Midterm 2 (Total Time: 50 min)

Question	**Max. Points**
Q1	30
Q2	10
Q4	10
Q3	25
Total	**75**

1. Do not turn the page until you are instructed to do so.
2. Fill in your name, and for subject please indicate "PY406 Spring 2026 Midterm 2". All other fields can be left blank.
3. **All of your work must be written in the blue exam book.** This document only contains the questions, and will **not** be graded.
4. Hand your blue exam booklet in at the end of the exam and before you leave the room.
5. This exam contains **8** pages (including this cover page) and **4** problems.
6. This is a closed book exam. No books, notes or calculators allowed.
7. A formula sheet is provided to you on the page behind this.
8. Answers without supporting work will receive no credit. **Conversely, supporting work will be given plenty of partial credit, even if the final answer is incorrect.**
9. You will be graded based solely on what the grader can understand of your solutions.
10. Points allocated to each question are indicated in brackets. **Please plan your time appropriately.**

PY406 Formula Sheet — Midterm 2

Vector Identities

Triple Products

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$$
$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

Product Rules

$$\begin{aligned}
\nabla(fg) &= f(\nabla g) + g(\nabla f)\\
\nabla(\mathbf{A}\cdot\mathbf{B}) &= \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A})\\
&\quad + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}\\
\nabla\cdot(f\mathbf{A}) &= f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)\\
\nabla\cdot(\mathbf{A}\times\mathbf{B}) &= \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})\\
\nabla\times(f\mathbf{A}) &= f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)\\
\nabla\times(\mathbf{A}\times\mathbf{B}) &= (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B}\\
&\quad + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})
\end{aligned}$$

Second Derivatives

$$\begin{aligned}
\nabla\cdot(\nabla\times\mathbf{A}) &= 0\\
\nabla\times(\nabla f) &= 0\\
\nabla\times(\nabla\times\mathbf{A}) &= \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}
\end{aligned}$$

Vector Calculus with Delta Functions

$$\nabla^2\frac{1}{|\mathbf{r}-\mathbf{r}'|} = -4\pi\,\delta^3(\mathbf{r}-\mathbf{r}')$$
$$\nabla\cdot\left(\frac{\hat{\mathbf{r}}}{r^2}\right) = 4\pi\,\delta^3(\mathbf{r})$$

Fundamental Theorems

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathrm{d}\mathbf{l}\cdot(\nabla f) = f(\mathbf{b}) - f(\mathbf{a})$$
$$\int_{\mathcal{V}} \mathrm{d}^3\boldsymbol{r}\,(\nabla\cdot\mathbf{A}) = \oint_{\partial\mathcal{V}} \mathrm{d}\mathbf{S}\cdot\mathbf{A}$$
$$\int_{\mathcal{S}} \mathrm{d}\mathbf{S}\cdot(\nabla\times\mathbf{A}) = \oint_{\partial\mathcal{S}} \mathrm{d}\mathbf{l}\cdot\mathbf{A}$$

Cartesian Coordinates

$\mathrm{d}\mathbf{l} = \mathrm{d}x\,\hat{x} + \mathrm{d}y\,\hat{y} + \mathrm{d}z\,\hat{z}$, $\mathrm{d}^3\boldsymbol{r} = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$

$$\begin{aligned}
\nabla f &= \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}\\
\nabla\cdot\mathbf{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\\
\nabla\times\mathbf{A} &= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{y}\\
&\quad + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z}\\
\nabla^2 f &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}
\end{aligned}$$

Cylindrical Coordinates

$\mathrm{d}\mathbf{l} = \mathrm{d}s\,\hat{s} + s\,\mathrm{d}\phi\,\hat{\phi} + \mathrm{d}z\,\hat{z}$, $\mathrm{d}^3\boldsymbol{r} = s\,\mathrm{d}s\,\mathrm{d}\phi\,\mathrm{d}z$
$\hat{s}\times\hat{\theta} = \hat{z}$

$$\begin{aligned}
\hat{s} &= \cos\phi\,\hat{x} + \sin\phi\,\hat{y}\\
\hat{\phi} &= -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}
\end{aligned}$$

$$\begin{aligned}
\nabla f &= \frac{\partial f}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial f}{\partial\phi}\hat{\phi} + \frac{\partial f}{\partial z}\hat{z}\\
\nabla\cdot\mathbf{A} &= \frac{1}{s}\frac{\partial(sA_s)}{\partial s} + \frac{1}{s}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}\\
\nabla\times\mathbf{A} &= \left(\frac{1}{s}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{s}\\
&\quad + \left(\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}\right)\hat{\phi}\\
&\quad + \frac{1}{s}\left(\frac{\partial(sA_\phi)}{\partial s} - \frac{\partial A_s}{\partial\phi}\right)\hat{z}\\
\nabla^2 f &= \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial f}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 f}{\partial\phi^2} + \frac{\partial^2 f}{\partial z^2}
\end{aligned}$$

Spherical Coordinates

$\mathrm{d}\mathbf{l} = \mathrm{d}r\,\hat{r} + r\,\mathrm{d}\theta\,\hat{\theta} + r\sin\theta\,\mathrm{d}\phi\,\hat{\phi}$, $\mathrm{d}^3\boldsymbol{r} = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$
$\hat{r}\times\hat{\theta} = \hat{\phi}$

$$\begin{aligned}
\hat{r} &= \cos\phi\sin\theta\,\hat{x} + \sin\phi\sin\theta\,\hat{y} + \cos\theta\,\hat{z}\\
\hat{\theta} &= \cos\phi\cos\theta\,\hat{x} + \sin\phi\cos\theta\,\hat{y} - \sin\theta\,\hat{z}\\
\hat{\phi} &= -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}
\end{aligned}$$

$$\begin{aligned}
\nabla f &= \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\hat{\phi}\\
\nabla\cdot\mathbf{A} &= \frac{1}{r^2}\frac{\partial(r^2A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\,A_\theta)}{\partial\theta}\\
&\quad + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}\\
\nabla\times\mathbf{A} &= \frac{1}{r\sin\theta}\left(\frac{\partial(\sin\theta\,A_\phi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right)\hat{r}\\
&\quad + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial(rA_\phi)}{\partial r}\right)\hat{\theta}\\
&\quad + \frac{1}{r}\left(\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)\hat{\phi}\\
\nabla^2 f &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)\\
&\quad + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}
\end{aligned}$$

Index Notation

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is even permutation of } (1,2,3)\\ -1 & \text{if } (i,j,k) \text{ is odd permutation of } (1,2,3)\\ 0 & \text{if any two indices are equal}\end{cases}$$

$$\begin{aligned}
\epsilon_{ijk}\epsilon^{ilm} &= \delta^l_j\delta^m_k - \delta^m_j\delta^l_k\\
\mathbf{A}\cdot\mathbf{B} &= A^iB_i\\
(\mathbf{A}\times\mathbf{B})_i &= \epsilon_{ijk}A^jB^k
\end{aligned}$$

Maxwell's Equations

Differential form ($\mathbf{E}$ and $\mathbf{B}$)

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$
$$\nabla\cdot\mathbf{B} = 0 \qquad \nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

Differential form ($\mathbf{D}$, $\mathbf{E}$ and $\mathbf{B}$, $\mathbf{H}$)

$$\nabla\cdot\mathbf{D} = \rho_f \qquad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$
$$\nabla\cdot\mathbf{B} = 0 \qquad \nabla\times\mathbf{H} = \mathbf{J}_f + \frac{\partial\mathbf{D}}{\partial t}$$

where

$$\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P} \qquad \mathbf{H} \equiv \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}$$

$\mathbf{P}$ is the polarization and $\mathbf{M}$ is the magnetization, defined via

$$\rho_b = -\nabla \cdot \mathbf{P} \qquad \mathbf{J}_b = \nabla \times \mathbf{M}$$

In linear media: $\mathbf{D} = \epsilon \mathbf{E}$, $\mathbf{B} = \mu \mathbf{H}$.

Lorentz Force Law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \qquad \text{(point charge)}$$
$$\mathbf{f} = \rho \mathbf{E} + \mathbf{J} \times \mathbf{B} \qquad \text{(continuous)}$$

Boundary Conditions

With no free charges or currents on the surface,

$$\epsilon_1 E_1^\perp = D_1 = D_2 = \epsilon_2 E_2^\perp \,, \qquad B_1^\perp = B_2^\perp \,,$$
$$\mathbf{E}_1^\parallel = \mathbf{E}_2^\parallel \,, \qquad \frac{1}{\mu_1}\mathbf{B}_1^\parallel = \mathbf{H}_1^\parallel = \mathbf{H}_2^\parallel = \frac{1}{\mu_2}\mathbf{B}_2^\parallel \,,$$

Conservation Laws

Charge

Differential form:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

Energy

Total energy for electrostatic charge distribution $\rho(\boldsymbol{r})$:

$$U = \frac{1}{2}\int \mathrm{d}^3\boldsymbol{r}\, \rho(\boldsymbol{r}) V(\boldsymbol{r})$$

Energy density of EM fields and Poynting vector:

$$u_{\mathrm{EM}} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) \qquad \mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$$

Mechanical work done per unit time on charges:

$$\frac{\partial u_{\mathrm{mech}}}{\partial t} = \mathbf{J} \cdot \mathbf{E}$$

Differential form:

$$\frac{\partial}{\partial t}\left(u_{\mathrm{EM}} + u_{\mathrm{mech}}\right) + \nabla \cdot \mathbf{S} = 0$$

Momentum

Momentum density:

$$\boldsymbol{\wp}_{\mathrm{EM}} = \mu_0 \epsilon_0 \mathbf{S} = \epsilon_0 (\mathbf{E} \times \mathbf{B})$$

Force density on charges:

$$\boldsymbol{\wp}_{\mathrm{mech}} = \mathbf{f} = \rho \mathbf{E} + \mathbf{J} \times \mathbf{B}$$

Maxwell stress tensor:

$$T_{ij} = \epsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2\right) + \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij} B^2\right)$$

Differential form:

$$\frac{\partial}{\partial t}\left(\wp^i_{\mathrm{EM}} + \wp^i_{\mathrm{mech}}\right) - \partial_j T^{ij} = 0$$

Integral form:

$$F^i + \epsilon_0 \mu_0 \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}} \mathrm{d}^3\boldsymbol{r}\, S^i - \oint_{\partial \mathcal{V}} \mathrm{d}A_j\, T^{ij} = 0$$

Angular Momentum

Angular momentum density of EM fields:

$$\boldsymbol{\ell}_{\mathrm{EM}} = \mathbf{r} \times \boldsymbol{\wp}_{\mathrm{EM}} = \epsilon_0\, \mathbf{r} \times (\mathbf{E} \times \mathbf{B})$$

Electromagnetic Waves

Wave Equation

For electromagnetic waves in linear media,

$$\nabla^2 \mathbf{E} - \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

with μ and ϵ being the permeability and permittivity of the medium, respectively. The wave equation also holds for $\mathbf{B}$ for EM waves. The wave speed and refractive index are

$$v = \frac{1}{\sqrt{\mu\epsilon}} \,, \qquad n \equiv \frac{c}{v}$$

Multiplying Complex Exponentials

Since only the real part of complex exponentials is physical, we need to be careful about multiplying them. For f and g which are real sinusoids represented in complex exponential form as F and G,

$$f \cdot g = \frac{1}{2}(F \cdot G + F \cdot G^*) \,,$$

where taking the real part on the RHS is implied. For time-averages of two sinusoids with the same frequency ω,

$$\langle f \cdot g \rangle = \frac{1}{2} F \cdot G^*$$

Monochromatic Plane Waves

$$\mathbf{E} = E_0\, \hat{\mathbf{n}}\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$
$$\mathbf{B} = \frac{E_0}{v}\,(\hat{\mathbf{k}} \times \hat{\mathbf{n}})\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

with $\hat{\mathbf{n}}$ the polarization direction, $\hat{\mathbf{k}}$ the propagation direction, $\omega = vk$. Time-averaged quantities:

$$\langle u \rangle = \frac{1}{2}\epsilon |E_0|^2 \qquad \langle \mathbf{S} \rangle = \frac{1}{2} v\epsilon |E_0|^2\, \hat{\mathbf{k}} \qquad \langle \boldsymbol{\wp} \rangle = \frac{1}{2}\frac{\epsilon}{v}|E_0|^2\, \hat{\mathbf{k}}$$

Incidence at Interface

For oblique incidence from medium 1 with μ_1, ϵ_1 to medium 2 with μ_2, ϵ_2, with angles of incidence, reflection and transmission $\theta_I, \theta_R, \theta_T$ defined with respect to the normal to the interface, we have the following results:

Law of reflection and Snell's law:

$$\theta_I = \theta_R \qquad n_1 \sin\theta_I = n_2 \sin\theta_T \,.$$

Defining

$$\alpha \equiv \frac{\cos\theta_T}{\cos\theta_I} \,, \qquad \beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} \tag{1}$$

Fresnel equations for $\mathbf{E} \perp$ plane of incidence:

$$E_{0,R} = \left(\frac{1 - \alpha\beta}{1 + \alpha\beta}\right) E_{0,I}$$
$$E_{0,T} = \left(\frac{2}{1 + \alpha\beta}\right) E_{0,I}$$

Fresnel equations for $\mathbf{E} \parallel$ plane of incidence:

$$E_{0,R} = \left(\frac{\alpha - \beta}{\alpha + \beta}\right) E_{0,I}$$
$$E_{0,T} = \left(\frac{2}{\alpha + \beta}\right) E_{0,I}$$

In $\mathbf{E} \parallel$ plane of incidence, the Brewster angle θ_B is the incident angle at which there is no reflected wave. It is approximately given by

$$\tan\theta_B = \frac{n_2}{n_1}$$

Reflection and transmission coefficients:

$$R = \left(\frac{E_{0,R}}{E_{0,I}}\right)^2 \qquad T = \frac{n_2\cos\theta_T}{n_1\cos\theta_I}\left(\frac{E_{0,T}}{E_{0,I}}\right)^2$$

EM Waves in Conductors

In a conductor with conductivity σ, Ohm's law gives $\mathbf{J}_f = \sigma\mathbf{E}$. The wave equation becomes

$$\nabla^2\mathbf{E} = \mu\epsilon\frac{\partial^2\mathbf{E}}{\partial t^2} + \mu\sigma\frac{\partial\mathbf{E}}{\partial t}$$

and similar for $\mathbf{B}$. Plane wave solution: $\mathbf{E} = E_0\hat{\mathbf{n}}e^{i(\tilde{k}z-\omega t)}$ with complex wave number

$$\tilde{k} \equiv k + i\kappa$$

where

$$k = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}+1\right]^{1/2}$$
$$\kappa = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}-1\right]^{1/2}$$

Skin depth:

$$d = \frac{1}{\kappa}$$

The magnetic field is

$$\mathbf{B} = \frac{\tilde{k}}{\omega}(\hat{\mathbf{k}}\times\mathbf{E})$$

For reflection and transmission at the interface of a linear medium (medium 1) and a conductor (medium 2), the same expressions apply, except that we replace β with

$$\tilde{\beta} = \frac{\mu_1 v_1}{\mu_2\omega}\tilde{k}_2\,.$$

Waveguides

Fields in waveguides with a constant cross section oriented along the z-axis are of the form

$$\mathbf{E}(\boldsymbol{r},t) = \mathbf{E}_0(x,y)e^{i(kz-\omega t)}$$
$$\mathbf{B}(\boldsymbol{r},t) = \mathbf{B}_0(x,y)e^{i(kz-\omega t)}\,,$$

with $\mathbf{E}_0$ and $\mathbf{B}_0$ satisfying the Helmholtz equation,

$$\left(\nabla_T^2 - k^2 + \frac{\omega^2}{c^2}\right)\mathbf{E}_0 = 0\,, \tag{2}$$

and likewise for $\mathbf{B}_0$, where ∇_T^2 is the transverse Laplacian.

The x- and y-components of $\mathbf{E}$ and $\mathbf{B}$ can be expressed in terms of the z-components as follows:

$$E_x = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial E_z}{\partial x} + \omega\frac{\partial B_z}{\partial y}\right)$$
$$E_y = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial E_z}{\partial y} - \omega\frac{\partial B_z}{\partial x}\right)$$
$$B_x = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial B_z}{\partial x} - \frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\right)$$
$$B_y = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial B_z}{\partial y} + \frac{\omega}{c^2}\frac{\partial E_z}{\partial x}\right)$$

For a rectangular waveguide of sides a and b with $a > b$, the TE_{mn}-mode solution is given by

$$B_{0,z}(x,y) = B_0\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\,.$$

The TM_{mn}-mode solution is given by

$$E_{0,z}(x,y) = E_0\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\,.$$

The dispersion relation is

$$k^2 = \frac{\omega^2}{c^2} - \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)\,.$$

Potentials

The fields in terms of potentials:

$$\mathbf{B} = \nabla\times\mathbf{A} \qquad \mathbf{E} = -\nabla V - \frac{\partial\mathbf{A}}{\partial t}$$

Gauge Transformations

For any scalar $\lambda(\boldsymbol{r},t)$, the gauge transformation:

$$\mathbf{A}' = \mathbf{A} + \nabla\lambda \qquad V' = V - \frac{\partial\lambda}{\partial t}$$

gives the same $\mathbf{E}$ and $\mathbf{B}$.

1 A Spy Beacon (30 points)

A secret underwater beacon is placed on the floor of a lake, with the surface of the lake forming a plane interface with air on the outside. The beacon emits a polarized laser beam upward at an angle toward the water-air interface. Let the refractive index of water be $n = 4/3$. For parts (a)–(c), treat water as a dielectric (i.e. $\sigma = 0$).

(a) *(5 points)* The beam hits the water-air interface at $30°$ to the normal, going from water to air. At what angle of incidence should a detector receive the signal? You may leave your answer in terms of trigonometric functions, and you may find the following useful: $\sin 30° = 1/2$, $\cos 30° = \sqrt{3}/2$, $\tan 30° = 1/\sqrt{3}$)

> Using Snell's law, we have:
>
> $$\sin\theta_T = \frac{4}{3}\sin(30°)$$
>
> $$\theta_T = \sin^{-1}\left(\frac{2}{3}\right) \tag{3}$$

(b) *(5 points)* Suppose the sensor is polarized such that the electric field is parallel to the plane of incidence. What should the angle of incidence be in order for there to be no reflection at the interface? You may leave your answer in terms of trigonometric functions.

> The Brewster's angle is going from water to air:
>
> $$\tan\theta_B = \frac{n_2}{n_1} = \frac{3}{4} \tag{4}$$
>
> $$\theta_B = \tan^{-1}\left(\frac{3}{4}\right) \tag{5}$$

(c) *(5 points)* A curious graduate student changes the sensor polarization such that the electric field is now orthogonal to the plane of incidence. Show that for no reflection to occur at the interface, the angle of incidence θ_I and the angle of transmission θ_T must satisfy

$$\cos\theta_I = \frac{1}{n}\cos\theta_T\,. \tag{6}$$

> In this case, the Fresnel equations show that we require
>
> $$1 - \alpha\beta = 0 \tag{7}$$
>
> with
>
> $$\alpha = \frac{\cos\theta_T}{\cos\theta_I} \quad \text{and} \quad \beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{1}{n}. \tag{8}$$
>
> This can be rewritten as:
>
> $$1 = \alpha\beta = \frac{\cos\theta_T}{\cos\theta_I}\frac{1}{n} \tag{9}$$
>
> Rearranging, we arrive at the required expression.

(d) *(5 points)* Hence, show that there is no possible incident angle θ_I for which there is no reflection in this case.

Rewriting the expression in terms of $\sin\theta_I$ and $\sin\theta_T$, we have

$$\sqrt{1-\sin^2\theta_I} = \frac{1}{n}\sqrt{1-\sin^2\theta_T} = \frac{1}{n}\sqrt{1-n^2\sin^2\theta_I}\,, \tag{10}$$

where the last step follows from Snell's law. Squaring both sides, we find

$$n^2 - n^2\sin^2\theta_I = 1 - n^2\sin^2\theta_I\,, \tag{11}$$

which is only satisfied if $n = 1$. Since we are considering water with $n = 4/3$, this can never be satisfied, and so there is no angle θ_I for which there is any reflection (in fact, it satisfied only in the trivial case where there is no boundary, i.e. with $n_1 = n_2$).

(e) *(10 points)* A navy general wants to put this beacon at the bottom of the Atlantic Ocean at a depth of 1 km, directed vertically upward. The conductivity of seawater is much higher than freshwater, and for this problem can be taken to be $\sigma = (1/2\pi)\,\Omega^{-1}\,\mathrm{m}^{-1}$. The operating frequency of the beacon is $\omega = 10^7\,\mathrm{s}^{-1}$. Suppose the beacon can emit a beam with some power P_0. Estimate the power that you would receive at the surface of the ocean, and discuss the feasibility of receiving signals from this beacon. You may assume that $\sigma \gg \epsilon\omega$, $\mu = \mu_0 = 4\pi\times10^{-7}\,\mathrm{H\,m^{-1}}$ in SI units, and you may leave your final answer as a power of e.

Since we may assume $\sigma \gg \epsilon\omega$, we have

$$\kappa = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}-1\right]^{1/2} \approx \omega\sqrt{\frac{\mu\epsilon}{2}}\sqrt{\frac{\sigma}{\epsilon\omega}} = \sqrt{\frac{\omega\mu_0\sigma}{2}} \tag{12}$$

Plugging in numbers, we get

$$\kappa = \sqrt{\frac{1}{2}(10^7)(4\pi\times10^{-7})\frac{1}{2\pi}}\,\mathrm{m}^{-1} = 1\,\mathrm{m}^{-1} \tag{13}$$

Over a distance $d = 1\,\mathrm{km}$, the power at the surface is attenuated as:

$$P = P_0 e^{-2\kappa d} = P_0 e^{-2000}\,, \tag{14}$$

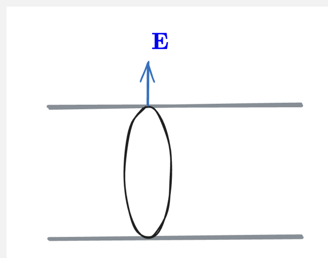
which for all practical values of P_0 is surely too small to be detected.

2 Circular Waveguides (10 points)

Consider a waveguide with a circular cross section. Note that the waveguide has cylindrical symmetry about the axis. As with rectangular waveguides, different TE and TM modes exist for electromagnetic waves propagating down such a waveguide.

(a) *(5 points)* Consider a mode where the electric field on the conducting surface of the waveguide is nonzero. Draw a circle to represent the cross section of the waveguide, and indicate the direction of the transverse electric field at the conducting surface for such a mode.

On a conducting surface, we require $\mathbf{E}_{\parallel} = 0$. This implies that the only non-zero component of the electric field should point along the radial direction: $\mathbf{E} = E_0\,\hat{\mathbf{r}}$.



(b) *(5 points)* Which components of the magnetic field, given in cylindrical coordinates, can be nonzero? Explain your answer.

For the magnetic field, we require $B_{\perp} = 0$ on the conducting surface. On the cylindrical surface, this implies that the radial component is zero. The other two components, B_z and B_{ϕ}, are allowed to be non-zero.

3 A Square Waveguide (10 points)

A radio astronomer would like to design a square waveguide that would allow a 600 MHz signal to propagate down the waveguide. Determine the minimum side length a for the signal to propagate in a TM-mode. Take $c = 3 \times 10^8\,\mathrm{m\,s^{-1}}$.

The cutoff frequencies for a square waveguide are given by:

$$\omega_{mn} = \frac{c\pi}{a}\sqrt{m^2 + n^2} \tag{15}$$

For a non-trivial TM-mode, we require $m, n \geq 1$. This implies that the minimum side length obtained by setting

$$a = \frac{c\pi\sqrt{2}}{\omega_{11}} = \frac{3 \times 10^8 \times \pi \times \sqrt{2}}{2\pi \times 600 \times 10^6}\,\mathrm{m} = \frac{1}{2\sqrt{2}}\,\mathrm{m} \tag{16}$$

4 Potentials of a Solenoidal Magnetic Field (25 points)

Consider an infinitely solenoid of radius R carrying a time-dependent current that produces a uniform magnetic field inside the solenoid, but zero magnetic field outside:

$$\mathbf{B}(t) = \begin{cases} B(t)\hat{\mathbf{z}}\,, & r < R\,, \\ 0\,, & r \geq R\,. \end{cases} \tag{17}$$

(a) *(10 points)* Show that the electric field is given by

$$\mathbf{E} = \begin{cases} -\frac{1}{2}r\dot{B}\hat{\boldsymbol{\phi}}\,, & r < R\,, \\ -\frac{R^2}{2r}\dot{B}\hat{\boldsymbol{\phi}}\,, & r \geq R\,, \end{cases} \tag{18}$$

where $\dot{B} \equiv \mathrm{d}B/\mathrm{d}t$.

By cylindrical symmetry, we know that that the induced electric field must be in the $\hat{\phi}$ direction, $\mathbf{E} = E(t)\,\hat{\phi}$, and have the same magnitude at equal radial distances from the axis. We apply Faraday's law to a circle with axis parallel to the z-axis. For a point inside the solenoid, we get

$$\oint_{\text{circle}} \mathbf{E}\cdot \mathrm{d}\boldsymbol{\ell} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_0^r \mathbf{B}\cdot \mathrm{d}\mathbf{A} \tag{19}$$

$$2\pi r E = -\pi r^2 \dot{B} \tag{20}$$

$$E = -\frac{r}{2}\dot{B}\,. \tag{21}$$

Similarly, for a point outside the solenoid, we get:

$$\oint_{\text{circle}} \mathbf{E}\cdot \mathrm{d}\boldsymbol{\ell} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_0^R \mathbf{B}\cdot \mathrm{d}\mathbf{A} \tag{22}$$

$$2\pi r E = -\pi R^2 \dot{B} \tag{23}$$

$$E = -\frac{R^2}{2r}\dot{B}\,. \tag{24}$$

(b) *(10 points)* Show that *inside the solenoid*, a suitable choice of scalar and vector potentials to describe the fields is

$$V = \frac{1}{2}\dot{B}xy\,, \qquad \mathbf{A} = -By\hat{\mathbf{x}}\,. \tag{25}$$

For the given gauge fields, the electric and magnetic field inside the solenoid is given by:

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{26}$$

$$= -\partial_y(-By)\,\hat{\mathbf{z}} = B\,\hat{\mathbf{z}} \tag{27}$$

$$\mathbf{E} = -\nabla V - \frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} \tag{28}$$

$$= -\frac{1}{2}\dot{B}\,(y\,\hat{\mathbf{x}} + x\,\hat{\mathbf{y}}) - \dot{B}y\,\hat{\mathbf{x}} \tag{29}$$

$$= \frac{1}{2}\dot{B}\,(y\,\hat{\mathbf{x}} - x\,\hat{\mathbf{y}}) = -\frac{1}{2}\dot{B}r\,\hat{\phi} \tag{30}$$

These expressions are consistent with the fields considered.

(c) *(5 points)* For the fields *inside the solenoid*, find a gauge transformation such that $V = 0$, and obtain the expression for $\mathbf{A}$.

The required gauge transformation can be chosen by imposing

$$V' = V - \frac{\partial \lambda}{\partial t} = 0 \tag{31}$$

$$\text{or} \quad \lambda = \frac{1}{2} Bxy \tag{32}$$

This gives the expression for the vector potential:

$$\mathbf{A}' = \mathbf{A} + \nabla \lambda \tag{33}$$

$$= -By\,\hat{\mathbf{x}} + \frac{1}{2} B\,(y\,\hat{\mathbf{x}} + x\,\hat{\mathbf{y}}) \tag{34}$$

$$= \frac{1}{2} B\,(-y\,\hat{\mathbf{x}} + x\,\hat{\mathbf{y}}) = \frac{1}{2} Br\,\hat{\phi} \tag{35}$$