24-4 Image Formation by Thin Lenses

Lenses, which are important for correcting vision, for microscopes, and for many telescopes, rely on the refraction of light to form images. As with mirrors, we draw ray diagrams to help us to understand how such images are formed. Let's first begin by looking at what a lens does to a set of parallel rays of light, such as the five rays in Figure 24.16.

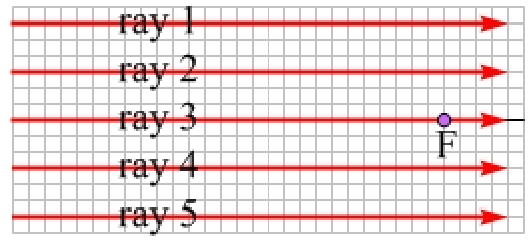

Figure 24.16: Five parallel rays of light.

How can we change the direction of the five rays of light in Figure 24.16 so that they all pass through the point labeled F? Ray 3 already passes through point F, so we don't need to change its direction at all. We can deflect ray 2 with a triangular piece of glass, as shown in Figure 24.17. Passing from air into the glass prism, the ray deflects toward the normal at that surface, while when it emerges back into the air the deflection is away from the normal at the second surface. We can follow a similar process for ray 1, except that we need to produce a larger change in direction for ray 1 compared to ray 2. The glass prism we use for ray 1 thus has its sides at a greater angle from the vertical. For ray 4, we use an identical prism to that used for ray 2, except that we invert it, and for ray 5 the prism is identical to the prism for ray 1, but inverted.

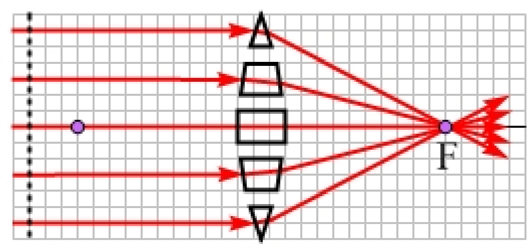

Figure 24.17: We can deflect four of the five rays, so that all five rays meet at point F, by using prisms of the appropriate shapes to produce the deflection required for each beam.

Now, not only do we want all five rays to converge on point F, but we want them to take equal times to travel from the vertical dashed line in Figure 24.17 to point F. Remember that the light travels more slowly in glass than in air. Because rays 1 and 5 travel the greatest distance, they need to pass through the least amount of glass. Ray 3 travels the shortest distance, so we need to delay ray 3 by having it pass through the thickest piece of glass. We can do this without deflecting ray 3 by using a piece of glass with vertical sides, so that ray 3 is incident along the normal. Rather than using various individual glass rectangles and prisms to do the job, we can use a single piece of glass that is thickest in the middle. This piece of glass gets thinner, and its surfaces curve farther away from the vertical, as you move away from ray 3 (that is, as you move away from the principal axis). This is shown in Figure 24.18 – we use a lens. Point F is a focal point of the lens. Lenses allow light to pass through from left-to-right or from right-to-left, so a lens has two focal points, one on each side of the lens.

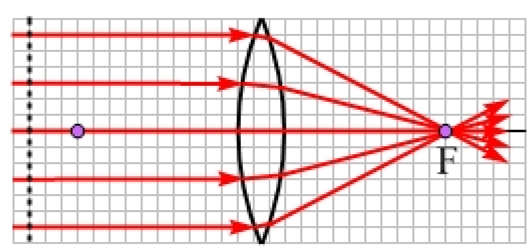

Figure 24.18: A convex lens with surfaces that are spherical arcs brings all parallel rays to one of the focal points, and ensures that the parallel rays take the same time to travel from the vertical line to this focal point.

How does a concave lens, which is thinner in the center than at the edges, affect parallel rays? As shown in Figure 24.19, such lenses generally diverge parallel rays away from the focal point that is on the same side of the lens that the light comes from.

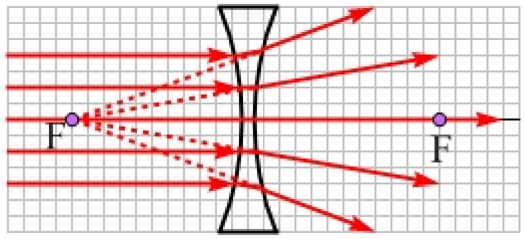

Figure 24.19: The influence of a diverging lens on a set of parallel rays.

EXPLORATION 24.4 – Using a ray diagram to find the location of an image

When drawing ray diagrams for lenses, we follow a process similar to that for mirrors.

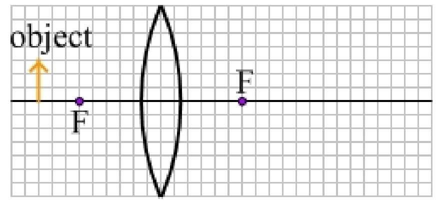

Figure 24.20: An object located some distance in front of a converging lens.

Step 1 – ***Figure 24.20 shows an object in front of a converging lens. Sketch two rays of light, which travel in different directions, that leave the tip (the top) of the object and pass through the lens. Show the direction of these rays after they are refracted by the lens.*** Figure 24.21 shows three rays that leave the tip of the object and which are incident on the lens. Because the ray in red is parallel to the principal axis, it is refracted by the lens to pass through the focal point that is to the right of the lens. The ray in green travels along the line connecting the tip of the object to the focal point on the left of the lens. This ray is refracted so that it is parallel to the principal axis. We know this because of the reversibility of light rays – if we reversed the direction of the ray, it would come from the right and be refracted by the lens to pass through the focal point on the right. Finally, the ray in blue passes through the center of the lens without changing direction. This is an approximation, which is valid as long as the lens is thin so the ray enters and exits the lens very close to the principal axis.

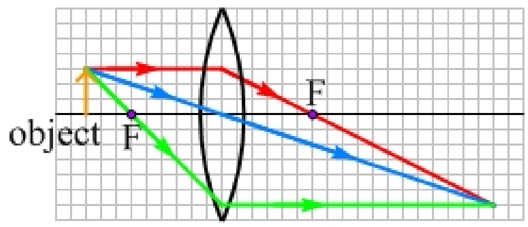

Figure 24.21: Three of the rays that are refracted by the lens. In general, we draw a ray changing direction once, inside the lens. Each ray really changes direction twice, once at each air-glass interface. We are drawing the ray's path incorrectly within the lens, but it is correct outside of the lens, which is where it really matters.

Step 2 – ***The point where the refracted rays meet is where the tip of the image is located. Use this information to sketch the image of the object.*** In this situation, the refracted rays meet at a point to the right of the lens, below the principal axis. Thus, we draw an inverted image between the point where the image of the tip is (where the rays meet) and the principal axis. This is a real image, because the rays pass through the image. All the rays take the same time to travel from the tip of the object to the tip of the image.

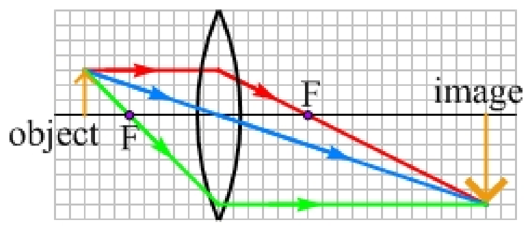

Figure 24.22: Because the rays originate at the tip of the object, the point where the refracted rays meet is the location of the tip of the image of the object. The base of the object is on the principal axis, so the base of the image is on the principal axis, too.

Key idea for ray diagrams: The location of the image of any point on an object, when the image is created by a lens, can be found by drawing rays of light that leave that point on the object and are refracted by the lens. The point where the refracted rays meet (or where they appear to diverge from) is where the image of that point is. **Related End-of-Chapter Exercises: 13, 49**

Essential Question 24.4: Are the three rays in Figure 24.22 the only rays that pass through the tip of the image? Explain.

Answer to Essential Question 24.4: The three rays in Figure 24.21 are easy to draw, because we know what they do after passing through the lens. However, as shown in Figure 24.23, all rays that leave the tip of the object and which are refracted by the lens will converge at the tip of the image.

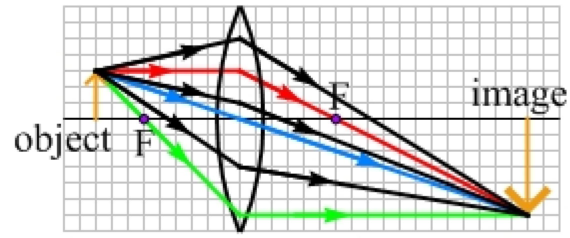

Figure 24.23: All rays that leave the tip of the object and pass through the lens will converge at the tip of the image.

24-5 Lens Concepts

In general, a lens has a larger index of refraction than the medium that surrounds it. In that case, a lens that is thicker in the center than the ends is a converging lens (it converges parallel rays toward a focal point), while a lens that is thinner in the middle than at the ends is a diverging lens (it diverges parallel rays away from a focal point). Like a concave mirror, converging lenses can produce a real image or a virtual image, and the image can be larger, smaller, or the same size as the object. Like a convex mirror, diverging lenses can only produce a virtual image that is smaller than the object.

As with mirrors, the focal point of a lens is defined by what the lens does to a set of rays of light that are parallel to one another and to the principal axis of the mirror. As we discussed in Section 24-4, a converging lens generally refracts the rays so they converge to pass through a focal point, *F*. A diverging lens, in contrast, refracts parallel rays so that they diverge away from a focal point.

Because of dispersion (the fact that the index of refraction of the lens material depends on wavelength), a lens generally has slightly different focal points for different colors of light. This range of focal points is a defect called **chromatic aberration**.

Focal length of a lens with spherical surfaces: The focal length of a lens depends on the curvature of the two surfaces of the lens, the index of refraction of the lens material, and on the index of refraction of the surrounding medium. The focal length is given by:

$$\frac{1}{f} = \left(n_{lens} - n_{medium}\right)\left(\frac{1}{R_1} + \frac{1}{R_2}\right),$$ (Equation 24.5: **The lensmaker's equation**)

where the two *R*'s represent the radii of curvature of the two lens surfaces. A radius is positive if the surface is convex, and negative if the surface is concave.

Companies that make eyeglasses exploit the lensmaker's equation in creating lenses of the desired focal length. By choosing a lens material that has a high index of refraction, a smaller radius of curvature (and thus a thinner and lighter lens) can be used, compared to a lens made from glass with a smaller index of refraction.

The factor of ($n_{lens} - n_{medium}$) in Equation 24.5 has an interesting implication. First, consider the diagram in Figure 24.24, which shows a familiar situation of a lens, made from material with an index of refraction larger than that of air, surrounded by air. This lens causes the parallel rays to change direction so that they pass through the focal point on the right, and ($n_{lens} - n_{medium}$) is positive, so the focal length of the lens is positive.

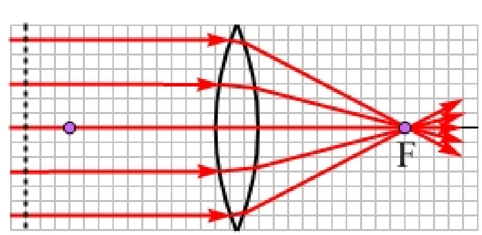

Figure 24.24: A ray diagram for a set of parallel rays encountering a convex lens, made of plastic, surrounded by air.

What happens to the rays if the medium surrounding the lens has the same index of refraction as the lens material (perhaps we immerse the lens in some kind of oil)? In this case, the factor of ($n_{lens} - n_{medium}$) is zero, so the focal length is infinite. An infinite focal length means that the lens does not change the direction of the parallel rays at all! Going further, if the lens is immersed in a medium that has a larger index of refraction than the lens material, ($n_{lens} - n_{medium}$) is negative and so is the focal length: parallel rays would be *diverged* in this situation.

Thus, depending on the situation, a lens with a convex shape can be a converging lens, a diverging lens, or neither. To minimize any ambiguity, for the rest of this chapter we will refer to lenses by their function (converging or diverging) rather than their shape (convex or concave).

EXPLORATION 24.5 – Ray diagram for a diverging lens

We will follow a process similar to that of mirrors to draw a ray diagram for a diverging lens, starting with the situation in Figure 24.25. The ray diagram will show us where the image of an object is.

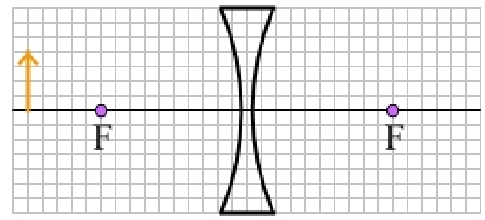

Figure 24.25: An object in front of a diverging lens.

Step 1 – *Draw a ray of light that leaves the tip of the object (the top of the arrow) and goes parallel to the principal axis (this is known as the parallel ray). Show how this ray is refracted by the lens.* For a diverging lens, all parallel rays appear to diverge from the focal point on the side of the lens that the light comes from, so we draw the refracted ray (see Figure 24.26) refracting along a line that takes it directly away from that focal point.

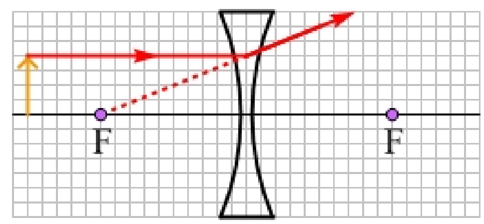

Figure 24.26: The parallel ray refracts to travel directly away from the focal point on the side of the lens the light came from.

Step 2 – *Sketch a second ray that leaves the tip of the object and is refracted by the lens. Using the refracted rays, draw the image.* One useful ray, shown in blue in Figure 24.27, passes through the lens without changing direction. This is something of an approximation, but the thinner the lens, the more accurate this is. Another useful ray, in green, goes straight toward the focal point on the right of the lens. This ray refracts so as to emerge from the lens going parallel to the principal axis. The refracted rays diverge to the right of the lens, but we can extend them back to meet on the left side of the lens, showing us where the tip of the image is.

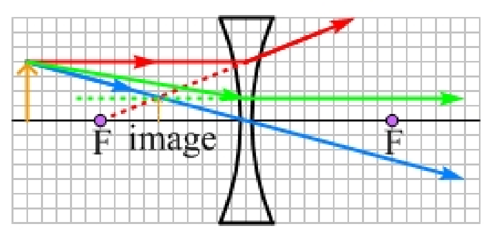

Figure 24.27: In addition to the parallel ray, two other rays are easy to draw the refracted rays for. The ray (in blue) that passes through the center of the lens is undeflected, approximately. The ray in green travels directly toward the focal point on the far side of the lens, and is refracted so it emerges from the lens traveling parallel to the principal axis. If you look at the object through the lens, your brain interprets the light as coming from the image.

Key idea: When a number of rays leave the same point on an object and are refracted by a lens, the corresponding point on the image is located at the intersection of the refracted rays.
Related End-of-Chapter Exercises: 11, 12, 52.

Essential Question 24.5: Starting with Figure 24.27, show a few more rays of light leaving the tip of the object and being refracted by the lens. How do you know how to draw the refracted rays?

Answer to Essential Question 24.5: To draw the refracted rays properly, we know that when we extend the refracted rays back, they will pass through the tip of the image, which we located in Figure 24.27. Three additional rays are shown in Figure 24.28.

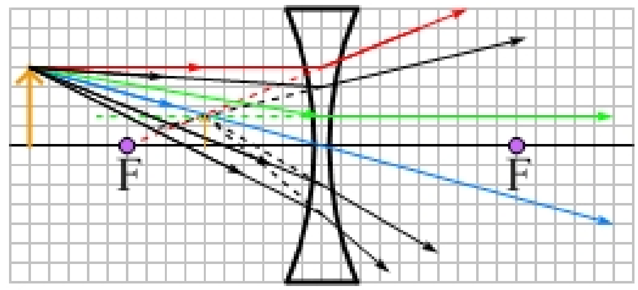

Figure 24.28: For all rays of light that leave the tip of the object and reflect from the mirror, the refracted rays can be extended back to pass through the tip of the image.

24-6 A Quantitative Approach: The Thin-Lens Equation

Even though mirrors and lenses form images using completely different principles (the law of reflection versus Snell's law), we use the same equation to relate focal length, object distance, and image distance, for both mirrors and lenses. This surprising result comes from the fact that the formation of images with both mirrors and lenses can be understood using the geometry of similar triangles. Let's look at how that works for lenses.

Let's look at the ray diagram we drew in Figure 24.22 of section 24-4, shown again here in Figure 24.29.

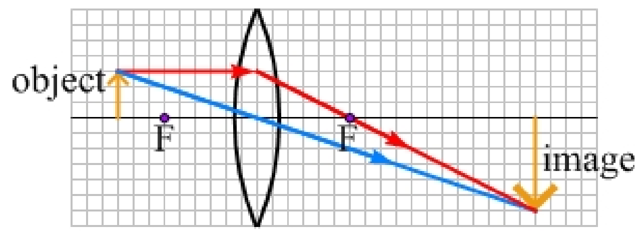

Figure 24.29: The ray diagram we constructed in section 24-4, for an object in front of a converging lens.

Remove the red rays, and examine the two triangles in Figure 24.30, one shaded green and one shaded yellow, bounded by the blue rays, the principal axis, and the object and image. The two triangles are similar, because the three angles in one triangle are the same as the three angles in the other triangle. We can now define the following variables: d_o is the object distance, the distance of the object from the center of the mirror; d_i is the image distance, the distance of the image from the center of the mirror; h_o is the height of the object; h_i is the height of the image.

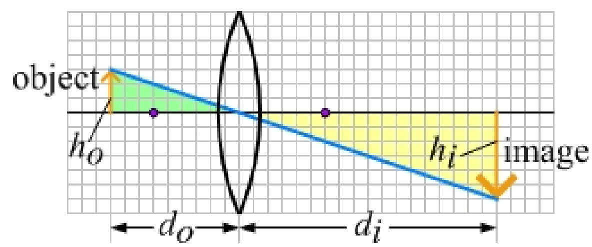

Figure 24.30: Similar triangles, bounded by the principal axis, the object and image, and the blue ray that passes through the center of the lens.

Using the fact that the ratios of the lengths of corresponding sides in similar triangles are equal, we find that:

$$-\frac{h_i}{h_o} = \frac{d_i}{d_o} . \qquad \text{(Equation 24.6)}$$

The image height is negative because the image is inverted, which is why we need the minus sign in the equation. Let's now return to Figure 24.29, and remove the blue rays. This gives us the shaded similar triangles shown in Figure 24.31.

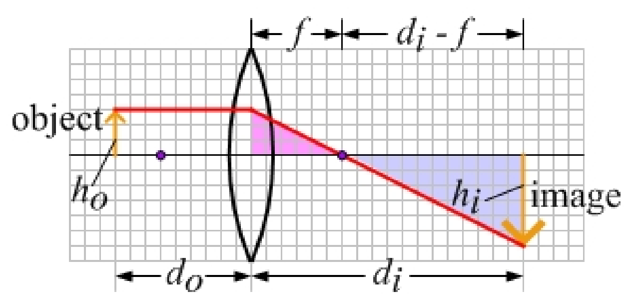

Figure 24.31: Similar triangles, with two sides bounded by the principal axis and the red ray, and a third side that is equal to the object height (pink triangle) or the image height (blue triangle).

Again, using the fact that the ratios of the lengths of corresponding sides in similar triangles are equal, we find that: $\frac{d_i - f}{f} = -\frac{h_i}{h_o}$.

Simplifying the left side, and bringing in equation 24.3, we get: $\frac{d_i}{f} - 1 = \frac{d_i}{d_o}$.

Dividing both sides by d_i gives: $\frac{1}{f} - \frac{1}{d_i} = \frac{1}{d_o}$, which is generally written as:

$$\frac{1}{f} = \frac{1}{d_o} + \frac{1}{d_i}.$$ (Equation 24.7: **The thin-lens equation**)

The mnemonic "If I do I di" can help you to remember the thin-lens equation.

Often, we know the focal length f and the object distance d_o, so equation 24.4 can be solved for d_i, the image distance:

$$d_i = \frac{d_o \times f}{d_o - f}$$ (Equation 24.8: **The thin-lens equation, solved for the image distance**)

Sign conventions

We derived the lens equation above by using a specific case involving a convex lens. The equation can be applied to all situations involving a convex lens or a concave lens if we use the following sign conventions.

The focal length is positive for a converging lens, and negative for a diverging lens.

The image distance is positive, and the image is real, if the image is on the side of the lens the light passes through to, and negative, and the image is virtual, if the image is on the side the light comes from.

The image height is positive when the image is above the principal axis, and negative when the image is below the principal axis. A similar rule applies to the object height.

Magnification

The magnification, m, is defined as the ratio of the height of the image (h_i) to the height of the object (h_o). Making use of Equation 24.6, we can write the magnification as:

$$m = \frac{h_i}{h_o} = -\frac{d_i}{d_o}.$$ (Equation 24.9: **Magnification**)

The relative sizes of the image and object are as follows:

- The image is larger than the object if $|m| > 1$.
- The image and object have the same size if $|m| = 1$.
- The image is smaller than the object if $|m| < 1$.

The sign of the magnification tells us whether the image is upright (+) or inverted (–) compared to the object.

Related End-of-Chapter Exercises: 15 – 18.

Essential Question 24.6: As you are analyzing a thin-lens situation, you write an equation that states: $\frac{1}{f} = \frac{1}{+12 \text{ cm}} + \frac{1}{+24 \text{ cm}}$. What is the value of 1/f in this situation? What is f?

Answer to Essential Question 24.6: To add fractions, you need to find a common denominator.
$\frac{1}{f} = \frac{1}{+12\text{ cm}} + \frac{1}{+24\text{ cm}} = \frac{2}{+24\text{ cm}} + \frac{1}{+24\text{ cm}} = \frac{3}{+24\text{ cm}}$. This gives $f = \frac{+24\text{ cm}}{3} = 8.0\text{ cm}$.

24-7 Analyzing a Converging Lens

In section 24-4, we drew one ray diagram for a converging lens. Let's investigate the range of ray diagrams we can draw for such a lens. Note the similarities between a converging lens and a concave mirror. This section is very much a parallel of section 23-6, in which we analyzed the range of images formed by a concave mirror.

EXPLORATION 24.7 – Ray diagrams for a converging lens

Step 1 – ***Draw a ray diagram for an object located 40 cm from a converging lens that has a focal length of +10 cm. Verify the image location on your diagram with the thin-lens equation.*** Two rays are shown in Figure 24.32. One is the parallel ray, which leaves the tip of the object, travels parallel to the principal axis, and is refracted by the lens to pass through the focal point on the far side of the lens. The second ray passes straight through the center of the lens, undeflected.

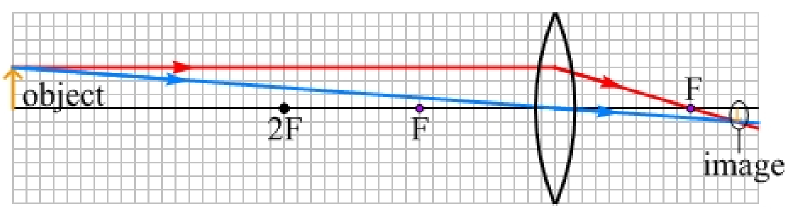

Figure 24.32: A ray diagram for the situation in which the object is far from the mirror. The squares in the grid measure 1.0 cm × 1.0 cm.

Applying the thin-lens equation, in the form of equation 24.5, to find the image distance:

$$d_i = \frac{d_o \times f}{d_o - f} = \frac{(40\text{ cm})\times(+10\text{ cm})}{(40\text{ cm})-(+10\text{ cm})} = \frac{+400\text{ cm}^2}{30\text{ cm}} = +13.3\text{ cm}.$$

This image distance is consistent with the ray diagram in Figure 24.32.

Step 2 – ***Repeat step 1, with the object now twice the focal length from the lens.*** We draw the same two rays again, with the parallel ray (in red) being refracted so that it passes through the focal point on the far side of the lens, and the second ray (in blue) passing undeflected (approximately) through the center of the lens. As shown in Figure 24.33, this situation is a special case. When the object is located at twice the focal length from the lens, the image is inverted, also at twice the focal length from the lens (on the other side of the lens), and the same size as the object because the object and image are the same distance from the lens.

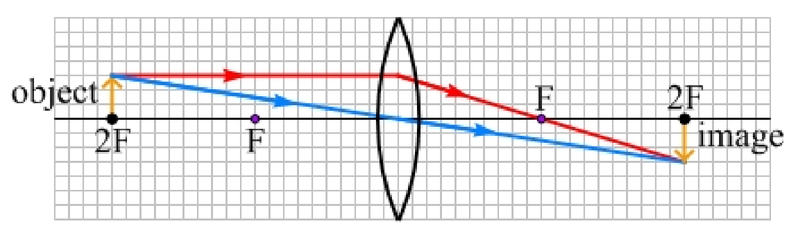

Figure 24.33: When the object is at twice the focal length from the lens, so is the image.

Applying the thin-lens equation to find the image distance, we get:

$$d_i = \frac{d_o \times f}{d_o - f} = \frac{(20\text{ cm})\times(+10\text{ cm})}{(20\text{ cm})-(+10\text{ cm})} = \frac{+200\text{ cm}^2}{10\text{ cm}} = +20\text{ cm},$$ matching the ray diagram.

Step 3 – *Repeat step 1, with the object 15 cm from the mirror.* No matter what the object distance is, the parallel ray always does the same thing, being refracted by the lens to pass through the focal point on the far side. The path of the second ray, in blue, depends on the object's position. The ray diagram (Figure 24.34) shows that the image is real, inverted, larger than the object, and about twice as far from the lens as the object.

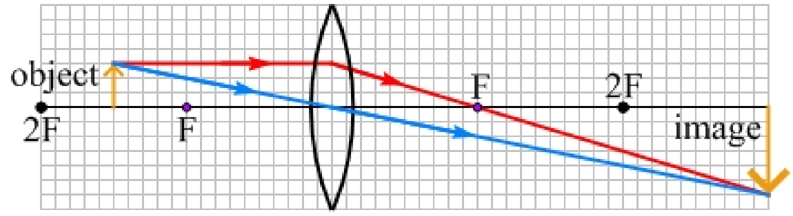

Figure 24.34: A ray diagram for a situation in which the object is between twice the focal length from the lens and the focal point.

Applying the thin-lens equation gives:

$$d_i = \frac{d_o \times f}{d_o - f} = \frac{(15\text{ cm})\times(+10\text{ cm})}{(15\text{ cm})-(+10\text{ cm})} = \frac{+150\text{ cm}^2}{5.0\text{ cm}} = +30\text{ cm}$$, matching the ray diagram.

Step 4 – *Repeat step 1, with the object at a focal point.* As shown in Figure 24.35, the two refracted rays are parallel to one another, and never meet. In such a case the image is formed at infinity.

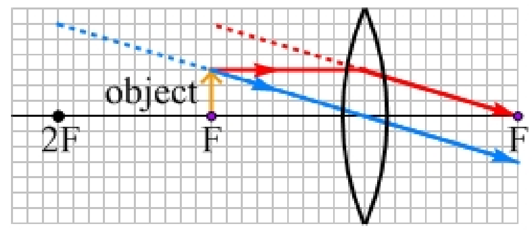

Figure 24.35: A ray diagram for a situation in which the object is at the focal point.

Applying the thin-lens equation gives:

$$d_i = \frac{d_o \times f}{d_o - f} = \frac{(10\text{ cm})\times(+10\text{ cm})}{(10\text{ cm})-(+10\text{ cm})} = \frac{+100\text{ cm}^2}{0\text{ cm}} = +\infty$$, which agrees with the ray diagram.

Step 5 – *Repeat step 1, with the object 5.0 cm from the lens.* When the object is closer to the lens than the focal point, the refracted rays diverge to the right of the lens, and they must be extended back to meet on the left of the lens. The result is a virtual, upright image that is larger than the object, as shown in Figure 24.36. If you look at the object through the lens, your brain interprets the light as coming from the image.

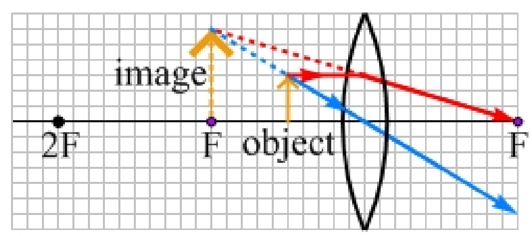

Figure 24.36: A ray diagram for a situation in when the object is between the lens and its focal point.

Applying the thin-lens equation gives:

$$d_i = \frac{d_o \times f}{d_o - f} = \frac{(5.0\text{ cm})\times(+10\text{ cm})}{(5.0\text{ cm})-(+10\text{ cm})} = \frac{+50\text{ cm}^2}{-5.0\text{ cm}} = -10\text{ cm}.$$

Recalling the sign convention that a negative image distance is consistent with a virtual image, the result from the thin-lens equation is consistent with the ray diagram.

Key idea for converging lenses: Depending on where the object is relative to the focal point of a converging lens, the lens can form an image of the object that is real or virtual. If the image is real, it can be larger than, smaller than, or the same size as the object. If the image is virtual, the image is larger than the object. **Related End-of-Chapter Exercises: 44, 50, 53.**

Essential Question 24.7: When an object is placed 20 cm from a lens, the image formed by the lens is real. What kind of lens is it? What, if anything, can you say about the lens' focal length?

Answer to Essential Question 24.7: The lens must be converging, because a diverging lens cannot produce an image that is larger than the object. A converging lens produces a real image only when the object distance is larger than the focal length, so the focal length in this case must be positive but less than 20 cm.

24-8 An Example Problem Involving a Lens

Let's begin by discussing a general approach we can use to solve problems involving a lens. We will then apply the method to a particular situation.

A general method for solving problems involving a lens

1. Sketch a ray diagram, showing rays leaving the tip of the object and being refracted by the lens. Where the refracted rays meet is where the tip of the image is located. The ray diagram gives us qualitative information about the location and size of the image and about the characteristics of the image.

2. Apply the thin-lens equation and/or the magnification equation. Make sure that the signs you use match those listed in the sign convention in section 24-5. The equations provide quantitative information about the location and size of the image and about the image characteristics.

3. Check the results of applying the equations with your ray diagram, to see if the equations and the ray diagram give consistent results.

Rays that are easy to draw the reflections for

To locate an image on a ray diagram, you need a minimum of two rays. If you draw more than two rays, however, you can check the image location you find with the first two rays. You can draw any number of rays being refracted by the lens, but some are easier to draw than others because we know exactly where the refracted rays go for these rays. Such rays are shown on Figure 24.37, and include:

1. The ray, in red, that goes parallel to the principal axis, and refracts to pass through the focal point on the far side of the lens (converging lens), or away from the focal point on the near side of the lens (diverging lens).

2. The ray, in blue, that passes straight through the center of the lens without being deflected.

3. The ray, in green, that travels along the straight line connecting the tip of the object and the focal point not associated with the first ray. This ray is refracted by the lens to go parallel to the principal axis.

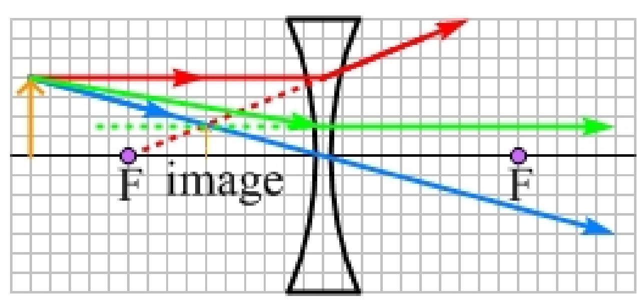

Figure 24.37: An example of the three rays that are easy to draw the refracted rays for. When you look at the object through the lens, your brain interprets the light as traveling in straight lines as if it emanated from the image.

EXAMPLE 24.8 – Applying the general method

When you look at a cat through a lens that has its focal points at distances of 24 cm on either side of the lens, you see an image of the cat that is 1.5 times as large as the cat. How far is the cat from the lens? Sketch a ray diagram to check your calculations.

SOLUTION

In this case, let's first apply the equations and then draw the ray diagram. The lens is clearly a converging lens, because only converging lenses produce images that are larger than the object. One possibility is that the lens produces a virtual, upright image, so the sign of the magnification is positive. Applying the magnification equation, we get:

$m = +1.5 = -\frac{d_i}{d_o}$, which tells us that $\frac{1}{d_i} = -\frac{1}{1.5d_o}$.

Applying the thin-lens equation:

$$\frac{1}{f} = \frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{d_o} - \frac{1}{1.5d_o} = \frac{3}{3d_o} - \frac{2}{3d_o} = \frac{1}{3d_o}.$$

Thus, we find that $3d_o = f = +24$ cm, so $d_o = +8.0$ cm and we can show that $d_i = -12$ cm.

The ray diagram for this situation is shown in Figure 24.38, confirming the calculations.

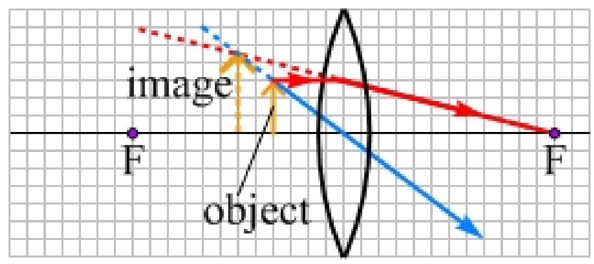

Figure 24.38: A ray diagram for the solution involving a virtual image. Each box on the grid measures 2 cm × 2 cm.

The solution above is only one of the possible answers. The image could also be real and inverted, so the sign of the magnification is negative. Applying the magnification equation, we get:

$m = -1.5 = -\frac{d_i}{d_o}$, which tells us that $\frac{1}{d_i} = +\frac{1}{1.5d_o}$.

Applying the mirror equation:

$$\frac{1}{f} = \frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{d_o} + \frac{1}{1.5d_o} = \frac{3}{3d_o} + \frac{2}{3d_o} = \frac{5}{3d_o}.$$

Thus, we find that $d_o = \frac{5}{3} f = +40$ cm, and we can show that $d_i = +60$ cm.

The ray diagram for this situation is shown in Figure 24.39, again confirming the calculations above. These two rays, and all rays that travel from the tip of the object to the tip of the image, take the same time to get there.

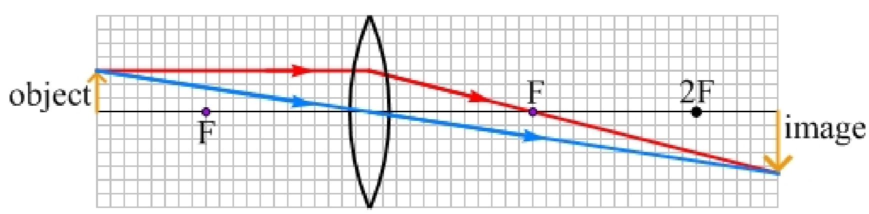

Figure 24.39: A ray diagram for the situation involving a real image. Each box on the grid measures 2 cm × 2 cm.

Related End-of-Chapter Exercises: 19 – 22.

Essential Question 24.8: Return to the situation described in Example 24.8. Would there still be two solutions if the image was smaller than the object? Explain.