} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} . \tag{67}$$

In S: Again $y' = y$, and the boost along z gives:

$$E^y = \gamma E'^y = \frac{\lambda}{2\pi\epsilon_0 y} , \tag{68}$$

$$B^x = -\gamma v E'^y = -\frac{\mu_0 \lambda v}{2\pi y} . \tag{69}$$

The field tensor is:

$$F^{\mu\nu} = \begin{pmatrix} 0 & 0 & E^y & 0 \\ 0 & 0 & 0 & 0 \\ -E^y & 0 & 0 & B^x \\ 0 & 0 & -B^x & 0 \end{pmatrix} . \tag{70}$$

The dual tensor is:

$$G^{\mu\nu} = \begin{pmatrix} 0 & B^x & 0 & 0 \\ -B^x & 0 & 0 & E^y \\ 0 & 0 & 0 & 0 \\ 0 & -E^y & 0 & 0 \end{pmatrix} . \tag{71}$$

(d) Use Maxwell's equations to determine $\mathbf{E}$ and $\mathbf{B}$ in the lab frame, and check that your results are consistent with the previous part.

In the lab frame S, the wire has linear charge density λ and current $I = \lambda v$ in the $+z$ direction.

By Gauss's law, at distance s from the wire:

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{\mathbf{s}}\,. \tag{72}$$

By Ampère's law:

$$\mathbf{B} = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}} = \frac{\mu_0\lambda v}{2\pi s}\hat{\boldsymbol{\phi}}\,. \tag{73}$$

At the point $(x,0,0)$: $s = x$, $\hat{\mathbf{s}} = \hat{\mathbf{x}}$, $\hat{\boldsymbol{\phi}} = \hat{\mathbf{y}}$:

$$E^x = \frac{\lambda}{2\pi\epsilon_0 x}\,, \quad B^y = \frac{\mu_0\lambda v}{2\pi x}\,. \tag{74}$$

At the point $(0,y,0)$: $s = y$, $\hat{\mathbf{s}} = \hat{\mathbf{y}}$, $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}$:

$$E^y = \frac{\lambda}{2\pi\epsilon_0 y}\,, \quad B^x = -\frac{\mu_0\lambda v}{2\pi y}\,. \tag{75}$$

These are fully consistent with the results obtained from boosting the field tensors in parts (b) and (c).

6 Tensor Invariants (10 points)

Let $F^{\mu\nu}$ be the electromagnetic field tensor, and $G^{\mu\nu}$ be the dual tensor.

(a) Evaluate the following Lorentz scalars: $F^{\mu\nu}F_{\mu\nu}$, $F^{\mu\nu}G_{\mu\nu}$ and $G^{\mu\nu}G_{\mu\nu}$, in terms of $\mathbf{E}$ and $\mathbf{B}$.

Recall the field tensor and dual tensor (in natural units):

$$F^{\mu\nu} = \begin{pmatrix} 0 & E^x & E^y & E^z \\ -E^x & 0 & B^z & -B^y \\ -E^y & -B^z & 0 & B^x \\ -E^z & B^y & -B^x & 0 \end{pmatrix}, \quad G^{\mu\nu} = \begin{pmatrix} 0 & B^x & B^y & B^z \\ -B^x & 0 & -E^z & E^y \\ -B^y & E^z & 0 & -E^x \\ -B^z & -E^y & E^x & 0 \end{pmatrix}. \tag{76}$$

Computing each contraction:

$$F^{\mu\nu}F_{\mu\nu} = 2(B^2 - E^2)\,, \tag{77}$$
$$F^{\mu\nu}G_{\mu\nu} = -4\,\mathbf{E}\cdot\mathbf{B}\,. \tag{78}$$

For the dual:

$$G^{\mu\nu}G_{\mu\nu} = 2(E^2 - B^2) = -F^{\mu\nu}F_{\mu\nu}\,. \tag{79}$$

This follows since G is obtained from F by the replacement $\mathbf{E} \to \mathbf{B}$, $\mathbf{B} \to -\mathbf{E}$.

(b) Suppose that in one inertial frame, the magnetic field at a particular point is $\mathbf{B} = 0$, with $\mathbf{E} \neq 0$. Is it possible to find another inertial frame such that the electric field is zero at the same point?

No. The quantity $F^{\mu\nu}F_{\mu\nu} = 2(B^2 - E^2)$ is a Lorentz invariant. In the frame where $\mathbf{B} = 0$:

$$F^{\mu\nu}F_{\mu\nu} = -2E^2 < 0\,. \tag{80}$$

In any other frame with $\mathbf{E}' = 0$, we would need:

$$F^{\mu\nu} F_{\mu\nu} = 2B'^2 > 0\,, \tag{81}$$

which contradicts the invariance of $F^{\mu\nu}F_{\mu\nu}$. Therefore, it is impossible to find a frame in which the electric field vanishes.

7 Maxwell's Equations (10 points)

Let $F^{\mu\nu}$ and $G^{\mu\nu}$ be the electromagnetic field tensor and the dual tensor respectively.

(a) Show that

$$\partial_\nu F^{\mu\nu} = \mu_0 J^\mu\,, \quad \partial_\nu G^{\mu\nu} = 0 \tag{82}$$

is equivalent to Maxwell's equations. Which equations come from which expression?

From $\partial_\nu F^{\mu\nu} = \mu_0 J^\mu$: For $\mu = 0$, we get Gauss's law:

$$\partial_\nu F^{0\nu} = \partial_i F^{0i} = -\partial_i E_i = -\nabla \cdot \mathbf{E} = \mu_0 J^0 = -\frac{\rho}{\epsilon_0}\,. \tag{83}$$

For $\mu = i$ (spatial), we get Ampère Maxwell's eqauation :

$$\partial_\nu F^{i\nu} = \partial_0 F^{i0} + \partial_j F^{ij} = \frac{\partial E_i}{\partial t} + (\nabla \times \mathbf{B})_i = \mu_0 J^i\,. \tag{84}$$

From $\partial_\nu G^{\mu\nu} = 0$: For $\mu = 0$, we get the Gauss law for magnetism:

$$\partial_\nu G^{0\nu} = -\partial_i B_i = -\nabla \cdot \mathbf{B} = 0\,. \tag{85}$$

For $\mu = i$, we get Faraday's law:

$$\partial_\nu G^{i\nu} = \partial_0 G^{i0} + \partial_j G^{ij} = \frac{\partial B_i}{\partial t} + (-\nabla \times \mathbf{E})_i = 0\,. \tag{86}$$

(b) Given that $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, show that $\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$.

Substituting $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$:

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} \tag{87}$$
$$= \partial_\lambda(\partial_\mu A_\nu - \partial_\nu A_\mu) + \partial_\mu(\partial_\nu A_\lambda - \partial_\lambda A_\nu) + \partial_\nu(\partial_\lambda A_\mu - \partial_\mu A_\lambda) \tag{88}$$

Since partial derivatives commute ($\partial_\mu \partial_\nu = \partial_\nu \partial_\mu$), the six terms cancel in pairs:

$$(\partial_\lambda \partial_\mu A_\nu - \partial_\mu \partial_\lambda A_\nu) + (\partial_\nu \partial_\lambda A_\mu - \partial_\lambda \partial_\nu A_\mu) + (\partial_\mu \partial_\nu A_\lambda - \partial_\nu \partial_\mu A_\lambda) = 0\,. \tag{89}$$

(c) Show that $\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$ is equivalent to the same set of Maxwell's equations that you would get from $\partial_\nu G^{\mu\nu} = 0$.

The Bianchi identity $\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$ has nontrivial content only when λ, μ, ν are all different. Since there are four spacetime indices, we choose 3 out of 4, giving $\binom{4}{3} = 4$ independent equations.

When all indices are chosen to be spatial:

$$\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} = \partial_1 B^x + \partial_2 B^y + \partial_3 B^z = \nabla \cdot \mathbf{B} = 0\,. \tag{90}$$

This is the no-monopoles equation.
When one of the indices is chosen to be temporal:

$$\partial_0 F_{12} + \partial_1 F_{20} + \partial_2 F_{01} = -\frac{\partial B^z}{\partial t} - \partial_1 E^y + \partial_2 E^x = 0\,. \tag{91}$$

This gives $(\nabla \times \mathbf{E})_z = -\partial B^z/\partial t$, which is the z-component of Faraday's law. The cases $(0, 2, 3)$ and $(0, 3, 1)$ similarly give the x- and y-components.
These are exactly the two homogeneous Maxwell equations, i.e. the same equations obtained from $\partial_\nu G^{\mu\nu} = 0$.