8-1 Newton's Law of Universal Gravitation

One of the most famous stories of all time is the story of Isaac Newton sitting under an apple tree and being hit on the head by a falling apple. It was this event, so the story goes, that led Newton to realize that the same force that brought the apple down on his head was also responsible for keeping the Moon in its orbit around the Earth, and for keeping all the planets of the solar system, including our own planet Earth, in orbit around the Sun. This force is the force of gravity.

It is hard to over-state the impact of Newton's work on gravity. Prior to Newton, it was widely thought that there was one set of physical laws that explained how things worked on Earth (explaining why apples fall down, for instance), and a completely different set of physical laws that explained the motion of the stars in the heavens. Armed with the insight that events on Earth, as well as the behavior of stars, can be explained by a relatively simple equation (see the box below), humankind awoke to the understanding that our fates are not determined by the whims of gods, but depend, in fact, on the way we interact with the Earth, and in the way the Earth interacts with the Moon and the Sun. This simple, yet powerful idea, that we have some control over our own lives, helped trigger a real enlightenment in many areas of arts and sciences.

The force of gravity does not require the interacting objects to be in contact with one another. The force of gravity is an attractive force that is proportional to the product of the masses of the interacting objects, and inversely proportional to the square of the distance between them.

> A gravitational interaction involves the attractive force that any object with mass exerts on any other object with mass. The general equation to determine the gravitational force an object of mass *M* exerts on an object of mass *m* when the distance between their centers-of-mass is *r* is:
>
> $$\vec{F}_G = -\frac{GmM}{r^2}\hat{r}$$ (Equation 8.1: **Newton's Law of Universal Gravitation**)
>
> where $G = 6.67\times10^{-11}\text{N m}^2/\text{kg}^2$ is known as the universal gravitational constant. The magnitude of the force is equal to GmM/r^2 while the direction is given by $-\hat{r}$, which means that the force is attractive, directed back toward the object exerting the force.

At the surface of the Earth, should we use $\vec{F}_G = m\vec{g}$ or Newton's Law of Universal Gravitation instead? Why is *g* equal to 9.8 N/kg at the surface of the Earth, anyway? The two equations must be equivalent to one another, at least at the surface of the Earth, because they represent the same gravitational interaction. If we set the expressions equal to one another we get:

$$mg = \frac{GmM}{r^2} \quad \text{which gives} \quad g = \frac{GM}{r^2}.$$

At the surface of the Earth *M* is the mass of the Earth, $M_E = 5.98\times10^{24}$ kg, and *r* is the radius of the Earth, $R_E = 6.37\times10^6$ m. So, the magnitude of g at the Earth's surface is:

$$g_E = \frac{GM_E}{R_E^2} = \frac{(6.67\times10^{-11}\text{ N m}^2/\text{kg}^2)(5.98\times10^{24}\text{ kg})}{\left(6.37\times10^6\text{ m}\right)^2} = 9.83\text{ N/kg}.$$

For any object at the surface of the Earth, when we use Newton's Law of Universal Gravitation, the factors *G*, M_E, and R_E are all constants, so, until this point in the book, we have simply been replacing the constant value of GM_E/R_E^2 by *g* = 9.8 N/kg.

EXAMPLE 8.1 – A two-dimensional situation

Three balls, of mass *m*, 2*m*, and 3*m*, are placed at the corners of a square measuring L on each side, as shown in Figure 8.5. Assume this set of three balls is not interacting with anything else in the universe. What is the magnitude and direction of the net gravitational force on the ball of mass *m*?

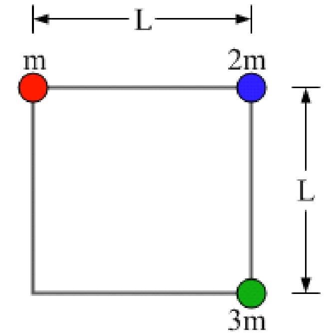

Figure 8.1: Three balls placed at the corners of a square.

SOLUTION

Let's begin by attaching force vectors to the ball of mass *m*. In Figure 8.6 each vector is color-coded based on the object exerting the force. The length of each vector is proportional to the magnitude of the force it represents.

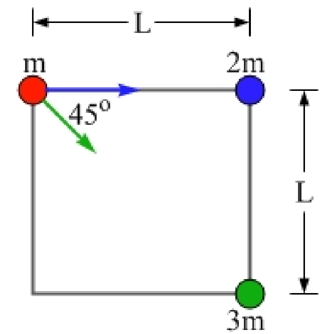

Figure 8.2: Attaching force vectors to the ball of mass *m*.

We can find the two individual forces acting on the ball of mass *m* using Newton's Law of Universal Gravitation. Let's define +x to the right and +y up.

From the ball of mass 2*m*: $\vec{F}_{21} = \frac{Gm(2m)}{L^2}$ to the right.

From the ball of mass 3*m*: $\vec{F}_{31} = \frac{Gm(3m)}{L^2 + L^2}$ at 45° below the x-axis.

Finding the net force is a vector-addition problem.

In the *x*-direction, we get: $\vec{F}_{1x} = \vec{F}_{21x} + \vec{F}_{31x} = +\frac{2Gm^2}{L^2} + \frac{3Gm^2}{2L^2}\cos 45^\circ = \left(2 + \frac{3}{2\sqrt{2}}\right)\frac{Gm^2}{L^2}$.

In the *y*-direction, we get: $\vec{F}_{1y} = \vec{F}_{21y} + \vec{F}_{31y} = 0 - \frac{3Gm^2}{2L^2}\sin 45^\circ = \left(-\frac{3}{2\sqrt{2}}\right)\frac{Gm^2}{L^2}$.

The Pythagorean theorem gives the magnitude of the net force on the ball of mass *m*:

$$F_1 = \sqrt{F_{1x}^2 + F_{1y}^2} = \sqrt{\left(4 + \frac{6}{\sqrt{2}} + \frac{9}{8} + \frac{9}{8}\right)}\frac{Gm^2}{L^2} = 3.24\frac{Gm^2}{L^2}.$$

The angle is given by: $\tan\theta = \frac{F_{1y}}{F_{1x}} = \frac{\frac{3}{2\sqrt{2}}}{\frac{4\sqrt{2}+3}{2\sqrt{2}}} = \frac{3}{4\sqrt{2}+3}$.

So, the angle is 19.1° below the *x*-axis.

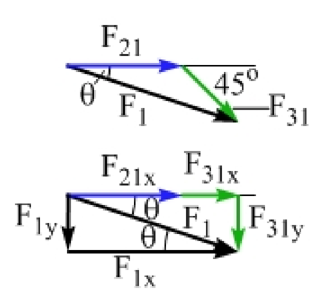

Figure 8.3: The triangle representing the vector addition problem being solved above.

Related End-of-Chapter Exercises: 16, 56 – 58.

Essential Question 8.1: The Sun has a much larger mass than the Earth. Which object exerts a larger gravitational force on the other, the Sun or the Earth?

Answer to Essential Question 8.1: Newton's third law tells us that the gravitational force the Sun exerts on the Earth is equal in magnitude (and opposite in direction) to the gravitational force the Earth exerts on the Sun. This follows from Equation 8.1, because, whether we look at the force exerted by the Sun or the Earth, the factors going into the equation are the same.

8-2 The Principle of Superposition

EXPLORATION 8.2 – Three objects in a line

Three balls, of mass m, $2m$, and $3m$, are equally spaced along a line. The spacing between the balls is r. We can arrange the balls in three different ways, as shown in Figure 8.2. In each case the balls are in an isolated region of space very far from anything else.

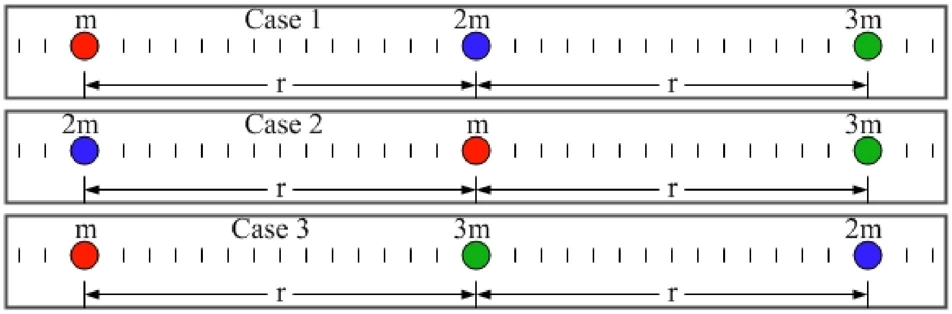

Figure 8.2: Three different arrangements of three balls of mass m, $2m$, and $3m$ placed on a line with a distance r between neighboring balls.

Step 1 – ***How many forces does each ball experience in each case?*** Each ball experiences two gravitational forces, one from each of the other balls. We can neglect any other interactions.

Step 2 – ***Consider Case 1. Is the force that the ball of mass m exerts on the ball of mass 3m affected by the fact that the ball of mass 2m lies between the other two balls?***

Interestingly, no. To find the net force on any object, we simply add the individual forces acting on an object as vectors. This is known as **the principle of superposition**, and it applies to many different physical situations. In case 1, for instance, we find the force the ball of mass m applies to the ball of mass $3m$ as if the ball of mass $2m$ is not present. The net force on the ball of mass $3m$ is the vector addition of that force and the force on the $3m$ ball from the ball of mass $2m$.

Step 3 – ***In which case does the ball of mass 2m experience the largest-magnitude net force? Argue qualitatively.*** Let's attach arrows to the ball of mass $2m$, as in Figure 8.3, to represent the two forces the ball experiences in each case. The length of each arrow is proportional to the force.

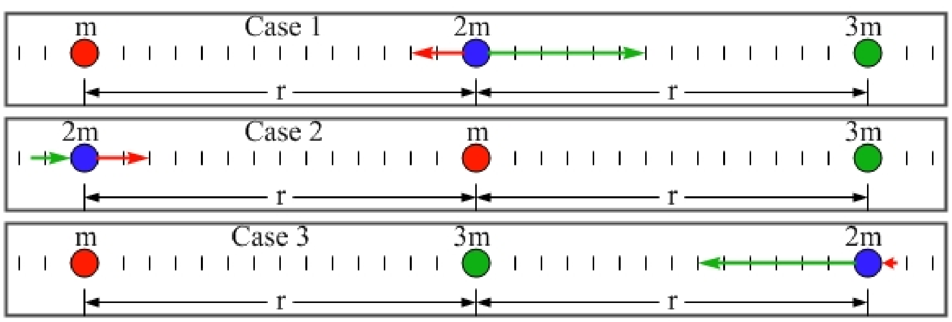

Figure 8.3: Attaching force vectors to the ball of mass $2m$. The vectors are color-coded based on the color of the object exerting the force. The length of each vector is drawn in units of Gm^2/r^2.

In case 1, the two forces partly cancel, and, in case 2, the forces add but give a smaller net force than that in case 3. Thus, the ball of mass 2*m* experiences the largest-magnitude net force in Case 3.

Step 4 – ***Calculate the force experienced by the ball of mass 2m in each case.***

To do this, we will make extensive use of Newton's Universal Law of Gravitation. Let's define right to be the positive direction, and use the notation $\vec{F}_{21}$ for the force that the ball of mass 2*m* experiences from the ball of mass *m*. In each case:

$$\vec{F}_{2,net} = \vec{F}_{21} + \vec{F}_{23}$$

$$\text{Case 1: } \vec{F}_{2,net} = \vec{F}_{21} + \vec{F}_{23} = -\frac{Gm(2m)}{r^2} + \frac{G(2m)(3m)}{r^2} = -\frac{2Gm^2}{r^2} + \frac{6Gm^2}{r^2} = +\frac{4Gm^2}{r^2}$$

$$\text{Case 2: } \vec{F}_{2,net} = \vec{F}_{21} + \vec{F}_{23} = +\frac{Gm(2m)}{r^2} + \frac{G(2m)(3m)}{(2r)^2} = +\frac{2Gm^2}{r^2} + \frac{3Gm^2}{2r^2} = +\frac{7Gm^2}{2r^2}$$

$$\text{Case 3: } \vec{F}_{2,net} = \vec{F}_{21} + \vec{F}_{23} = -\frac{Gm(2m)}{(2r)^2} - \frac{G(2m)(3m)}{r^2} = -\frac{Gm^2}{2r^2} - \frac{6Gm^2}{r^2} = -\frac{13Gm^2}{2r^2}$$

This approach confirms that the ball of mass 2*m* experiences the largest-magnitude net force in case 3.

Step 5 - ***Rank the three cases, from largest to smallest, based on the magnitude of the net force exerted on the ball in the middle of the set of three balls.*** Let's extend our pictorial method by attaching force vectors to each ball in each case, as in Figure 8.4.

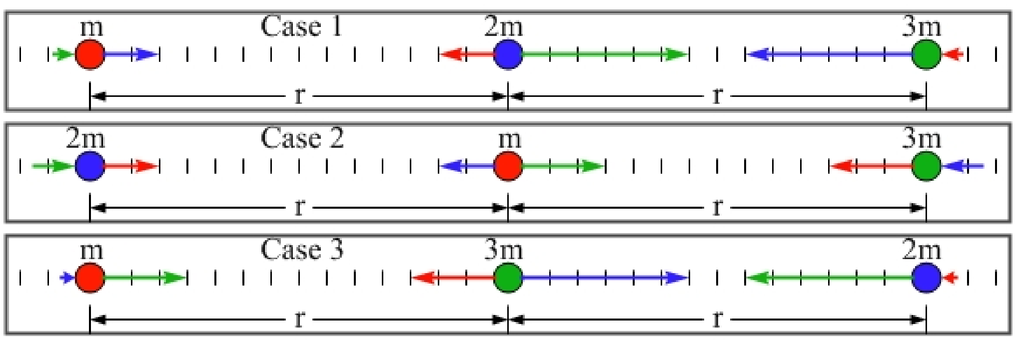

Figure 8.4: Attaching force vectors to the balls in each case. The force vectors are color-coded according to the ball applying the force. The length of each vector is drawn in units of Gm^2/r^2.

Again, when considering the net force on the middle ball, we need to add the individual forces as vectors. Referring to Figure 8.4, ranking the cases based on the magnitude of the net force exerted on the middle ball gives **Case 1 > Case 3 > Case 2**.

Key idea about the principle of superposition: The net force acting on an object can be found using the principle of superposition, adding all the individual forces together as vectors and remembering that each individual force is unaffected by the presence of other forces.
Related End of Chapter Exercises: 15 and 27.

Essential Question 8.2: In the Exploration above, which ball experiences the largest-magnitude net force in (i) Case 1 (ii) Case 2 (iii) Case 3?

Answer to Essential Question 8.2: We could determine the net force on each object quantitatively, but Figure 8.4 shows that the object experiencing the largest-magnitude net force is the object of mass 3*m* in cases 1 and 2, and the object of mass 2*m* in case 3.

In general, in the case of three objects of different mass arranged in a line the object experiencing the largest net force will be one of the objects at the end of the line, the one with the larger mass. The object in the middle will not have the largest net force because the two forces it experiences are in opposite directions.

8-3 Gravitational Field

Let's discuss the concept of a gravitational field, which is represented by $\vec{g}$. So far, we have referred to $\vec{g}$ as "the acceleration due to gravity", but a more appropriate name is "the strength of the local gravitational field."

A field is something that has a magnitude and direction at all points in space. One way to define the gravitational field at a particular point is in terms of the gravitational force that an object of mass *m* would experience if it were placed at that point:

$$\vec{g} = \frac{\vec{F}_G}{m} . \qquad \text{(Equation 8.2: \textbf{Gravitational field})}$$

The units for gravitational field are N/kg, or m/s^2.

A special case is the gravitational field outside an object of mass *M*, such as the Earth, that is produced by that object:

$$\vec{g} = -\frac{GM}{r^2}\hat{r} , \qquad \text{(Equation 8.3: \textbf{Gravitational field from a point mass})}$$

where *r* is the distance from the center of the object to the point. The magnitude of the field is GM / r^2, while the direction is given by $-\hat{r}$, which means that the field is directed back toward the object producing the field.

One way to think about a gravitational field is the following: it is a measure of how an object, or a set of objects, with mass influences the space around it.

Visualizing the gravitational field

It can be useful to draw a picture that represents the gravitational field near an object, or a set of objects, so we can see at a glance what the field in the region is like. In general there are two ways to do this, by using either field lines or field vectors. The field-line representation is shown in Figure 8.7. If Figure 8.7 (a) represents the field at the surface of the Earth, Figure 8.7 (b) could represent the field at the surface of another planet where *g* is twice as large as it is at the surface of the Earth. In both these cases we have a **uniform field**, because the field lines are equally spaced and parallel. In Figure 8.7 (c) we have zoomed out far from a planet to get a wider perspective on how the planet affects the space around it, while in Figure 8.7 (d) we have done the same thing for a different planet with half the mass, but the same radius, as the planet in (c).

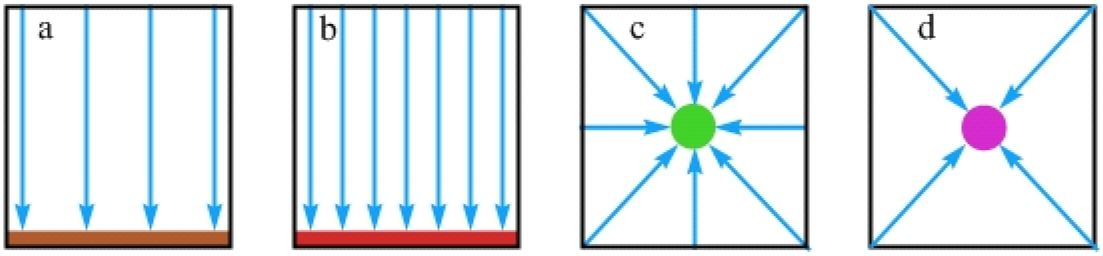

Figure 8.7: Field-line diagrams for various situations. Diagrams *a* and *b* represent uniform gravitational fields, with the field in *b* two times larger than that in *a*. Diagrams *c* and *d* represent non-uniform fields, such as the fields near a planet. The field at the surface of the planet in *c* is two times larger than that at the surface of the planet in *d*.

Question: How is the direction of the gravitational field at a particular point shown on a field-line diagram? What indicates the relative strength of the gravitational field at a particular point on the field-line diagram?

Answer: Each field line has a direction marked on it with an arrow that shows the direction of the gravitational field at all points along the field line. The relative strength of the gravitational field is indicated by the density of the field lines (i.e., by how close the lines are). The more lines there are in a given area the larger the field.

A second method of representing a field is to use field vectors. A field vector diagram has the nice feature of reinforcing the idea that every point in space has a gravitational field associated with it, because a grid made up of equally spaced dots is superimposed on the picture and a vector is attached to each of these grid points. All the vectors are the same length. The situations represented by the field-line patterns in Figure 8.7 are now re-drawn in Figure 8.8 using the field-vector representation.

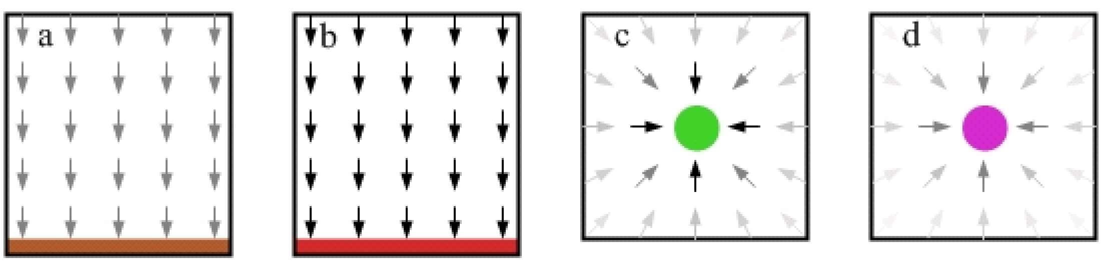

Figure 8.8: Field-vector diagrams for various situations. In figures *a* and *b* the field is uniform and directed down. The field vectors are darker in figure *b*, reflecting the fact that the field has a larger magnitude in figure *b* than in figure *a*. Figures *c* and *d* represent non-uniform fields, such as those found near a planet. Again, the fact that each field vector in figure *c* is darker than its counterpart in figure *d* tells us that the field at any point in figure *c* has a larger magnitude than the field at an equivalent point in figure *d*.

Related End of Chapter Exercises: 18, 36.

Essential Question 8.3: How is the direction of the gravitational field at a particular point shown on a field-vector diagram? What indicates the relative strength of the gravitational field at a particular point on the field-vector diagram?

Answer to Essential Question 8.3: The direction of the gravitational field at a particular point is represented by the direction of the field vector at that point (or the ones near it if the point does not correspond exactly to the location of a field vector). The relative strength of the field is indicated by the darkness of the arrow. The larger the field's magnitude, the darker the arrow.

8-4 Gravitational Potential Energy

The expression we have been using for gravitational potential energy up to this point, $U_G = mgh$, applies when the gravitational field is uniform. In general, the equation for gravitational potential energy is:

$$U_G = -\frac{GmM}{r}.$$ (Equation 8.4: **Gravitational potential energy, in general**)

This gives the energy associated with the gravitational interaction between two objects, of mass *m* and *M*, separated by a distance *r*. The minus sign tells us the objects attract one another.

Consider the differences between the *mgh* equation for gravitational potential energy and the more general form. First, when using Equation 8.4 we are no longer free to define the potential energy to be zero at some convenient point. Instead, the gravitational potential energy is zero when the two objects are infinitely far apart. Second, when using Equation 8.4 we find that the gravitational potential energy is always negative, which is certainly not what we found with *mgh*. That should not worry us, however, because **what is critical is how potential energy changes** as objects move with respect to one another. If you drop your pen and it falls to the floor, for instance, both forms of the gravitational potential energy equation give consistent results for the change in the pen's gravitational potential energy.

Equation 8.4 also reinforces the idea that, when two objects are interacting via gravity, neither object has its own gravitational potential energy. Instead, gravitational potential energy is associated with the interaction between the objects.

EXPLORATION 8.4 – Calculate the total potential energy in a system

Three balls, of mass *m*, 2*m*, and 3*m*, are placed in a line, as shown in Figure 8.10. What is the total gravitational potential energy of this system?

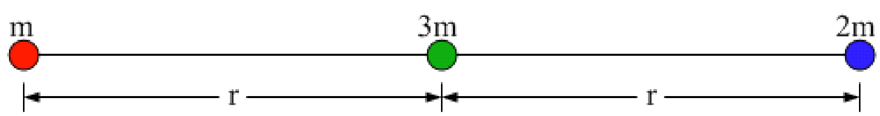

Figure 8.10: Three equally spaced balls placed in a line.

To determine the total potential energy of the system, consider the number of interacting pairs. In this case there are three ways to pair up the objects, so there are three terms to add together to find the total potential energy. Because energy is a scalar, we do not have to worry about direction. Using a subscript of 1 for the ball of mass *m*, 2 for the ball of mass 2*m*, and 3 for the ball of mass 3*m*, we get:

$$U_{Total} = U_{13} + U_{23} + U_{12} = -\frac{Gm(3m)}{r} - \frac{G(2m)(3m)}{r} - \frac{Gm(2m)}{2r} = -\frac{10Gm^2}{r}.$$

Key ideas for gravitational potential energy: Potential energy is a scalar. The total gravitational potential energy of a system of objects can be found by adding up the energy associated with each interacting pair of objects. **Related End-of-Chapter Exercises: 25, 29, 40.**

EXAMPLE 8.4 – Applying conservation ideas

A ball of mass 1.0 kg and a ball of mass 3.0 kg are initially separated by 4.0 m in a region of space in which they interact only with one another. When the balls are released from rest, they accelerate toward one another. When they are separated by 2.0 m, how fast is each ball going?

SOLUTION

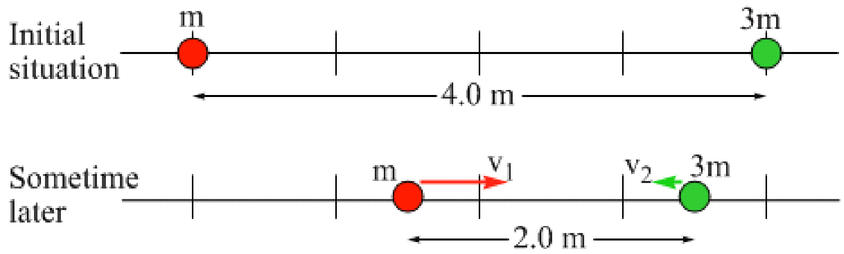

Figure 8.11: The initial situation shows the balls at rest. The force of gravity causes them to accelerate toward one another.

Figure 8.11 shows the balls at the beginning and when they are separated by 2.0 m. Analyzing forces, we find that the force on each ball increases as the distance between the balls decreases. This makes it difficult to apply a force analysis. Energy conservation is a simpler approach. Our energy equation is:

$$U_i + K_i + W_{nc} = U_f + K_f \ .$$

In this case, there are no non-conservative forces acting, and in the initial state the kinetic energy is zero because both objects are at rest. This gives $U_i = U_f + K_f$. The final kinetic energy represents the kinetic energy of the system, the sum of the kinetic energies of the two objects.

Let's solve this generally, using a mass of *m* and a final speed of v_1 for the 1.0 kg ball, and a mass of 3*m* and a final speed of v_2 for the 3.0 kg ball. The energy equation becomes:

$$-\frac{Gm(3m)}{4.0 \text{ m}} = -\frac{Gm(3m)}{2.0 \text{ m}} + \frac{1}{2}mv_1^2 + \frac{1}{2}(3m)v_2^2 \ .$$

Canceling factors of m gives: $-\frac{3Gm}{4.0 \text{ m}} = -\frac{3Gm}{2.0 \text{ m}} + \frac{1}{2}v_1^2 + \frac{3}{2}v_2^2$.

Multiplying through by 2, and combining terms, gives: $+\frac{3Gm}{2.0 \text{ m}} = v_1^2 + 3v_2^2$.

Because there is no net external force, the system's momentum is conserved. There is no initial momentum. For the net momentum to remain zero, the two momenta must always be equal-and-opposite. Defining right to be positive, momentum conservation gives:

$0 = +mv_1 - 3mv_2$, which we can simplify to $v_1 = 3v_2$.

Substituting this into the expression we obtained from applying energy conservation:

$$+\frac{3Gm}{2.0 \text{ m}} = (3v_2)^2 + 3v_2^2 = 12v_2^2$$

This gives $v_2 = \sqrt{\frac{Gm}{8.0 \text{ m}}}$, and $v_1 = 3v_2 = 3\sqrt{\frac{Gm}{8.0 \text{ m}}}$.

Using *m* = 1.0 kg, we get $v_2 = 2.9 \times 10^{-6}$ m/s and $v_1 = 8.7 \times 10^{-6}$ m/s .

Related End-of-Chapter Exercises: Problems 43 – 45.

Essential Question 8.4: Return to the previous Example. If you repeat the experiment with balls of mass 2.0 kg and 6.0 kg instead, would the final speeds change? If so, how?

Answer to Essential Question 8.4: If we double each mass, the analysis above still works. Plugging m = 2.0 kg into our speed equations shows that the speeds increase by a factor of $\sqrt{2}$.

8-5 Example Problems

EXAMPLE 8.5A – Where is the field zero?

Locations where the net gravitational field is zero are special, because an object placed where the field is zero experiences no net gravitational force. Let's place a ball of mass m at the origin, and place a second ball of mass $9m$ on the x-axis at $x = +4a$. Find all the locations near the balls where the net gravitational field associated with these balls is zero.

SOLUTION

A diagram of the situation is shown in Figure 8.9. Let's now approach the problem conceptually. At every point near the balls there are two gravitational fields, one from each ball. The net field is zero only where the two fields are equal-and-opposite. These fields are in exactly opposite directions only at locations on the x-axis between the balls. If we get too close to the first ball it dominates, and if we get too close to the second ball it dominates; there is just one location between the balls where the fields exactly balance.

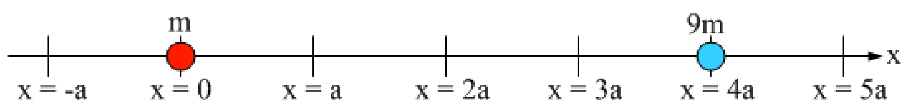

Figure 8.9: The two balls in Example 8.5A.

An equivalent approach is to use forces. Imagine having a third ball (we generally call this a **test mass**) and placing it near the other two balls. The third ball experiences two forces, one from each of the original balls, and these forces have to exactly balance. This happens at one location between the original two balls.

Whether we think about fields or forces, the approach is equivalent. The special place where the net field is zero is closer to the ball with the smaller mass. To make up for a factor of 9, representing the ratio of the two masses, we need to have a factor of 3 (which gets squared to 9) in the distances. In other words, we need to be three times further from the ball with a mass of $9m$ than we are from the ball of mass m for the fields to be of equal magnitude. This occurs at x = $+a$.

We can also get this answer using a quantitative approach. Using the subscript 1 for the ball of mass m, and 2 for the ball of mass $9m$, we can express the net field as:

$$\vec{g}_{net} = \vec{g}_1 + \vec{g}_2 = 0.$$

Define right to be positive. If the point we're looking for is between the balls a distance x from the ball of mass m, it is $(4a - x)$ from the ball of mass $9m$. Using the definition of $\vec{g}$ gives:

$$+\frac{Gm}{x^2} - \frac{G(9m)}{(4a-x)^2} = 0.$$

Canceling factors of G and m, and re-arranging gives: $\frac{1}{x^2} = \frac{9}{(4a-x)^2}$.

Cross-multiplying leads to: $(4a-x)^2 = 9x^2$.

We could use the quadratic equation to solve for x, but let's instead take the square root of both sides of the equation. When we take a square root the result can be either plus or minus:

$$4a - x = \pm 3x.$$

Using the positive sign, we get $4a = +4x$, so $x = +a$. This is the correct solution, lying between the balls and closer to the ball with the smaller mass. Because it is three times farther from the ball of mass 9*m* than the ball of mass *m*, and because the distance is squared in the equation for field, this exactly balances the factor of 9 in the masses.

Using a minus sign gives a second solution, $x = -2a$. This location is three times farther (6*a*) from the ball of mass 9*m* than from the ball of mass *m* (2*a*). Thus at *x* = -2*a* the two fields have the same magnitude, but they point in the same direction so they add rather than canceling.

Related End-of-Chapter Exercises: 13, 14, 20.

EXAMPLE 8.5B – Escape from Earth

When you throw a ball up into the air, it comes back down. How fast would you have to launch a ball so that it never came back down, but instead it escaped from the Earth? The minimum speed required to do this is known as the escape speed.

SOLUTION

A diagram is shown in Figure 8.12. Let's assume the ball starts at the surface of the Earth and that we can neglect air resistance (this would be fine if we were escaping from the Moon, but it is a poor assumption if we're escaping from Earth - let's not worry about that, however). We'll also assume the Earth is the only object in the Universe. So, this is an interesting calculation but the result will only be a rough approximation of reality.

Let's apply the energy conservation equation:

$$U_i + K_i + W_{nc} = U_f + K_f .$$

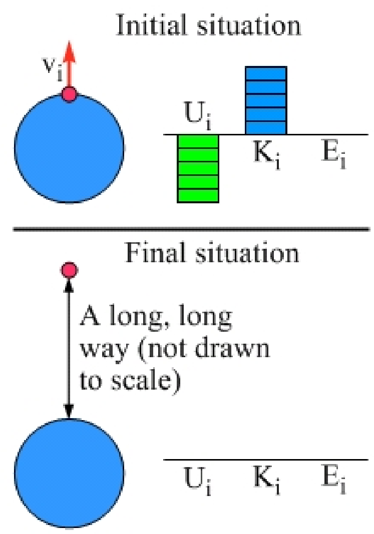

Figure 8.12: Energy bar graphs are shown in addition to the pictures showing the initial and final situations.

We're neglecting any work done by non-conservative forces, so W_{nc} = 0. The final gravitational potential energy is negligible, because the distance between the ball and Earth is very large (we can assume it to be infinite). What about the final kinetic energy? Because we're looking for the minimum initial speed let's use the minimum possible speed of the ball when it is very far from Earth, which we can assume to be zero. This leads to an equation in which everything on the right-hand side is zero:

$$U_i + K_i = 0 .$$

$$-\frac{GmM_E}{R_E} + \frac{1}{2}mv_{escape}^2 = 0 .$$

The mass of the ball does not matter, because it cancels out. This gives:

$$v_{escape} = \sqrt{\frac{2GM_E}{R_E}} = \sqrt{\frac{2(6.67\times10^{-11}\ \mathrm{N\,m^2/kg^2})(5.97\times10^{24}\ \mathrm{kg})}{6.37\times10^6\,\mathrm{m}}} = 11.2\ \mathrm{km/s} .$$

This is rather fast, and explains why objects we throw up in the air come down again!

Related End-of-Chapter Exercises: 41, 42.

Essential Question 8.5: Let's say we were on a different planet that had the same mass as Earth but twice Earth's radius. How would the escape speed compare to that on Earth?

Answer to Essential Question 8.5: Since $v_{escape} = \sqrt{\frac{2GM_E}{R_E}}$, keeping the mass the same while doubling the radius reduces the escape speed by a factor of $\sqrt{2}$.

8-6 Orbits

Imagine that we have an object of mass *m* in a circular orbit around an object of mass *M*. An example could be a satellite orbiting the Earth. What is the total energy associated with this object in its circular orbit?

The total energy is the sum of the potential energy plus the kinetic energy:

$$E = U + K = -\frac{GmM}{r} + \frac{1}{2}mv^2 .$$

This is a lovely equation, but it doesn't tell us much. Let's consider forces to see if we can shed more light on what's going on. For the object of mass *m* to experience uniform circular motion about the larger mass it must experience a net force directed toward the center of the circle (i.e., toward the object of mass *M*). This is the gravitational force exerted by the object of mass *M*. Applying Newton's Second Law gives:

$\Sigma\vec{F} = m\vec{a} = \frac{mv^2}{r}$, directed toward the center.

$\frac{GmM}{r^2} = \frac{mv^2}{r}$, which tells us that $mv^2 = \frac{GmM}{r}$.

Substituting this result into the energy expression gives:

$$E = -\frac{GmM}{r} + \frac{GmM}{2r} = -\frac{GmM}{2r} .$$

This result is generally true for the case of a lighter object traveling in a circular orbit around a more massive object. We can make a few observations about this. First, the magnitude of the total energy equals the kinetic energy; the kinetic energy has half the magnitude of the gravitational potential energy; and the total energy is half of the gravitational potential energy. All this is true when the orbit is circular. Second, the total energy is negative, which is true for a **bound system** (a system in which the components remain together). Systems in which the total energy is positive tend to fly apart.

What happens when an object has a velocity other than that necessary to travel in a circular orbit? One way to think of this is to start the orbiting object off at the same place, with a velocity directed perpendicular to the line connecting the two objects, and simply vary the speed. If the speed necessary to maintain a circular orbit is denoted by $v_{\square}$, let's consider what happens if the speed is 20% less than $v_{\square}$; 20% larger than $v_{\square}$; the special case of $\sqrt{2}v_{\square}$; and $1.5v_{\square}$. The orbits followed by the object in these cases are shown in Figure 8.13.

Unless the object's initial speed is too small, causing it to eventually collide with the more massive object, an initial speed that is less than $v_{\square}$ will produce an elliptical orbit where the initial point turns out to be the furthest the object ever gets from the more massive object. The initial point is special because at that point the object's velocity is perpendicular to the gravitational force the object experiences.

If the initial speed is larger than $v_{\square}$ the result depends on how much larger it is. When the initial speed is $\sqrt{2}v_{\square}$ that is the escape speed, and is thus a special case. The shape of the orbit is parabolic, and this path marks the boundary between the elliptical paths in which the object remains in orbit and the higher-speed hyperbolic paths in which the object escapes from the gravitational pull of the massive object.

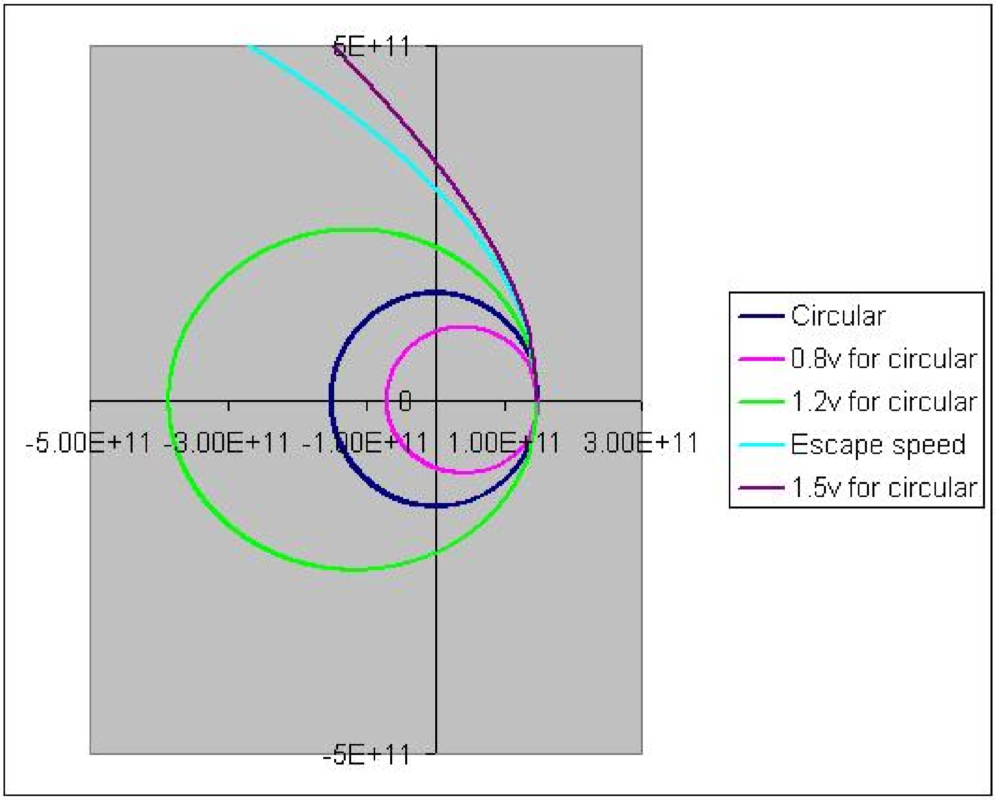

Figure 8.13: The orbits resulting from starting at a particular spot, the right-most point on each orbit, with initial velocities directed the same way (up in the figure) but with different initial speeds. The dark blue orbit represents the almost-circular orbit of the Earth, where the distances on each axis are in units of meters and the Sun is not shown but is located at the intersection of the axes. If the Earth's speed were suddenly reduced by 20% the Earth would instead follow the light purple orbit, coming rather close to the Sun. If instead the Earth's speed were increased by 20% the resulting elliptical orbit would take us quite a long way from the Sun before coming back again. Increasing the Earth's speed to $\sqrt{2}$ times its current speed (an increase of a little more than 40%) the Earth would be moving at the escape speed and we would follow the light blue parabolic orbit to infinity (and beyond). Any initial speed larger than this would result in a hyperbolic orbit to infinity, such as that shown in the dark purple. Note that the speeds given in the key to the right of the graph represent initial speeds, the speed the Earth would have at the right-most point in the orbit to follow the corresponding path.

Related End-of-Chapter Exercises: 47, 59, and 60.

Essential Question 8.6: Is linear momentum conserved for any of these orbits? If so, which?