Springs

We are used to dealing with constant forces. Springs are more complicated - not only does the magnitude of the spring force vary, the direction of the force depends on whether the spring is being stretched or compressed.

Measuring all distances from the equilibrium length of the spring, the force from an ideal spring is given by:

Hooke's Law: $\vec{F} = -k\,\vec{x}$

k is the spring constant, a measure of the stiffness of the spring in units of N/m.

The minus sign means that the spring force is opposite in direction to the displacement.

Simulation

Force

$$\vec{F} = -k\,\vec{x}$$

A spring hangs vertically down from a support. When you hang a 100-gram mass from the bottom end of the spring and stop any motion of the system the spring is stretched by 10 cm. Determine the spring constant. Hint: sketch a free-body diagram.

If we hang another 100-gram mass on the spring the spring stretches further. Is the additional stretch more than, less than, or equal to 10 cm?

Energy for springs

The elastic potential energy of a spring is given by:

$$U = \frac{1}{2}kx^2$$

A block connected to a horizontal spring sits on a frictionless table. The system is released from rest, with the spring initially compressed. What happens to the energy stored in the spring? What is the maximum speed of the block?

As the block oscillates, the energy goes back and forth between elastic potential energy and kinetic energy. Simulation

$$U_{max} = K_{max} \qquad \Rightarrow \qquad \frac{1}{2}kx_{max}^2 = \frac{1}{2}mv_{max}^2$$

$$v_{max} = x_{max}\left(\sqrt{\frac{k}{m}}\right)$$

Energy conservation

$$U = \frac{1}{2}kx^2$$

A block with a mass of 1.0 kg is released from rest on a frictionless incline. At the bottom of the incline, which is 1.8 m vertically below where the block started, the block slides across a horizontal frictionless surface before encountering a spring that has a spring constant of 100 N/m. What is the maximum compression of the spring? Hint: use energy conservation.

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$$U_i = U_f$$

$$mgh = \frac{1}{2} k x_{\text{max}}^2$$

$$x_{\text{max}} = \sqrt{\frac{2mgh}{k}} = \sqrt{\frac{2 \times 1.0\ \text{kg} \times 10\ \text{N/kg} \times 1.8\ \text{m}}{100\ \text{N/m}}} = \sqrt{0.36\ \text{m}^2} = 0.60\ \text{m}$$

Acceleration?

What is the block’s maximum acceleration?

Acceleration

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The acceleration is maximum when the force is maximized, which is when the spring is most compressed. Just worrying about the magnitudes:

$$F_{\mathrm{max}} = kx = 100\ \mathrm{N/m} \times 0.60\ \mathrm{m} = 60\ \mathrm{N}$$

$$a_{\mathrm{max}} = \frac{F_{\mathrm{max}}}{m} = \frac{60\ \mathrm{N}}{1.0\ \mathrm{kg}} = 60\ \mathrm{m/s^2}$$

Energy graphs

Have a look at graphs of kinetic energy and elastic potential energy, first as a function of time and then as a function of position.

Which color goes with kinetic energy, and which with elastic potential energy?

Simulation

$$U = \frac{1}{2}kx^2$$

Energy graphs

Which graph is which?

1. The red one is the kinetic energy; the blue one is the potential energy.
2. The blue one is the kinetic energy; the red one is the potential energy.
3. The graphs are interchangeable so you can't tell which is which.

Understanding oscillations

We have an object attached to a spring. The object is on a horizontal frictionless surface. We move the object so the spring is stretched, and then we release it. The object oscillates back and forth in what we call ***simple harmonic motion***, in which no energy is lost. How do we find the object's maximum speed?

Understanding oscillations

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$$U_{\max} = K_{\max} \qquad \Rightarrow \qquad \frac{1}{2}kA^2 = \frac{1}{2}mv^2_{\max}$$

A is called the amplitude – the maximum distance from equilibrium. The key lesson to take away from this is that you already know a lot about analyzing simple harmonic motion situations - apply energy conservation, especially when you want to relate a speed to a position.

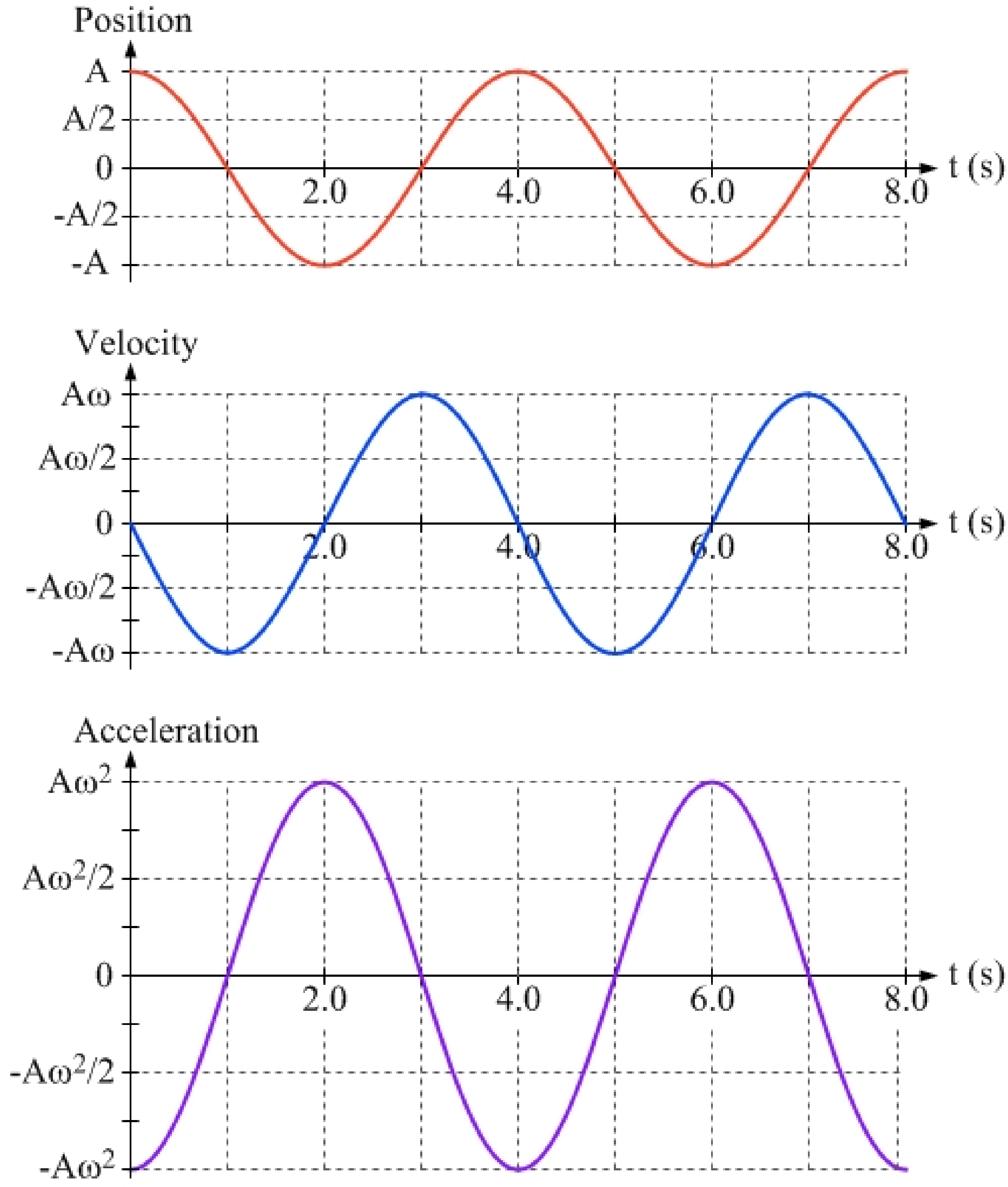
Splitting the energy

An object attached to a spring is pulled a distance *A* from the equilibrium position and released from rest. It then experiences simple harmonic motion. When the object is *A*/2 from the equilibrium position how is the energy divided between spring potential energy and the kinetic energy of the object? Assume mechanical energy is conserved.

1. The energy is 25% spring potential energy and 75% kinetic.
2. The energy is 50% spring potential energy and 50% kinetic.
3. The energy is 75% spring potential energy and 25% kinetic.
4. One of the above, but it depends whether the object is moving toward or away from the equilibrium position.

Motion Graphs

If we graph position, velocity, and acceleration of the object on the spring, as a function of time, we get the following. The period of these oscillations happens to be 4.0 seconds.



Equations

$$x = A\cos(\omega t)$$

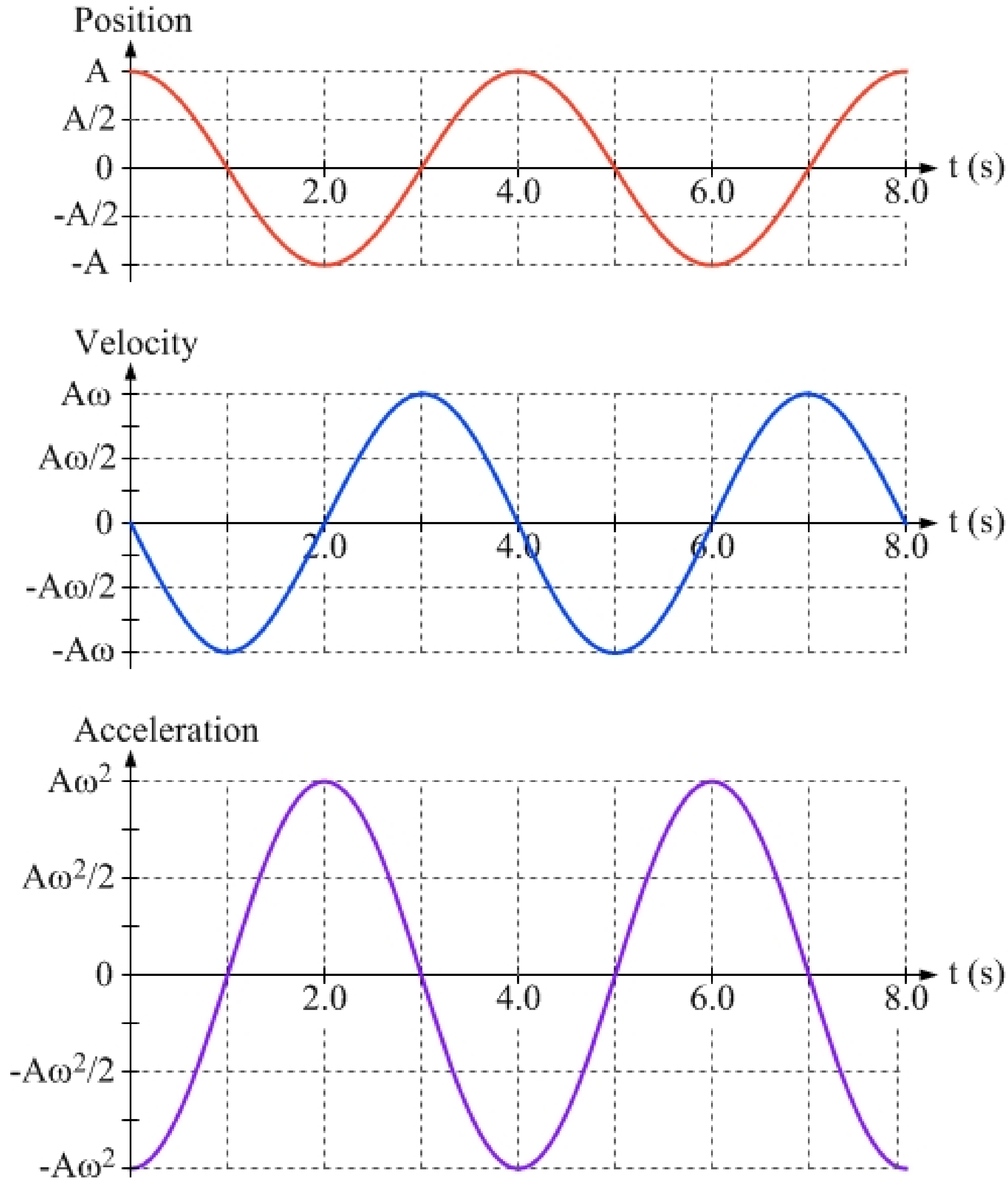
$$v = -A\omega\sin(\omega t)$$

$$v_{max} = A\omega$$

$$a = -A\omega^2\cos(\omega t)$$

$$a_{max} = A\omega^2$$

ω is called the angular frequency.



What determines angular frequency?

In general, we have $\vec{a} = -\omega^2 \vec{x}$.

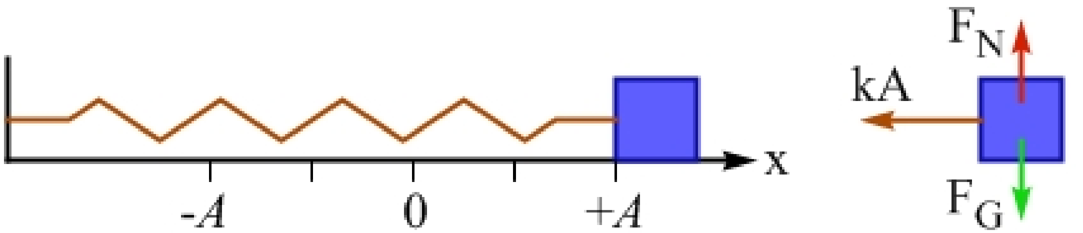
In a specific case, the angular frequency is determined by the forces involved. For an object on a spring, we have:

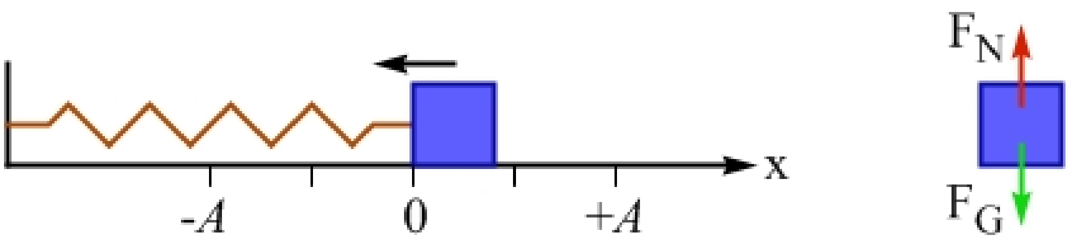
$$\sum \vec{F} = m\vec{a}$$

$$-k\vec{x} = m\vec{a}$$

$$\vec{a} = -\frac{k}{m}\vec{x}$$

$$\omega^2 = \frac{k}{m} \quad \Rightarrow \quad \omega = \sqrt{\frac{k}{m}}$$





A change in frequency

Simulation

The first set of graphs is for an angular frequency ω = 1 rad/s. The second set of graphs is for ω = 0.6 rad/s. This change of ω is accomplished either by decreasing the spring constant or by increasing the mass. Which change did we make in this case?

1. We decreased the spring constant
2. We increased the mass
3. We could have done one or the other, you can't tell the difference

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A change in frequency

We can’t tell the difference. All we can tell is that the angular frequency has changed, but the graphs give us no information about whether the spring constant or the mass is different.

A change in frequency, II

Simulation

The first set of graphs is for an angular frequency ω = 1 rad/s. The second set of graphs is for ω = 0.8 rad/s. This change of ω is accomplished either by decreasing the spring constant or by increasing the mass. Which change did we make in this case?

1. We decreased the spring constant
2. We increased the mass
3. We could have done one or the other, you can't tell the difference

A change in frequency, II

The first set of graphs is for an angular frequency ω = 1 rad/s. The second set of graphs is for ω = 0.8 rad/s. This change of ω is accomplished either by decreasing the spring constant or by increasing the mass. Which change did we make in this case?

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3. We could have done one or the other, you can't tell the difference

A change in frequency

With the energy graphs, we can tell the difference. All the energy is in the spring, initially, with the spring energy given by:

$$U_i = \frac{1}{2}kA^2$$

The graphs tell us that the energy stored in the spring is smaller, but we have not changed the amplitude. Thus, we must have changed the spring constant, *k*.

Thinking about time

An object attached to a spring is pulled a distance *A* from the equilibrium position and released from rest. It then experiences simple harmonic motion with a period *T*. The time taken to travel between the equilibrium position and a point *A* from equilibrium is *T*/4. How much time is taken to travel between points *A*/2 from equilibrium and *A* from equilibrium? Assume the points are on the same side of the equilibrium position, and that mechanical energy is conserved.

1. T/8
2. More than T/8
3. Less than T/8
4. It depends whether the object is moving toward or away from the equilibrium position

Using the time equations

An object attached to a spring is pulled a distance *A* from the equilibrium position and released from rest. It then experiences simple harmonic motion with a period *T*. The time taken to travel between the equilibrium position and a point *A* from equilibrium is *T*/4. How much time is taken to travel between points *A*/2 from equilibrium and *A* from equilibrium? Assume the points are on the same side of the equilibrium position, and that mechanical energy is conserved.

Let's say the object is *A* from equilibrium at $t = 0$, so the equation $x = A\cos(\omega t)$ applies.

Now just solve for the time *t* when the object is *A*/2 from equilibrium.

Using the time equations

Solve for *t* in the equation:

$$\frac{A}{2} = A\cos(\omega t) \quad \Rightarrow \quad \frac{1}{2} = \cos(\omega t)$$

Here we can use $\omega = \frac{2\pi}{T}$, so we need to solve:

$$\frac{1}{2} = \cos(\frac{2\pi t}{T})$$

Take the inverse cosine of both sides.

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Take the inverse cosine of both sides. We need to work in radians!

$$\frac{\pi}{3} = \frac{2\pi t}{T} \quad \Rightarrow \quad t = \frac{T}{6}$$

This is more than T/8, because the object travels at a small average speed when it is far from equilibrium.

General features of simple harmonic motion

A system experiencing simple harmonic motion has:

- No loss of mechanical energy.

- A restoring force or torque that is proportional, and opposite in direction, to the displacement from equilibrium.

The motion is described by an equation of the form:

$$x = A\cos(\omega t)$$

where ω is the angular frequency of the system.

The period of oscillation is $T = \dfrac{1}{f} = \dfrac{2\pi}{\omega}$

Connecting SHM and circular motion

Compare the motion of an object experiencing simple harmonic motion (SHM) to that of an object undergoing uniform circular motion. Simulation.

The equation of motion for the object on the spring is the same as that for the x-component of the circular motion,

$$x = A\cos(\omega t)$$

Amplitude does not affect frequency!

For simple harmonic motion, a neat feature is that the oscillation frequency is completely independent of the amplitude of the oscillation. Simulation.

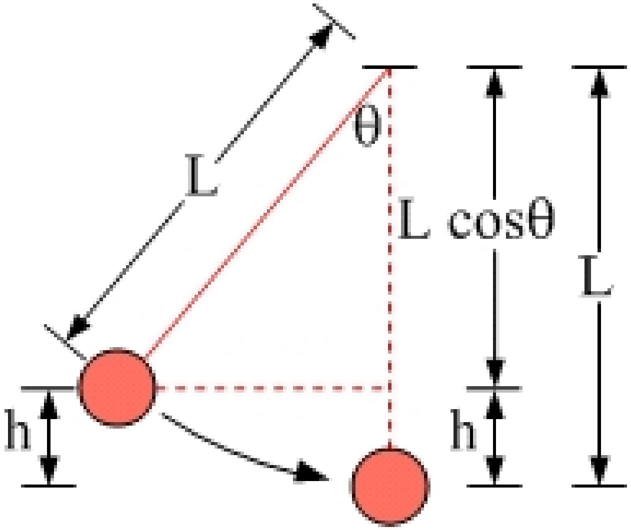
A pendulum question

A simple pendulum is a ball on a string or light rod. We have two simple pendula of equal lengths. One has a heavy object attached to the string, and the other has a light object. Which has the longer period of oscillation?

1. The heavy one
2. The light one
3. Neither, they're equal

Analyze it using energy

Pull back the ball so it is a vertical distance *h* above the equilibrium position.



If you release the ball from rest, what is its speed when it passes through equilibrium?

Energy conservation: $mgh = \frac{1}{2}mv^2$

We get our familiar result $v = \sqrt{2gh}$

Does the ball's mass matter? No.
Simulation

Free-body diagrams for a simple pendulum

Sketch a free-body diagram for a pendulum when you release it from rest, after displacing it to the left.

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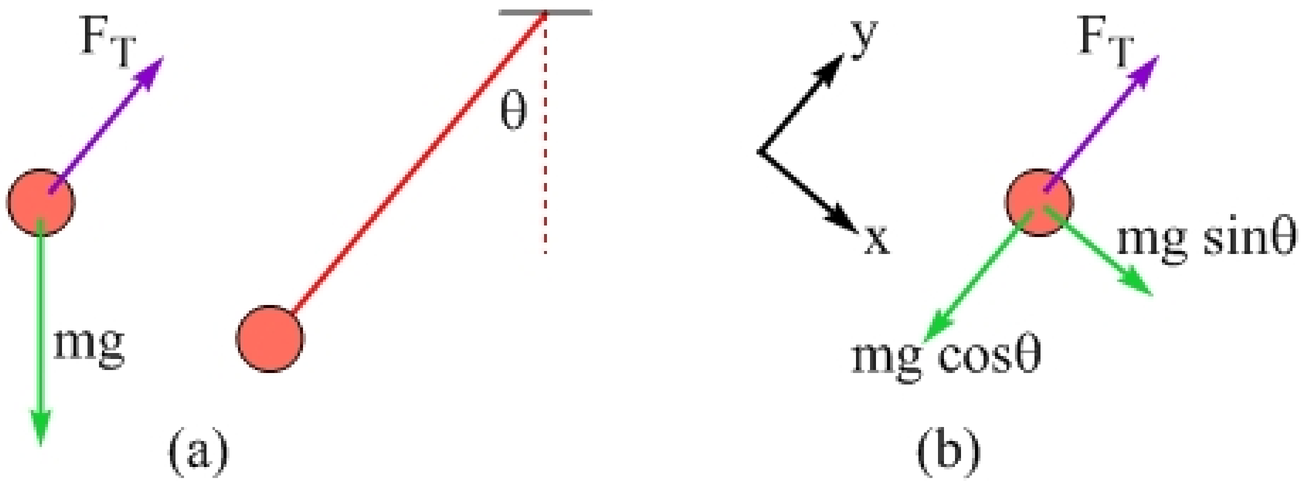

Free-body diagrams for a simple pendulum

Sketch a free-body diagram for the pendulum as it passes through equilibrium.

How should we analyze the pendulum?

Free-body diagrams for a simple pendulum

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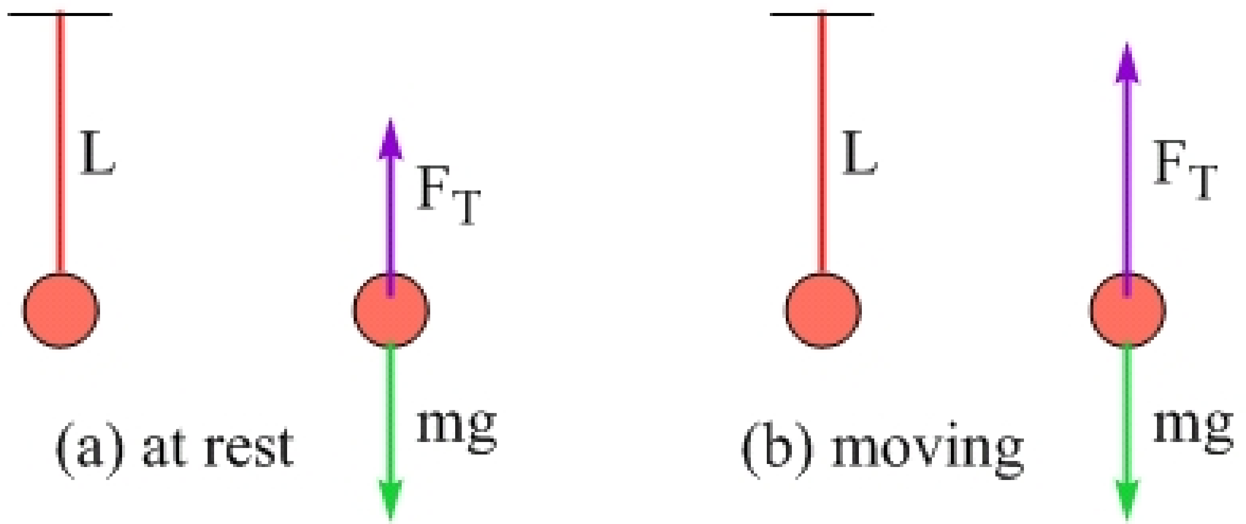


How should we analyze the pendulum? Let’s try torque.

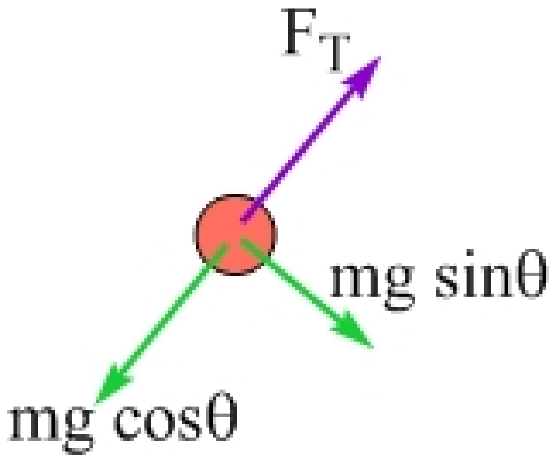
Analyzing the pendulum

Take torques around the support point.

$$\sum \vec{\tau} = I\vec{\alpha}$$

$$-Lmg\sin\theta = mL^2\alpha$$

$$\alpha = -\frac{g}{L}\sin\theta$$



For small angles we can say that $\sin\theta \approx \theta$

$$\alpha \approx -\frac{g}{L}\theta$$ which has the SHM form $$\alpha = -\omega^2\theta$$

So, the angular frequency is $\omega = \sqrt{\frac{g}{L}}$

Simulation