Final (Total Time: 120 min)

Question	Max. Points
Q1	35
Q2	40
Q3	35
Q4	60
Total	**170**

1. Do not turn the page until you are instructed to do so.
2. Fill in your name, and for subject please indicate "PY406 Spring 2026 Final". All other fields can be left blank.
3. **All of your work must be written in the blue exam book.** This document only contains the questions, and will **not** be graded.
4. Hand your blue exam booklet in at the end of the exam and before you leave the room.
5. This exam contains **10** pages (including this cover page) and **4** problems.
6. The total number of points is 170, with 15 bonus points.
7. This is a closed book exam. No books, notes or calculators allowed.
8. A formula sheet is provided to you on the page behind this.
9. Answers without supporting work will receive no credit. **Conversely, supporting work will be given plenty of partial credit, even if the final answer is incorrect.**
10. You will be graded based solely on what the grader can understand of your solutions.
11. Points allocated to each question are indicated in brackets. **Please plan your time appropriately.**
12. **Please ask for clarification promptly**. You could very well be confused by a typographical error.

PY406 Formula Sheet — Final

Vector Identities

Triple Products

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$$
$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

Product Rules

$$\begin{aligned}
\nabla(fg) &= f(\nabla g) + g(\nabla f) \\
\nabla(\mathbf{A}\cdot\mathbf{B}) &= \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) \\
&\quad + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} \\
\nabla\cdot(f\mathbf{A}) &= f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f) \\
\nabla\cdot(\mathbf{A}\times\mathbf{B}) &= \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B}) \\
\nabla\times(f\mathbf{A}) &= f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f) \\
\nabla\times(\mathbf{A}\times\mathbf{B}) &= (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} \\
&\quad + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})
\end{aligned}$$

Second Derivatives

$$\begin{aligned}
\nabla\cdot(\nabla\times\mathbf{A}) &= 0 \\
\nabla\times(\nabla f) &= 0 \\
\nabla\times(\nabla\times\mathbf{A}) &= \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}
\end{aligned}$$

Vector Calculus with Delta Functions

$$\begin{aligned}
\nabla^2\frac{1}{|\mathbf{r}-\mathbf{r}'|} &= -4\pi\,\delta^3(\mathbf{r}-\mathbf{r}') \\
\nabla\cdot\left(\frac{\hat{\mathbf{r}}}{r^2}\right) &= 4\pi\,\delta^3(\mathbf{r})
\end{aligned}$$

Fundamental Theorems

$$\begin{aligned}
\int_{\mathbf{a}}^{\mathbf{b}} \mathrm{d}\mathbf{l}\cdot(\nabla f) &= f(\mathbf{b}) - f(\mathbf{a}) \\
\int_{\mathcal{V}} \mathrm{d}^3\boldsymbol{r}\,(\nabla\cdot\mathbf{A}) &= \oint_{\partial\mathcal{V}} \mathrm{d}\mathbf{S}\cdot\mathbf{A} \\
\int_{\mathcal{S}} \mathrm{d}\mathbf{S}\cdot(\nabla\times\mathbf{A}) &= \oint_{\partial\mathcal{S}} \mathrm{d}\mathbf{l}\cdot\mathbf{A}
\end{aligned}$$

Cartesian Coordinates

$\mathrm{d}\mathbf{l} = \mathrm{d}x\,\hat{x} + \mathrm{d}y\,\hat{y} + \mathrm{d}z\,\hat{z}$, $\mathrm{d}^3\boldsymbol{r} = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$

$$\begin{aligned}
\nabla f &= \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z} \\
\nabla\cdot\mathbf{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \\
\nabla\times\mathbf{A} &= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{y} \\
&\quad + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z} \\
\nabla^2 f &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}
\end{aligned}$$

Cylindrical Coordinates

$\mathrm{d}\mathbf{l} = \mathrm{d}s\,\hat{s} + s\,\mathrm{d}\phi\,\hat{\phi} + \mathrm{d}z\,\hat{z}$, $\mathrm{d}^3\boldsymbol{r} = s\,\mathrm{d}s\,\mathrm{d}\phi\,\mathrm{d}z$
$\hat{s}\times\hat{\theta} = \hat{z}$

$$\begin{aligned}
\hat{s} &= \cos\phi\,\hat{x} + \sin\phi\,\hat{y} \\
\hat{\phi} &= -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}
\end{aligned}$$

$$\begin{aligned}
\nabla f &= \frac{\partial f}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial f}{\partial\phi}\hat{\phi} + \frac{\partial f}{\partial z}\hat{z} \\
\nabla\cdot\mathbf{A} &= \frac{1}{s}\frac{\partial(sA_s)}{\partial s} + \frac{1}{s}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z} \\
\nabla\times\mathbf{A} &= \left(\frac{1}{s}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{s} \\
&\quad + \left(\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}\right)\hat{\phi} \\
&\quad + \frac{1}{s}\left(\frac{\partial(sA_\phi)}{\partial s} - \frac{\partial A_s}{\partial\phi}\right)\hat{z} \\
\nabla^2 f &= \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial f}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 f}{\partial\phi^2} + \frac{\partial^2 f}{\partial z^2}
\end{aligned}$$

Spherical Coordinates

$\mathrm{d}\mathbf{l} = \mathrm{d}r\,\hat{r} + r\,\mathrm{d}\theta\,\hat{\theta} + r\sin\theta\,\mathrm{d}\phi\,\hat{\phi}$, $\mathrm{d}^3\boldsymbol{r} = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$
$\hat{r}\times\hat{\theta} = \hat{\phi}$

$$\begin{aligned}
\hat{r} &= \cos\phi\sin\theta\,\hat{x} + \sin\phi\sin\theta\,\hat{y} + \cos\theta\,\hat{z} \\
\hat{\theta} &= \cos\phi\cos\theta\,\hat{x} + \sin\phi\cos\theta\,\hat{y} - \sin\theta\,\hat{z} \\
\hat{\phi} &= -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}
\end{aligned}$$

$$\begin{aligned}
\nabla f &= \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\hat{\phi} \\
\nabla\cdot\mathbf{A} &= \frac{1}{r^2}\frac{\partial(r^2A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\,A_\theta)}{\partial\theta} \\
&\quad + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \\
\nabla\times\mathbf{A} &= \frac{1}{r\sin\theta}\left(\frac{\partial(\sin\theta\,A_\phi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right)\hat{r} \\
&\quad + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial(rA_\phi)}{\partial r}\right)\hat{\theta} \\
&\quad + \frac{1}{r}\left(\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)\hat{\phi} \\
\nabla^2 f &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) \\
&\quad + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}
\end{aligned}$$

Index Notation

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is even permutation of } (1,2,3) \\ -1 & \text{if } (i,j,k) \text{ is odd permutation of } (1,2,3) \\ 0 & \text{if any two indices are equal} \end{cases}$$

$$\begin{aligned}
\epsilon_{ijk}\epsilon^{ilm} &= \delta_j^l\delta_k^m - \delta_j^m\delta_k^l \\
\mathbf{A}\cdot\mathbf{B} &= A^iB_i \\
(\mathbf{A}\times\mathbf{B})_i &= \epsilon_{ijk}A^jB^k
\end{aligned}$$

Maxwell's Equations

Differential form ($\mathbf{E}$ and $\mathbf{B}$)

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$
$$\nabla\cdot\mathbf{B} = 0 \qquad \nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

Differential form ($\mathbf{D}$, $\mathbf{E}$ and $\mathbf{B}$, $\mathbf{H}$)

$$\nabla\cdot\mathbf{D} = \rho_f \qquad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$
$$\nabla\cdot\mathbf{B} = 0 \qquad \nabla\times\mathbf{H} = \mathbf{J}_f + \frac{\partial\mathbf{D}}{\partial t}$$

where

$$\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P} \qquad \mathbf{H} \equiv \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}$$

$\mathbf{P}$ is the polarization and $\mathbf{M}$ is the magnetization, defined via

$$\rho_b = -\nabla \cdot \mathbf{P} \qquad \mathbf{J}_b = \nabla \times \mathbf{M}$$

In linear media: $\mathbf{D} = \epsilon \mathbf{E}$, $\mathbf{B} = \mu \mathbf{H}$.

Lorentz Force Law

$$\begin{aligned} \mathbf{F} &= q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) && \text{(point charge)} \\ \mathbf{f} &= \rho \mathbf{E} + \mathbf{J} \times \mathbf{B} && \text{(continuous)} \end{aligned}$$

Boundary Conditions

With no free charges or currents on the surface,

$$\epsilon_1 E_1^\perp = D_1 = D_2 = \epsilon_2 E_2^\perp \,, \qquad B_1^\perp = B_2^\perp \,,$$
$$\mathbf{E}_1^\parallel = \mathbf{E}_2^\parallel \,, \qquad \frac{1}{\mu_1}\mathbf{B}_1^\parallel = \mathbf{H}_1^\parallel = \mathbf{H}_2^\parallel = \frac{1}{\mu_2}\mathbf{B}_2^\parallel \,,$$

Conservation Laws

Charge

Differential form:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

Energy

Total energy for electrostatic charge distribution $\rho(\boldsymbol{r})$:

$$U = \frac{1}{2}\int \mathrm{d}^3\boldsymbol{r}\, \rho(\boldsymbol{r}) V(\boldsymbol{r})$$

Energy density of EM fields and Poynting vector:

$$u_{\mathrm{EM}} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) \qquad \mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$$

Mechanical work done per unit time on charges:

$$\frac{\partial u_{\mathrm{mech}}}{\partial t} = \mathbf{J} \cdot \mathbf{E}$$

Differential form:

$$\frac{\partial}{\partial t}\left(u_{\mathrm{EM}} + u_{\mathrm{mech}}\right) + \nabla \cdot \mathbf{S} = 0$$

Momentum

Momentum density:

$$\boldsymbol{\wp}_{\mathrm{EM}} = \mu_0 \epsilon_0 \,\mathbf{S} = \epsilon_0 (\mathbf{E} \times \mathbf{B})$$

Force density on charges:

$$\dot{\boldsymbol{\wp}}_{\mathrm{mech}} = \mathbf{f} = \rho \mathbf{E} + \mathbf{J} \times \mathbf{B}$$

Maxwell stress tensor:

$$T_{ij} = \epsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2\right) + \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij} B^2\right)$$

Differential form:

$$\frac{\partial}{\partial t}\left(\wp^i_{\mathrm{EM}} + \wp^i_{\mathrm{mech}}\right) - \partial_j T^{ij} = 0$$

Integral form:

$$F^i + \epsilon_0 \mu_0 \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}} \mathrm{d}^3\boldsymbol{r}\, S^i - \oint_{\partial \mathcal{V}} \mathrm{d}A_j \, T^{ij} = 0$$

Angular Momentum

Angular momentum density of EM fields:

$$\boldsymbol{\ell}_{\mathrm{EM}} = \mathbf{r} \times \boldsymbol{\wp}_{\mathrm{EM}} = \epsilon_0 \,\mathbf{r} \times (\mathbf{E} \times \mathbf{B})$$

Electromagnetic Waves

Wave Equation

For electromagnetic waves in linear media,

$$\nabla^2 \mathbf{E} - \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

with μ and ϵ being the permeability and permittivity of the medium, respectively. The wave equation also holds for $\mathbf{B}$ for EM waves. The wave speed and refractive index are

$$v = \frac{1}{\sqrt{\mu\epsilon}} \,, \qquad n \equiv \frac{c}{v}$$

Multiplying Complex Exponentials

Since only the real part of complex exponentials is physical, we need to be careful about multiplying them. For f and g which are real sinusoids represented in complex exponential form as F and G,

$$f \cdot g = \frac{1}{2}(F \cdot G + F \cdot G^*) \,,$$

where taking the real part on the RHS is implied. For time-averages of two sinusoids with the same frequency ω,

$$\langle f \cdot g \rangle = \frac{1}{2} F \cdot G^*$$

Monochromatic Plane Waves

$$\begin{aligned} \mathbf{E} &= E_0 \,\hat{\mathbf{n}}\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \\ \mathbf{B} &= \frac{E_0}{v}\,(\hat{\mathbf{k}} \times \hat{\mathbf{n}})\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \end{aligned}$$

with $\hat{\mathbf{n}}$ the polarization direction, $\hat{\mathbf{k}}$ the propagation direction, $\omega = vk$. Time-averaged quantities:

$$\langle u \rangle = \frac{1}{2}\epsilon |E_0|^2 \qquad \langle \mathbf{S} \rangle = \frac{1}{2} v\epsilon |E_0|^2 \,\hat{\mathbf{k}} \qquad \langle \boldsymbol{\wp} \rangle = \frac{1}{2}\frac{\epsilon}{v}|E_0|^2 \,\hat{\mathbf{k}}$$

Incidence at Interface

For oblique incidence from medium 1 with μ_1, ϵ_1 to medium 2 with μ_2, ϵ_2, with angles of incidence, reflection and transmission $\theta_I, \theta_R, \theta_T$ defined with respect to the normal to the interface, we have the following results:

Law of reflection and Snell's law:

$$\theta_I = \theta_R \qquad n_1 \sin\theta_I = n_2 \sin\theta_T \,.$$

Defining

$$\alpha \equiv \frac{\cos\theta_T}{\cos\theta_I} \,, \qquad \beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} \tag{1}$$

Fresnel equations for $\mathbf{E} \perp$ plane of incidence:

$$\begin{aligned} E_{0,R} &= \left(\frac{1 - \alpha\beta}{1 + \alpha\beta}\right) E_{0,I} \\ E_{0,T} &= \left(\frac{2}{1 + \alpha\beta}\right) E_{0,I} \end{aligned}$$

Fresnel equations for $\mathbf{E} \parallel$ plane of incidence:

$$E_{0,R} = \left(\frac{\alpha - \beta}{\alpha + \beta}\right) E_{0,I}$$
$$E_{0,T} = \left(\frac{2}{\alpha + \beta}\right) E_{0,I}$$

In $\mathbf{E} \parallel$ plane of incidence, the Brewster angle θ_B is the incident angle at which there is no reflected wave. It is approximately given by

$$\tan\theta_B = \frac{n_2}{n_1}$$

Reflection and transmission coefficients:

$$R = \left(\frac{E_{0,R}}{E_{0,I}}\right)^2 \qquad T = \frac{n_2\cos\theta_T}{n_1\cos\theta_I}\left(\frac{E_{0,T}}{E_{0,I}}\right)^2$$

EM Waves in Conductors

In a conductor with conductivity σ, Ohm's law gives $\mathbf{J}_f = \sigma\mathbf{E}$. The wave equation becomes

$$\nabla^2\mathbf{E} = \mu\epsilon\frac{\partial^2\mathbf{E}}{\partial t^2} + \mu\sigma\frac{\partial\mathbf{E}}{\partial t}$$

and similar for $\mathbf{B}$. Plane wave solution: $\mathbf{E} = E_0\hat{\mathbf{n}}e^{i(\tilde{k}z-\omega t)}$ with complex wave number

$$\tilde{k} \equiv k + i\kappa$$

where

$$k = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}+1\right]^{1/2}$$
$$\kappa = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}-1\right]^{1/2}$$

Skin depth:

$$d = \frac{1}{\kappa}$$

The magnetic field is

$$\mathbf{B} = \frac{\tilde{k}}{\omega}(\hat{\mathbf{k}}\times\mathbf{E})$$

For reflection and transmission at the interface of a linear medium (medium 1) and a conductor (medium 2), the same expressions apply, except that we replace β with

$$\tilde{\beta} = \frac{\mu_1 v_1}{\mu_2\omega}\tilde{k}_2\,.$$

Waveguides

- Fields in waveguides with a constant cross section oriented along the z-axis are of the form

$$\mathbf{E}(\boldsymbol{r},t) = \mathbf{E}_0(x,y)e^{i(kz-\omega t)}$$
$$\mathbf{B}(\boldsymbol{r},t) = \mathbf{B}_0(x,y)e^{i(kz-\omega t)}\,,$$

  with $\mathbf{E}_0$ and $\mathbf{B}_0$ satisfying the Helmholtz equation,

$$\left(\nabla_T^2 - k^2 + \frac{\omega^2}{c^2}\right)\mathbf{E}_0 = 0\,, \tag{2}$$

  and likewise for $\mathbf{B}_0$, where ∇_T^2 is the transverse Laplacian.

- The x- and y-components of $\mathbf{E}$ and $\mathbf{B}$ can be expressed in terms of the z-components as follows:

$$E_x = \frac{i}{(\omega/c)^2-k^2}\left(k\frac{\partial E_z}{\partial x} + \omega\frac{\partial B_z}{\partial y}\right)$$
$$E_y = \frac{i}{(\omega/c)^2-k^2}\left(k\frac{\partial E_z}{\partial y} - \omega\frac{\partial B_z}{\partial x}\right)$$
$$B_x = \frac{i}{(\omega/c)^2-k^2}\left(k\frac{\partial B_z}{\partial x} - \frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\right)$$
$$B_y = \frac{i}{(\omega/c)^2-k^2}\left(k\frac{\partial B_z}{\partial y} + \frac{\omega}{c^2}\frac{\partial E_z}{\partial x}\right)$$

- For a rectangular waveguide of sides a and b with $a > b$, the TE$_{mn}$-mode solution is given by

$$B_{0,z}(x,y) = B_0\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\,.$$

  The TM$_{mn}$-mode solution is given by

$$E_{0,z}(x,y) = E_0\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\,.$$

  The dispersion relation is

$$k^2 = \frac{\omega^2}{c^2} - \pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)\,.$$

Potentials

The fields in terms of potentials:

$$\mathbf{B} = \nabla\times\mathbf{A} \qquad \mathbf{E} = -\nabla V - \frac{\partial\mathbf{A}}{\partial t}$$

Gauge Transformations

- For any scalar $\lambda(\boldsymbol{r},t)$, the gauge transformation:

$$\mathbf{A}' = \mathbf{A} + \nabla\lambda \qquad V' = V - \frac{\partial\lambda}{\partial t}$$

  gives the same $\mathbf{E}$ and $\mathbf{B}$.

- Lorenz gauge condition:

$$\nabla\cdot\mathbf{A} = -\frac{\partial V}{\partial t}$$

- Inhomogeneous wave equations in Lorenz gauge:

$$\Box V = -\frac{\rho}{\epsilon_0}\,, \qquad \Box\mathbf{A} = -\mu_0\mathbf{J}\,,$$

  where

$$\Box \equiv \nabla^2 - \frac{\partial^2}{\partial t^2}$$

Retarded Potentials

The solutions to the inhomogeneous wave equations in Lorenz gauge are the retarded potentials:

$$V(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3\boldsymbol{r}'\,\frac{\rho(\boldsymbol{r}',t_r)}{|\boldsymbol{r}-\boldsymbol{r}'|}\,,$$
$$\mathbf{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\int d^3\boldsymbol{r}'\,\frac{\mathbf{J}(\boldsymbol{r}',t_r)}{|\boldsymbol{r}-\boldsymbol{r}'|}\,,$$

where

$$t_r = t - |\mathbf{r}-\mathbf{r}'|$$

is the retarded time.

Jefimenko's Equations

$$\mathbf{E}(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3\boldsymbol{r}' \left[\rho\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3} + \dot{\rho}\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^2} - \frac{\dot{\mathbf{J}}}{|\boldsymbol{r}-\boldsymbol{r}'|}\right],$$
$$\mathbf{B}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\int \mathrm{d}^3\boldsymbol{r}' \left[\frac{\boldsymbol{J}}{|\boldsymbol{r}-\boldsymbol{r}'|^2} + \frac{\dot{\mathbf{J}}}{|\boldsymbol{r}-\boldsymbol{r}'|}\right]\times\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|},$$

where the quantities in $[\cdots]$ are to be evaluated at retarded time t_r.

Radiation

Perfect Electric Dipole

For an oscillating electric dipole $\mathbf{p}(t) = p_0\cos(\omega t)\hat{\mathbf{z}}$ where $p_0 = q_0 d$, in the perfect dipole and far-field approximation $d \ll 1/k \ll r$ where $\omega = k$,

$$V(\boldsymbol{r},t) = -\frac{p_0 k}{4\pi\epsilon_0}\frac{\cos\theta}{r}\sin[\omega(t-r)],$$
$$\mathbf{A}(\boldsymbol{r},t) = -\frac{\mu_0 p_0\omega}{4\pi}\frac{\sin[\omega(t-r)]}{r}\hat{\mathbf{z}},$$
$$\mathbf{E}(\boldsymbol{r},t) = -\frac{\mu_0 p_0\omega^2}{4\pi}\frac{\sin\theta}{r}\cos[\omega(t-r)]\hat{\boldsymbol{\theta}},$$
$$\mathbf{B}(\boldsymbol{r},t) = -\frac{\mu_0 p_0\omega^2}{4\pi}\frac{\sin\theta}{r}\cos[\omega(t-r)]\hat{\boldsymbol{\phi}}.$$

Perfect Magnetic Dipole

For an oscillating electric dipole $\mathbf{m}(t) = m_0\cos(\omega t)\hat{\mathbf{z}}$ where $m_0 = I_0\pi b^2$, in the perfect dipole and far-field approximation $b \ll 1/k \ll r$ where $\omega = k$,

$$\mathbf{A}(\boldsymbol{r},t) = -\frac{\mu_0 m_0\omega}{4\pi}\frac{\sin[\omega(t-r)]}{r}\hat{\boldsymbol{\phi}},$$
$$\mathbf{E}(\boldsymbol{r},t) = -\frac{\mu_0 m_0\omega^2}{4\pi}\frac{\sin\theta}{r}\cos[\omega(t-r)]\hat{\boldsymbol{\phi}},$$
$$\mathbf{B}(\boldsymbol{r},t) = -\frac{\mu_0 m_0\omega^2}{4\pi}\frac{\sin\theta}{r}\cos[\omega(t-r)]\hat{\boldsymbol{\theta}}.$$

Arbitrary Source

For an arbitrary charge distribution $\rho(\mathbf{r},t)$ distributed over within some radius R of the origin, under the assumption $R \ll r$ and $R \ll 1/k$ where $1/k = 1/\omega$ is a typical timescale over which charges and currents change, the radiation field (discarding terms going as $1/r^2$ or weaker) is approximately

$$V(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0 r}\left[Q + \frac{\hat{\boldsymbol{r}}\cdot\mathbf{p}(t_0)}{r} + \hat{\boldsymbol{r}}\cdot\dot{\mathbf{p}}(t_0)\right],$$
$$\mathbf{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi r}\dot{\mathbf{p}}(t_0),$$
$$\mathbf{E}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi r}[\hat{\boldsymbol{r}}\times(\hat{\boldsymbol{r}}\times\ddot{\mathbf{p}}(t_0))],$$
$$\mathbf{B}(\boldsymbol{r},t) = -\frac{\mu_0}{4\pi r}[\hat{\boldsymbol{r}}\times\ddot{\mathbf{p}}(t_0)],$$

where $t_0 \equiv t - r$, and the electric dipole moment is defined as

$$\mathbf{p}(t) = \int \mathrm{d}^3\mathbf{r}'\,\mathbf{r}'\rho(\mathbf{r}',t),$$

Special Relativity

Mechanics

- The spacetime interval between two events is

  $$\Delta s^2 = -\Delta t^2 + \Delta x^2,$$

  and is Lorentz invariant.

- The Minkowski metric is

  $$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

  It takes two vectors and returns a scalar (like the dot product), with $v_\mu = \eta_{\mu\nu}v^\nu$.

- The Lorentz transformation $\Lambda^\mu{}_\nu$ maps the components of a 4-vector in one frame to the components in another frame, $x'^\mu = \Lambda^\mu{}_\nu x^\nu$.

- The Lorentz factor is defined as $\gamma = 1/\sqrt{1-v^2}$.

- The Lorentz transformation going from frame S to frame S' moving at velocity $v\hat{\mathbf{x}}$ relative to S is

  $$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

- **Time dilation**: For all inertial observers, the time interval $\Delta t'$ between two timelike separated events is always greater than or equal to the time interval measured by an inertial observer S for whom the two events occur at the same point Δt. For any frame S' moving with velocity v relative to S, $\Delta t' = \gamma\Delta t$.

- **Length contraction**. The proper length of an object L is the length measured in its rest frame, S. For any other frame S' moving at a speed v relative to S, the length of the object is $L' = L/\gamma$.

- **Four-velocity**. The four-velocity of a particle in motion is

  $$u^\mu \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau},$$

  where τ is the proper time, i.e. the time elapsed in the particle's frame. If the particle is moving with *three-velocity* $\mathbf{v}$, then the four-velocity is $u^\mu = (\gamma, \gamma\mathbf{v})$.

- **Velocity addition formula in 1D**. Let S' be moving with speed $-v$ with respect to S. For an object with speed u in S, the speed in S' is

  $$u' = \frac{v+u}{1+uv}.$$

  All velocities are three-velocities here.

- **Four-momentum**. A particle of mass m has four-momentum $p^\mu = (\gamma m, \gamma m\mathbf{v})$ where $\mathbf{v}$ is the three-velocity. Calling $E \equiv p^0$, and $p = |\mathbf{p}|$, we have

  $$E^2 = p^2 + m^2.$$

Electrodynamics

- The **four-current** $J^\mu = (\rho, \mathbf{J})$ is a four-vector, as is the **four-potential** $A^\mu = (V, \mathbf{A})$.

- The **four-derivative** is defined as

  $$\partial_\mu \equiv \left(\frac{\partial}{\partial t}, \nabla\right),$$

  and is the generalization of ∇.

- The **electromagnetic field tensor** is defined as $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, with the following components:

$$F^{\mu\nu} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & B^3 & -B^2 \\ -E^2 & -B^3 & 0 & B^1 \\ -E^3 & B^2 & -B^1 & 0 \end{pmatrix} .$$

  The tensor is antisymmetric.

- The **dual electromagnetic field tensor** is

$$G^{\mu\nu} = \begin{pmatrix} 0 & B^1 & B^2 & B^3 \\ -B^1 & 0 & -E^3 & E^2 \\ -B^2 & E^3 & 0 & -E^1 \\ -B^3 & -E^2 & E^1 & 0 \end{pmatrix} .$$

- **Maxwell's equations** can be written in covariant form as

$$\partial_\nu F^{\mu\nu} = \mu_0 J^\mu \, ,$$
$$\partial_\nu G^{\mu\nu} = 0 \, ,$$

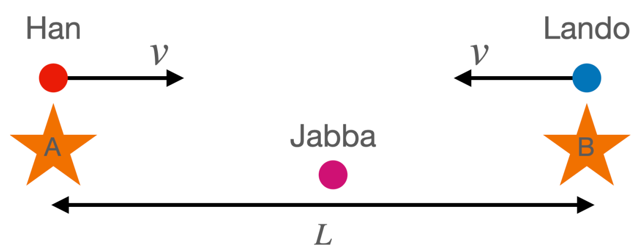
  with the first equation corresponding to the inhomogeneous Maxwell's equations with source terms, and the second to the homogeneous ones without source terms.

- $F^{\mu\nu}$ (and also $G^{\mu\nu}$) transform as $F'^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma F^{\rho\sigma}$. This can be computed very efficiently using matrix multiplication with $F' = \Lambda F \Lambda^T$.

- The electromagnetic invariants are $F^{\mu\nu} F_{\mu\nu} = 2(\mathbf{B}^2 - \mathbf{E}^2)$ and $F^{\mu\nu} G_{\mu\nu} = -4\mathbf{E} \cdot \mathbf{B}$.

1 The Kessel Run (35 points)

Smugglers Han Solo and Lando Calrissian are in a race to complete the Kessel Run, a trade route with a distance of 1 light-day between Star A and Star B, in a frame S where both stars are stationary. Jabba the Hutt is exactly between the two stars and is stationary with respect to frame S, and will pay whoever completes the journey first. Han is traveling from Star A to Star B with velocity v, while Lando is traveling in the opposite direction, from Star B to Star A, with velocity $-v$. They both leave at $t = 0$ in the S frame.



(a) *(5 points)* Determine the time taken to complete the Kessel Run in Han's frame, $\Delta t'_H$.

In Jabba's frame, the time taken by Han is L/v, traveling a distance L. By the invariance of the spacetime interval,

$$-\left(\frac{L}{v}\right)^2 + L^2 = -(\Delta t'_H)^2 \,, \tag{3}$$

i.e.

$$\Delta t'_H = \frac{L}{v}\sqrt{1-v^2} = \frac{L}{v\gamma} \,. \tag{4}$$

(b) *(5 points)* Determine the distance between Star A and B in Han's frame, L'.

In Han's frame, the distance between the two stars is length-contracted, with

$$L' = \frac{L}{\gamma} \,. \tag{5}$$

(c) *(10 points)* Let the time taken for Lando to complete the Kessel Run in Han's frame be $\Delta t'_L$, and the distance traveled by Lando be $\Delta x'_L$. Find $\Delta t'_L$ and $\Delta x'_L$, and show that

$$\frac{\Delta t'_H}{\Delta t'_L} = \frac{1-v^2}{1+v^2} \,. \tag{6}$$

Taking the origin to be at Star A in the S frame, Lando's journey takes $\Delta t_L = L/v$, with a displacement of $\Delta x_L = -L$. Applying the Lorentz transformation, to get into Han's frame, we find

$$\begin{pmatrix} \Delta t'_L \\ \Delta x'_L \end{pmatrix} = \begin{pmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{pmatrix} \begin{pmatrix} L/v \\ -L \end{pmatrix} = \begin{pmatrix} \gamma L\,(v + 1/v) \\ -2L\gamma \end{pmatrix} \tag{7}$$

Since $\Delta t'_H = L/(v\gamma)$, We have

$$\frac{\Delta t'_H}{\Delta t'_L} = \frac{L/(v\gamma)}{\gamma L(v+1/v)} = \frac{1}{\gamma^2(1+v^2)} = \frac{1-v^2}{1+v^2}\,, \tag{8}$$

as required.

(d) *(5 points)* Determine Lando's velocity, v'_L, in Han's frame (make sure to get the sign right). Write down a relation between $\Delta t'_L$, $\Delta x'_L$ and v'_L, and check that it is satisfied.

By the velocity addition rule, we have

$$v'_L = \frac{-v-v}{1+(-v)^2} = -\frac{2v}{1+v^2}\,. \tag{9}$$

We expect that

$$v'_L = \frac{\Delta x'_L}{\Delta t'_L}\,. \tag{10}$$

Indeed,

$$\frac{\Delta x'_L}{\Delta t'_L} = \frac{-2L\gamma}{\gamma L(v+1/v)} = -\frac{2v}{1+v^2}\,, \tag{11}$$

as expected.

(e) *(10 points)* Put the following events in order of occurrence in Han's frame: I) Han departs Star A, II) Lando departs Star B, III) Han arrives at Star B, and IV) Lando arrives at Star A. Explain your answer.

Let event I (Han departs Star A) have coordinates $(t,x)_{\mathrm{I}} = (t',x')_{\mathrm{I}} = (0,0)$. Event II has coordinates in the S frame of $(t,x)_{\mathrm{II}} = (0,L)$. Performing a Lorentz transformation, we find

$$\begin{pmatrix} t' \\ x' \end{pmatrix}_{\mathrm{II}} = \begin{pmatrix} \gamma & -\gamma v \\ -\gamma v & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ L \end{pmatrix} = \begin{pmatrix} -\gamma v L \\ \gamma L \end{pmatrix} \tag{12}$$

Event III we know from $\Delta t'_H$ and L': it has coordinates $(t',x')_{\mathrm{III}} = (L/(v\gamma), 0)$. Similarly, for Event IV, the coordinates are $(t,x)_{\mathrm{IV}} = (L/v, 0)$ in S, which by Lorentz transformation gives

$$\begin{pmatrix} t' \\ x' \end{pmatrix}_{\mathrm{IV}} = \begin{pmatrix} \gamma & -\gamma v \\ -\gamma v & \gamma \end{pmatrix} \begin{pmatrix} L/v \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma L/v \\ -\gamma L \end{pmatrix} \tag{13}$$

Looking at the time coordinates of these events, we find that the order of events in Han's frame is II, I, III, IV.

(f) *(BONUS: 5 points)* Jabba the Hutt sees both smugglers pass simultaneously. Do all observers agree that they arrived at Jabba the Hutt simultaneously?

Yes. These two events have the same spacetime coordinates, and under any coordinate system, will always both have the same coordinates.

2 Higgs Decay (40 points)

The Higgs boson was discovered by the ATLAS and CMS experiments at the Large Hadron Collider at CERN in 2012. It is produced in these detectors by colliding extremely high-energy proton beams traveling in opposite directions. Since the Higgs itself is extremely short-lived, you cannot detect it directly, but you can look for its decay products. In particular, the Higgs can decay into two massless photons, $H \to \gamma\gamma$.

Consider one such event, where the Higgs boson of mass m_H decays into two photons with four-momenta $p_1^\mu = (E_1, \mathbf{p}_1)$ and $p_2^\mu = (E_2, \mathbf{p}_2)$. For parts (a)–(c), you may **not** assume that the Higgs boson is produced at rest.

(a) *(5 points)* Write down a relation between E_1 and $|\mathbf{p}_1|$, and likewise for E_2 and $|\mathbf{p}_2|$. Explain your reasoning.

> The photon is massless, and so $E_1 = |\mathbf{p}_1|$. Likewise, $E_2 = |\mathbf{p}_2|$.

(b) *(5 points)* Write down the four-momentum for the Higgs particle before decay.

> By conservation of energy and momentum, the four-momentum of the Higgs before decay must be equal to the sum of the four-momenta of the two photons after decay, $p_H^\mu = (E_1 + E_2, \mathbf{p}_1 + \mathbf{p}_2)$.

(c) *(5 points)* Hence, determine m_H in terms of E_1, E_2, $|\mathbf{p}_1|$ and $|\mathbf{p}_2|$.

> Since $p_H^\mu p_{H,\mu} = -m_H^2$, we must have
>
> $$m_H^2 = (E_1 + E_2)^2 - (\mathbf{p}_1 + \mathbf{p}_2)^2 \,. \tag{14}$$

(d) *(5 points)* The Higgs lifetime is roughly 10^{-22} s. Each proton in the beam has total energy of roughly 10^4 GeV, while the mass of a proton is roughly 1 GeV. Suppose a Higgs boson was produced at rest in the lab frame. What is its lifetime in the beam frame?

> The beam energy is γm_p where m_p is the mass of the proton. Therefore, $\gamma = 10^4$. In the beam frame, the lifetime of the Higgs boson τ' undergoes time dilation, with
>
> $$\tau' = \gamma\tau \,, \tag{15}$$
>
> where τ is the lifetime in the Higgs boson's rest frame. Therefore, $\tau' = 10^{-18}$ s.

(e) *(10 points)* Suppose the photons have four-momenta $p_1^\mu = (p, p, 0, 0)$ and $p_2^\mu = (p, -p, 0, 0)$ in the lab frame, where one of the beams is traveling with velocity $+\beta\hat{\mathbf{x}}$. Show that in the beam frame, the ratio of the energies of each of the photons is

$$\frac{E_1'}{E_2'} = \frac{1-\beta}{1+\beta} \,. \tag{16}$$

> Performing the Lorentz transformation on these four-momenta gives
>
> $$p_1'^\mu = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ p \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma(1-\beta)p \\ \gamma(1-\beta)p \\ 0 \\ 0 \end{pmatrix} , \tag{17}$$

and

$$p_2'^\mu = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ -p \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma(1+\beta)p \\ -\gamma(1+\beta)p \\ 0 \\ 0 \end{pmatrix}, \tag{18}$$

We can therefore see that

$$\frac{E_1'}{E_2'} = \frac{\gamma(1-\beta)p}{\gamma(1+\beta)p} = \frac{1-\beta}{1+\beta}. \tag{19}$$

(f) *(10 points)* Now suppose $p_1^\mu = (p, 0, p, 0)$ and $p_2^\mu = (p, 0, -p, 0)$ in the lab frame, where one of the beams is traveling with velocity $+\beta\hat{\mathbf{x}}$. Show that in the beam frame, the angle between photon 2 and the beam is $\theta \simeq 1/\gamma$ in the limit that $\beta \to 1$, with $\gamma = 1/\sqrt{1-\beta^2}$.

Performing the required Lorentz transformation again,

$$p_2'^\mu = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ 0 \\ p \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma p \\ -\beta\gamma p \\ -p \\ 0 \end{pmatrix}, \tag{20}$$

Since the beam is on the $\hat{\mathbf{x}}'$ axis, the angle made by the photon and the beam is

$$\tan\theta = \frac{-p}{-\beta\gamma p} = \frac{1}{\beta\gamma} \tag{21}$$

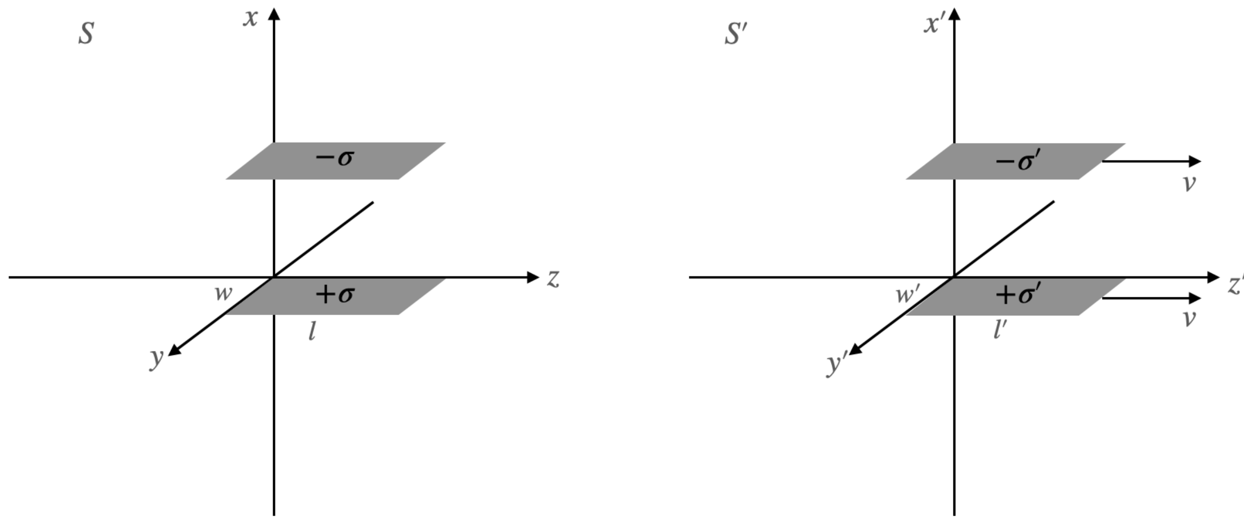
In the limit that $\beta \to 1$, we have

$$\tan\theta \simeq \frac{1}{\gamma} \simeq \theta, \tag{22}$$

using the small-angle approximation of $\tan\theta$ as $\gamma \to \infty$, as required.

3 A Moving Capacitor (35 points)

Consider a parallel-plate capacitor with length l and width w, with each plate carrying a charge density $\pm\sigma$ in the rest frame S of the capacitor. In another frame S', the capacitor is moving with speed v to the right. The set-up is shown in figure below. Throughout this problem, you may neglect any finite size effects of the capacitor on the fields.



(a) *(0 points)* Please indicate which metric convention you're using. If you do not respond to this part, the grader will assume that you're using the diag$(-1,+1,+1,+1)$ convention.

(b) *(5 points)* Show that the electric field between the plates in the S-frame is

$$\mathbf{E} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{x}}\,, \tag{23}$$

and write down the electromagnetic field tensor, $F^{\mu\nu}$.

Applying Gauss's law to a small cylindrical pillbox with radius r at the lower plate, with ends parallel to the plate. By symmetry, the electric field must be normal to both surfaces, with no flux through the curved part of the pillbox, giving

$$\oint \mathrm{d}\mathbf{S}\cdot\mathbf{E} = 2\times\pi r^2 E = \frac{Q_{\text{enc}}}{\epsilon_0} = \frac{\sigma\pi r^2}{\epsilon_0} \tag{24}$$

or

$$\mathbf{E}_+ = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{x}} \tag{25}$$

between the plates. Similarly, for the upper plate, we have

$$\mathbf{E}_- = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{x}} \tag{26}$$

between the plates. Therefore, the total field is

$$\mathbf{E} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{x}}\,, \tag{27}$$

as required. The field strength tensor is therefore

$$F^{\mu\nu} = \frac{\sigma}{\epsilon_0}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{28}$$

(c) *(10 points)* By performing a suitable Lorentz transformation, obtain the electromagnetic field tensor, $F'^{\mu\nu}$, in S'.

The field strength tensor transforms as

$$F'^{\mu\nu} = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}F^{\rho\sigma}\,, \tag{29}$$

which can efficiently be calculated via matrix multiplication as

$$F' = \Lambda F \Lambda^T\,. \tag{30}$$

Using a Lorentz transformation in the $-z$-direction with velocity v (since the plates are moving in the $+z$ direction), we have

$$\begin{aligned} F'^{\mu\nu} &= \frac{\sigma}{\epsilon_0}\begin{pmatrix} \gamma & 0 & 0 & v\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ v\gamma & 0 & 0 & \gamma \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \gamma & 0 & 0 & v\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ v\gamma & 0 & 0 & \gamma \end{pmatrix} \\ &= \frac{\sigma}{\epsilon_0}\begin{pmatrix} 0 & \gamma & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & v\gamma & 0 & 0 \end{pmatrix}\begin{pmatrix} \gamma & 0 & 0 & v\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ v\gamma & 0 & 0 & \gamma \end{pmatrix} \\ &= \frac{\sigma}{\epsilon_0}\begin{pmatrix} 0 & \gamma & 0 & 0 \\ -\gamma & 0 & 0 & -v\gamma \\ 0 & 0 & 0 & 0 \\ 0 & v\gamma & 0 & 0 \end{pmatrix} \end{aligned} \tag{31}$$

(d) *(5 points)* What is the charge density σ' in the S' frame? Use Maxwell's equations to obtain the electric field $\mathbf{E}'$, and check that it is consistent with $F'^{\mu\nu}$ in part (b).

The parallel plates in S' is length-contracted relative to S, and so $l' = l/\gamma$. This means that the the charge density goes up by a factor of γ, i.e.

$$\sigma' = \gamma\sigma\,. \tag{32}$$

Repeating the same calculation as in part (b), one would find that

$$\mathbf{E}' = \frac{\gamma\sigma}{\epsilon_0}\hat{\mathbf{x}}'\,, \tag{33}$$

in agreement with the 01 and 10 entries of $F'^{\mu\nu}$.

(e) *(5 points)* Using Maxwell's equations, obtain the magnetic field $\mathbf{B}'$ in the S' frame, and check that its value is consistent with part (b).

In the S' frame, draw a rectangular Amperian loop parallel to the xy-plane, going through the bottom plate, with dimensions $a \times b$, where the side of length a is parallel to the y-axis. Since the plate is moving to the right with speed v, there is an enclosed current given by $I_{\text{enc}} = \sigma' va$. Then

$$\oint \mathrm{d}l \cdot \mathbf{B} = \mu_0 I_{\text{enc}} \,. \tag{34}$$

By symmetry, we know that the contributions from both sides with length b must cancel, and that the magnetic field along the sides of length b must be parallel to the sides (which can also be seen by the right-hand rule). By extending the loop such that the side of length a below the plate goes to infinity, we can see that we must have

$$Ba = \mu_0 \sigma' va \implies \mathbf{B} = \mu_0 \gamma \sigma v \hat{\mathbf{y}}' \,. \tag{35}$$

This is consistent with the 13 and 31 entries of $F'^{\mu\nu}$.

(f) *(10 points)* Determine the Maxwell stress tensor T'_{ij} in the S' frame, and use it to compute the total electromagnetic force acting on the bottom plate. Check that this consistent with the electrostatic force on the bottom plate in S in the limit $v \ll 1$.

The Maxwell stress tensor is

$$\begin{aligned} T'_{ij} &= \epsilon_0 \left(E'_i E'_j - \frac{1}{2} \delta_{ij} E'^2 \right) + \frac{1}{\mu_0} \left(B'_i B'_j - \frac{1}{2} \delta_{ij} B'^2 \right) \\ &= \epsilon_0 \cdot \frac{\gamma^2 \sigma^2}{\epsilon_0^2} \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} + \frac{1}{\mu_0} \mu_0^2 \gamma^2 \sigma^2 v^2 \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} \\ &= \frac{\gamma^2 \sigma^2}{2\epsilon_0} \begin{pmatrix} 1 - v^2 & 0 & 0 \\ 0 & -1 + v^2 & 0 \\ 0 & 0 & -1 - v^2 \end{pmatrix} \\ &= \frac{\sigma^2}{2\epsilon_0} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\gamma^2 (1 + v^2) \end{pmatrix} \,. \end{aligned} \tag{36}$$

Using a surface that coats the lower plate, the Lorentz force acting on the bottom plate is therefore

$$\begin{aligned} F'^i &= \oint_{\text{plate}} \mathrm{d}A'_j \, T'^{ij} \\ F'^1 &= \int_0^{w'} \mathrm{d}y \int_0^{l'} \mathrm{d}z \, T'^{xx} \\ &= \frac{\sigma^2}{2\epsilon_0} w' l' \\ &= \frac{\sigma^2}{2\epsilon_0 \gamma} wl \,, \end{aligned} \tag{37}$$

with all other components being zero. In the limit of $\gamma \ll 1$, we have

$$F'^1 \simeq \frac{\sigma}{2\epsilon_0} (\sigma wl) \,, \tag{38}$$

which is exactly the electrostatic force in frame S.

(g) *(BONUS: 10 points)* Is it possible to find a frame in which the electric field is zero? How about a frame in which the electric and magnetic fields were not perpendicular?

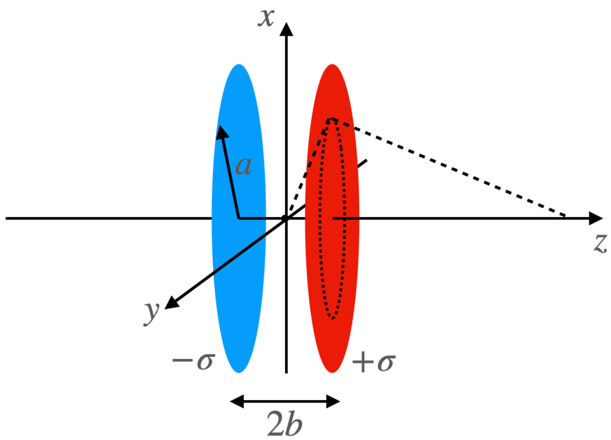
> Consider the invariant $F^{\mu\nu}F_{\mu\nu} = 2(\mathbf{B}^2 - \mathbf{E}^2)$. This is negative in the rest frame, and therefore is negative in all frames. Therefore, $\mathbf{E} \neq 0$, as otherwise this quantity would be positive. Similarly, consider the invariant $F^{\mu\nu}G_{\mu\nu} = -4\mathbf{E}\cdot\mathbf{B}$. It is zero in the rest frame, and therefore must be zero in all frames. This implies that there is no frame in which $\mathbf{E}\cdot\mathbf{B} \neq 0$, and therefore there is also no frame in which the two fields are not perpendicular.

4 Radiation from a Capacitor (60 points)

Consider a parallel plate capacitor, where each plate is a disc of radius a, separated by distance $2b$. One plate carries charge density $+\sigma(t)$ and is located at $+b\hat{\mathbf{z}}$, while the other carries charge density $-\sigma(t)$ and is located at $-b\hat{\mathbf{z}}$. The charge density varies as

$$\sigma(t) = \sigma_0 \cos(\omega t)\,. \tag{39}$$

The dashed lines in the figure indicate a useful way to think about the geometry throughout this problem.



(a) *(10 points)* Show that the dipole moment of the charge distribution is

$$\mathbf{p}(t) = 2\pi a^2 b\sigma_0 \cos(\omega t)\hat{\mathbf{z}}\,. \tag{40}$$

We can decompose the integral over the charge distribution on each of the discs into rings, each of which will contribute a dipole moment of $2\pi r \mathrm{d}r \cdot \sigma b\hat{\mathbf{z}}$ So the dipole moment is given by

$$\begin{aligned}\mathbf{p}(t) &= \int \mathrm{d}^3\boldsymbol{r}'\,\boldsymbol{r}'\rho(\boldsymbol{r}',t)\\ &= \int_0^a \mathrm{d}r'\,2\pi r'(-\sigma)(-b\hat{\mathbf{z}}) + \int_0^a \mathrm{d}r'\,2\pi r'(+\sigma)(+b\hat{\mathbf{z}})\\ &= 2b\pi a^2\sigma\hat{\mathbf{z}}\\ &= 2\pi a^2 b\sigma_0\cos(\omega t)\hat{\mathbf{z}}\,,\end{aligned} \tag{41}$$

as required.

(b) *(10 points)* In the far-field limit (i.e. at distances $r \gg a, b$, $\omega r \gg 1$ and keeping only radiation fields), write down $\mathbf{E}$ and $\mathbf{B}$ in spherical coordinates using complex notation. Hence, deduce the direction of the wavevector.

In the far-field limit, we know that

$$\mathbf{E} = \frac{\mu_0}{4\pi r}[\hat{\boldsymbol{r}} \times (\hat{\boldsymbol{r}} \times \ddot{\mathbf{p}}(t_0)]\,, \tag{42}$$

where $t_0 = t - r$, and r is the distance from the origin. First, we have

$$\ddot{\mathbf{p}}(t_0) = -2\pi a^2 b\sigma_0\omega^2\cos(\omega t_0)\hat{\mathbf{z}} \tag{43}$$

In spherical coordinates, we have

$$\hat{\boldsymbol{r}} \times (\hat{\boldsymbol{r}} \times \hat{\mathbf{z}}) = \hat{\boldsymbol{r}} \times \left(-\sin\theta\hat{\boldsymbol{\phi}}\right) = \sin\theta\,\hat{\boldsymbol{\theta}}\,. \tag{44}$$

In complex notation, we therefore have

$$\begin{aligned}\mathbf{E} &= \frac{\mu_0}{4\pi r}\left[-2\pi a^2 b\sigma_0\omega^2(\sin\theta\,\hat{\boldsymbol{\theta}})\right]e^{i\omega(t-r)} \\ &= -\frac{1}{2}\mu_0 a^2 b\sigma_0\omega^2\left(\frac{\sin\theta}{r}\right)e^{i\omega(t-r)}\hat{\boldsymbol{\theta}}\end{aligned} \tag{45}$$

For the magnetic field, we have

$$\begin{aligned}\mathbf{B} &= -\frac{\mu_0}{4\pi r}[\hat{\boldsymbol{r}} \times \ddot{\mathbf{p}}(t_0)] \\ &= -\frac{\mu_0}{4\pi r}[-2\pi a^2 b\sigma_0\omega^2\cos(\omega t_0)(-\sin\theta\hat{\boldsymbol{\phi}})]\end{aligned} \tag{46}$$

or

$$\mathbf{B} = -\frac{1}{2}\mu_0 a^2 b\sigma_0\omega^2\left(\frac{\sin\theta}{r}\right)e^{i\omega(t-r)}\hat{\boldsymbol{\phi}}\,. \tag{47}$$

Since the fields are proportional to $\exp(i\omega(t-r))$, the wavevector points radially outward.

(c) *(5 points)* In the far-field limit, is the electromagnetic wave transverse? Explain your answer.

Yes. Since the wavevector $\mathbf{k} \parallel \hat{\boldsymbol{r}}$, we can see that both $\mathbf{E}$ and $\mathbf{B}$ are perpendicular to $\mathbf{k}$.

(d) *(10 points)* Determine the Poynting vector $\mathbf{S}$ in the far-field, and the total time-averaged power emitted as radiation.

The Poynting vector is

$$\begin{aligned}\mathbf{S} &= \frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \\ &= \frac{1}{4}\mu_0 a^4 b^2 \sigma_0^2 \omega^4 \frac{\sin^2\theta}{r^2}\cos^2(\omega(t-r))\hat{\boldsymbol{r}}\,.\end{aligned} \tag{48}$$

The total time-averaged power emitted can be found by integrating the $\langle \mathbf{S} \rangle$ over a sphere of radius r,

$$\begin{aligned}P &= \oint \mathrm{d}\mathbf{A} \cdot \langle \mathbf{S} \rangle \\ &= \frac{1}{8}\mu_0 a^4 b^2 \sigma_0^2 \omega^4 \cdot\ 2\pi r^2 \int_{-1}^{1} \mathrm{d}(\cos\theta)\frac{\sin^2\theta}{r^2} \\ &= \frac{\pi}{4}\mu_0 a^4 b^2 \sigma_0^2 \omega^4 \int_{-1}^{1} d\mu\,(1-\mu^2) \\ &= \frac{\pi}{4}\mu_0 a^4 b^2 \sigma_0^2 \omega^4 \left(\mu - \frac{\mu^3}{3}\right)_{-1}^{1} \\ &= \frac{\pi}{3}\mu_0 a^4 b^2 \sigma_0^2 \omega^4\,.\end{aligned} \tag{49}$$

(e) *(15 points)* Show that the potential along the z-axis can be written *without approximations* as

$$V(z,t) = \frac{\sigma_0}{2\epsilon_0}\int \mathrm{d}s\, s\left[\frac{\cos[\omega(t-\rho_+)]}{\rho_+} - \frac{\cos[\omega(t-\rho_-)]}{\rho_-}\right], \tag{50}$$

where

$$\rho_\pm = \sqrt{s^2+b^2+z^2 \mp 2zb}\,, \tag{51}$$

and s is an integration variable.

The retarded potential is given by

$$V(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3\boldsymbol{r}'\,\frac{\rho(\boldsymbol{r}',t_r)}{|\boldsymbol{r}-\boldsymbol{r}'|}\,, \tag{52}$$

where $t_r = t - |\boldsymbol{r}-\boldsymbol{r}'|$. The distance from a point z along the z-axis and a point on (x,y,d) on the disc with charge $+\sigma$ is

$$\sqrt{x^2+y^2+(z-b)^2} = \sqrt{s^2+z^2-2zb+b^2} = \rho_+\,, \tag{53}$$

and similarly the distance from a point z along the z-axis and a point on $(x,y,-d)$ on the disc with charge $-\sigma$ is

$$\sqrt{x^2+y^2+(z+b)^2} = \sqrt{s^2+z^2+2zb+b^2} = \rho_-\,. \tag{54}$$

With this, we find that $t_r = t-\rho_\pm$ to each disc. The charge density can then be written as

$$\rho(\boldsymbol{r}',t_r) = \sigma_0\cos[\omega(t-\rho_+)]\delta(z-b) - \sigma_0\cos[\omega(t-\rho_-)]\delta(z+b)\,. \tag{55}$$

Putting this back into the integral for V, and integrating over z gives (in cylindrical coordinates)

$$\begin{aligned} V(\boldsymbol{r},t) &= \frac{1}{4\pi\epsilon_0}\int \mathrm{d}s\, 2\pi s\left[\sigma_0\frac{\cos[\omega(t-\rho_+)]}{\rho_+} - \sigma_0\frac{\cos[\omega(t-\rho_-)]}{\rho_-}\right] \\ &= \frac{\sigma_0}{2\epsilon_0}\int \mathrm{d}s\, s\left[\frac{\cos[\omega(t-\rho_+)]}{\rho_+} - \frac{\cos[\omega(t-\rho_-)]}{\rho_-}\right], \end{aligned} \tag{56}$$

as required.

(f) *(5 points)* A student in the class claims the following: "there are no currents here, and so we must have $\mathbf{A}=0$". Explain why this conclusion is incorrect.

There has to be current by charge conservation, since the charges on the plates are changing.

(g) *(5 points)* The same student now says, "Well, I can always choose a gauge in which $\mathbf{A}=0$". Is this statement true? Explain your answer.

This cannot be, since we already know that there is a magnetic field, and since $\nabla\times\mathbf{A}=\mathbf{B}$, we cannot have a choice of gauge where $\mathbf{A}=0$.