# Problem Set 10: Special Relativity I

## 1 Rotation Matrices (5 points)

Consider the following linear transformation acting on four-vectors,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1}$$

(a) Describe in words what this transformation corresponds to.

This is a spatial rotation in the $xy$-plane by an angle $\theta$. The time component ($\mu = 0$) and the $z$-component ($\mu = 3$) are left unchanged, while the $x$ and $y$ components mix according to the standard 2D rotation matrix:

$$x' = x\cos\theta + y\sin\theta\,, \qquad y' = -x\sin\theta + y\cos\theta\,. \tag{2}$$

This is a pure spatial rotation with no boost involved.

(b) Check using index notation that this matrix satisfies $\Lambda^\mu{}_\nu \eta_{\mu\rho} \Lambda^\rho{}_\sigma = \eta_{\nu\sigma}$.

The non-zero components of the transformation are:

$$\Lambda^0{}_0 = 1\,, \quad \Lambda^1{}_1 = \cos\theta\,, \quad \Lambda^1{}_2 = \sin\theta\,, \quad \Lambda^2{}_1 = -\sin\theta\,, \quad \Lambda^2{}_2 = \cos\theta\,, \quad \Lambda^3{}_3 = 1\,, \tag{3}$$

and the metric is $\eta_{\mu\rho} = \mathrm{diag}(-1, +1, +1, +1)$. We first contract the first two indices to form $M_{\nu\rho} \equiv \eta_{\mu\rho}\,\Lambda^\mu{}_\nu$. Since $\eta$ is diagonal, this simply lowers the first index with a sign:

$$M_{00} = -1\,, \quad M_{11} = \cos\theta\,, \quad M_{21} = \sin\theta\,, \quad M_{12} = -\sin\theta\,, \quad M_{22} = \cos\theta\,, \quad M_{33} = 1\,. \tag{4}$$

Now we compute $M_{\nu\rho}\,\Lambda^\rho{}_\sigma$ for each $(\nu, \sigma)$. For the diagonal components:

$$M_{00}\Lambda^0{}_0 = (-1)(1) = -1 = \eta_{00}\,, \tag{5}$$

$$M_{11}\Lambda^1{}_1 + M_{12}\Lambda^2{}_1 = \cos^2\theta + \sin^2\theta = 1 = \eta_{11}\,, \tag{6}$$

$$M_{21}\Lambda^1{}_2 + M_{22}\Lambda^2{}_2 = \sin^2\theta + \cos^2\theta = 1 = \eta_{22}\,, \tag{7}$$

$$M_{33}\Lambda^3{}_3 = (1)(1) = 1 = \eta_{33}\,. \tag{8}$$

For the only non-trivial off-diagonal components:

$$M_{11}\Lambda^1{}_2 + M_{12}\Lambda^2{}_2 = \cos\theta\sin\theta - \sin\theta\cos\theta = 0 = \eta_{12}\,, \tag{9}$$

$$M_{21}\Lambda^1{}_1 + M_{22}\Lambda^2{}_1 = \sin\theta\cos\theta - \cos\theta\sin\theta = 0 = \eta_{21}\,. \tag{10}$$

All remaining off-diagonal components vanish trivially since the relevant $M_{\nu\rho}$ or $\Lambda^\rho{}_\sigma$ are zero. Hence, $\Lambda^\mu{}_\nu \eta_{\mu\rho} \Lambda^\rho{}_\sigma = \eta_{\nu\sigma}$.

## 2 Lorentz Transformations (10 points)

Consider a rest frame $S$.

(a) Write out the matrix that describes a *Galilean transformation* for a frame moving in the positive $x$-direction with speed $v$.

In a Galilean transformation, time is absolute ($t' = t$) and the coordinates transform as $x' = x - vt$. In matrix form:

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -v & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \tag{11}$$

(b) Write out the matrix describing a Lorentz transformation into another frame moving with speed $+v$ along the $y$-axis.

This Lorentz transformation mixes the $t$ and $y$ components, leaving $x$ and $z$ unchanged. Using $\gamma = 1/\sqrt{1-v^2}$:

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & 0 & -\gamma v & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma v & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{12}$$

(c) Find the matrix describing a Lorentz transformation to a frame $S'$ with velocity $+v$ along the $x$-axis, followed by a Lorentz transformation from $S'$ to another frame $S''$ with velocity $+v$ along the $y$-axis. Does it matter in what order the transformations are carried out?

The first Lorentz transformation to $S'$ with velocity $v$ along $x$-axis is given by:

$$\bar{\Lambda}^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{13}$$

We then go to $S''$ by following the above transformation with the transformation we found in part(b):

$$\begin{aligned} \Lambda^\mu{}_\chi \bar{\Lambda}^\chi{}_\mu &= \begin{pmatrix} \gamma & 0 & -\gamma v & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma v & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \gamma^2 & -\gamma^2 v & -\gamma v & 0 \\ -\gamma v & \gamma & 0 & 0 \\ -\gamma^2 v & \gamma^2 v^2 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{14}$$

If we reversed the order, the corresponding Lorentz transformation is given by:

$$\begin{aligned} \bar{\Lambda}^\mu{}_\chi \Lambda^\chi{}_\mu &= \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & 0 & -\gamma v & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma v & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \gamma^2 & -\gamma v & -\gamma^2 v & 0 \\ -\gamma^2 v & \gamma & \gamma^2 v^2 & 0 \\ -\gamma v & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{15}$$

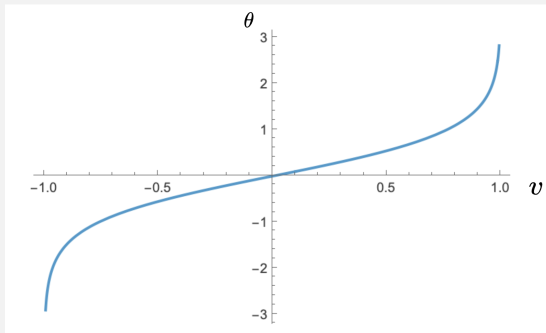
These two matrices are clearly not the same, so the order matters.

## 3 Rapidity and Generalized Rotations (10 points)

Let's define the **rapidity** as

$$\theta = \tanh^{-1} v\,. \tag{16}$$

(a) Make a plot of $\theta$ vs. $v$.




(b) Write down the Lorentz transformation $\Lambda$ going from frame $S$ to $S'$, where $S'$ is moving with velocity $-v_1\hat{x}$, in terms of $\theta_1 = \tanh^{-1} v_1$.

Using $\tanh\theta_1 = v_1$, we can write $\gamma_1 = \cosh\theta_1$ and $\gamma_1 v_1 = \sinh\theta_1$. The Lorentz transformation is:

$$\Lambda_1 = \begin{pmatrix} \cosh\theta_1 & \sinh\theta_1 & 0 & 0 \\ \sinh\theta_1 & \cosh\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{17}$$

(c) Now consider another frame $S''$, moving at velocity $-v_2\hat{x}'$ with respect to $S'$. Express the Lorentz transformation in going from $S$ to $S''$, in terms of $\theta_1$ and $\theta_2 = \tanh^{-1} v_2$.

The transformation from $S'$ to $S''$ has the same form as part (b), but with $\theta_2$:

$$\Lambda_2 = \begin{pmatrix} \cosh\theta_2 & \sinh\theta_2 & 0 & 0 \\ \sinh\theta_2 & \cosh\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{18}$$

The combined transformation from $S$ to $S''$ is $\Lambda = \Lambda_2\Lambda_1$:

$$\begin{aligned}\Lambda &= \begin{pmatrix} \cosh\theta_2 & \sinh\theta_2 & 0 & 0 \\ \sinh\theta_2 & \cosh\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh\theta_1 & \sinh\theta_1 & 0 & 0 \\ \sinh\theta_1 & \cosh\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cosh\theta_1\cosh\theta_2 + \sinh\theta_1\sinh\theta_2 & \sinh\theta_1\cosh\theta_2 + \cosh\theta_1\sinh\theta_2 & 0 & 0 \\ \sinh\theta_1\cosh\theta_2 + \cosh\theta_1\sinh\theta_2 & \cosh\theta_1\cosh\theta_2 + \sinh\theta_1\sinh\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\end{aligned} \tag{19}$$

(d) Show that rapidity adds, i.e. to transform from $S$ to $S''$, we should use a Lorentz transformation matrix $\Lambda$ in part (a), but with rapidity $\theta = \theta_1 + \theta_2$ instead.

Using the hyperbolic addition formulae:

$$\cosh(\theta_1 + \theta_2) = \cosh\theta_1 \cosh\theta_2 + \sinh\theta_1 \sinh\theta_2 \,, \tag{20}$$
$$\sinh(\theta_1 + \theta_2) = \sinh\theta_1 \cosh\theta_2 + \cosh\theta_1 \sinh\theta_2 \,, \tag{21}$$

we see that the combined matrix from part (c) becomes:

$$\Lambda = \begin{pmatrix} \cosh(\theta_1+\theta_2) & \sinh(\theta_1+\theta_2) & 0 & 0 \\ \sinh(\theta_1+\theta_2) & \cosh(\theta_1+\theta_2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{22}$$

This is exactly the matrix with rapidity $\theta = \theta_1 + \theta_2$. Therefore, rapidities add under composition of collinear boosts. This is the relativistic analogue of how angles add under composition of rotations about the same axis.

## 4 Getting to Alpha Centauri (5 points)

Alpha Centauri is one of the nearest stars to us, located approximately 4 light years away. Suppose we decide to launch a crewed spacecraft to Alpha Centauri, traveling at velocity $v$, and take the path to be a simple straight line path. Compute the following:

(a) The time taken, the distance traveled and the spacetime interval for the journey from the Earth's frame.

In the Earth frame, the distance is $\Delta x = 4$ light years and the spacecraft travels at speed $v$, so:

$$\Delta t = \frac{\Delta x}{v} = \frac{4}{v} \text{ years} \,. \tag{23}$$

The spacetime interval is :

$$\Delta s^2 = -(\Delta t)^2 + (\Delta x)^2 = -\frac{16}{v^2} + 16 = -16\left(\frac{1-v^2}{v^2}\right) \,. \tag{24}$$

Since $v < 1$, we have $\Delta s^2 < 0$, confirming that the journey is timelike, as expected for a massive particle.

(b) The spacetime interval, distance traveled and time taken for the journey in the spacecraft's frame.

The spacetime interval is a Lorentz invariant, so it is the same in both frames:

$$\Delta s^2 = -16\left(\frac{1-v^2}{v^2}\right) \,. \tag{25}$$

In the spacecraft frame, the spacecraft is at rest, so $\Delta x' = 0$. Therefore:

$$\Delta s^2 = -(\Delta t')^2 + 0 \implies \Delta t' = \frac{4}{v}\sqrt{1-v^2} = \frac{4}{\gamma v} \text{ years} \,. \tag{26}$$

This is simply the time-dilated result: $\Delta t' = \Delta t / \gamma$.

(c) What is the time taken and distance traveled when $v = 0.9$, $v = 0.99$ and $v = 0.999$?

Using the expressions from parts (a) and (b):
Earth frame: $\Delta t = 4/v$ years, $\Delta x = 4$ light years.
Spacecraft frame: $\Delta t' = 4\sqrt{1-v^2}/v$ years, $\Delta x' = 0$.
For $v = 0.9$: $\gamma \approx 2.294$,

$$\Delta t = \frac{4}{0.9} \approx 4.44 \text{ years}\,, \qquad \Delta t' = \frac{4\sqrt{1-0.81}}{0.9} \approx 1.94 \text{ years}\,. \tag{27}$$

For $v = 0.99$: $\gamma \approx 7.089$,

$$\Delta t = \frac{4}{0.99} \approx 4.04 \text{ years}\,, \qquad \Delta t' = \frac{4\sqrt{1-0.9801}}{0.99} \approx 0.570 \text{ years}\,. \tag{28}$$

For $v = 0.999$: $\gamma \approx 22.37$,

$$\Delta t = \frac{4}{0.999} \approx 4.004 \text{ years}\,, \qquad \Delta t' = \frac{4\sqrt{1-0.998001}}{0.999} \approx 0.179 \text{ years}\,. \tag{29}$$

As $v \to 1$, the Earth-frame travel time approaches 4 years (the light travel time), while the proper time experienced by the crew shrinks toward zero.

## 5 More Spaceships (5 points)

A rocket ship leaves Earth at a speed of 0.6. When a clock on the rocket says 1 hour has elapsed, the rocket ship sends a light signal back to Earth.

(a) According to Earth clocks, when was the signal sent?

The rocket clock measures proper time. The time dilation factor is $\gamma = 1/\sqrt{1-0.36} = 1/\sqrt{0.64} = 1/0.8 = 5/4$. One hour on the rocket clock corresponds to:

$$\Delta t_{\text{Earth}} = \gamma \cdot 1 \text{ hr} = \frac{5}{4} \text{ hr} = 1.25 \text{ hrs}\,. \tag{30}$$

So, according to Earth clocks, the signal was sent 1.25 hours after the rocket left.

(b) According to Earth clocks, how long after the rocket left did the signal arrive back on Earth?

At the moment the signal is sent (Earth time $t = 1.25$ hr), the rocket is at a distance:

$$d = v \cdot t = 0.6 \times 1.25 = 0.75 \text{ light-hours} \tag{31}$$

from Earth. The light signal travels back at speed 1, and it takes an additional 0.75 hr to reach Earth. The total time since the rocket left is:

$$\Delta t_{\text{total}} = 1.25 + 0.75 = 2.0 \text{ hrs}\,. \tag{32}$$

(c) According to the rocket observer, how long after the rocket left did the signal arrive back on Earth?

Both events, the rocket leaving and the signal arriving back on Earth, are at the same position in the Earth frame. Hence, the time interval we found in part(b) is the proper time for these events. For the rocket observer, these events would occur in a time dilated interval

$$\Delta t_{\text{rocket}} = \gamma \Delta t_{\text{total}} = \frac{5}{4} 2 = 2.5 \text{ hrs} . \tag{33}$$

## 6 Spacelike and Timelike (10 points)

Event $A$ happens at point $(t_A, x_A, y_A, z_A) = (15, 5, 3, 0)$; event $B$ occurs at $(t_B, x_B, y_B, z_B) = (5, 10, 8, 0)$, both coordinates in frame $S$.

(a) What is the invariant interval between $A$ and $B$?

The spacetime interval is:

$$\begin{aligned} \Delta s^2 &= -(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \\ &= -(15-5)^2 + (5-10)^2 + (3-8)^2 + 0 \\ &= -100 + 25 + 25 \\ &= -50 . \end{aligned} \tag{34}$$

Since $\Delta s^2 < 0$, the interval is **timelike**.

(b) Is there an inertial frame in which they occur simultaneously? If so, find its velocity (magnitude and direction) relative to $S$.

For two events to occur simultaneously in some frame, we need the interval to be spacelike ($\Delta s^2 > 0$). Since $\Delta s^2 = -50 < 0$, the interval is timelike, and there is no inertial frame in which the two events are simultaneous.

(c) Is there an inertial frame in which they occur at the same point? If so, find its velocity relative to $S$.

Since the interval is timelike, there does exist a frame in which the two events occur at the same spatial point. In that frame, the observer simply travels from $A$ to $B$ (or vice versa). The required velocity is:

$$\mathbf{v} = \frac{\Delta \mathbf{r}}{\Delta t} = \frac{(x_B - x_A,\, y_B - y_A,\, z_B - z_A)}{t_B - t_A} = \frac{(5, 5, 0)}{-10} = (-0.5, -0.5, 0) . \tag{35}$$

(d) Repeat parts (a)–(c) for $A = (1, 2, 0, 0)$ and $B = (3, 5, 0, 0)$.

The invariant interval is:

$$\Delta s^2 = -(3-1)^2 + (5-2)^2 = -4 + 9 = 5 . \tag{36}$$

Since $\Delta s^2 > 0$, the interval is **spacelike**.
There does exist a frame in which the two events are simultaneous. We need a velocity along the $x$-axis (since $\Delta y = \Delta z = 0$) such that $\Delta t' = 0$:

$$\Delta t' = \gamma(\Delta t - v\Delta x) = 0 \implies v = \frac{\Delta t}{\Delta x} = \frac{2}{3} . \tag{37}$$

So a frame moving at $v = 2/3$ in the $+x$-direction will see the two events as simultaneous.
Since the interval is spacelike, there is no inertial frame in which the two events occur at the same spatial location.