Angular momentum

The angular momentum of a spinning object is represented by *L*.

1. $\vec{L} = I\vec{\omega}$

2. Angular momentum is a vector, pointing in the direction of the angular velocity.

3. If there is no net torque acting on a system, the system's angular momentum is conserved.

4. A net torque produces a change in angular momentum that is equal to the torque multiplied by the time interval during which the torque was applied.

A figure skater

A spinning figure skater is an excellent example of angular momentum conservation. The skater starts spinning with her arms outstretched, and has a rotational inertia of I_i and an initial angular velocity of ω_i. When she moves her arms close to her body, she spins faster. Her moment of inertia decreases, so her angular velocity must increase to keep the angular momentum constant.

Conserving angular momentum:

$$\vec{L}_i = \vec{L}_f$$

$$I_i \vec{\omega}_i = I_f \vec{\omega}_f$$

In this process, what happens to the skater's kinetic energy?

A bicycle wheel

A person standing on a turntable while holding a bicycle wheel is an excellent place to observe angular momentum conservation in action. Initially, the bicycle wheel is rotating about a horizontal axis, and the person is at rest.

The initial angular momentum about a vertical axis is zero.

If the person re-positions the bicycle wheel so its rotation axis is vertical, the wheel exerts a torque on the person during the re-positioning that makes the person spin in the opposite direction. The angular momenta cancel, so $L = 0$ at all times about a vertical axis.

Flipping the bike wheel over makes the person spin in the opposite direction.

Jumping on a merry-go-round

Let's analyze a rotational collision.

Sarah, with mass m and velocity v, runs toward a playground merry-go-round, which is initially at rest, and jumps on at its edge. Sarah and the merry-go-round (mass M, radius R, and $I = cMR^2$) then spin together with a constant angular velocity ω_f. If Sarah's initial velocity is tangent to the circular merry-go-round, what is ω_f?

Simulation

What concept should we use to attack this problem?

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Simulation

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Conservation of angular momentum.

Jumping on a merry-go-round

The system clearly has angular momentum after the completely inelastic collision, but where is the angular momentum beforehand?

It's with Sarah. Much like a force gives rise to a torque, Sarah's linear momentum can be converted to an angular momentum relative to an axis through the center of the merry-go-round.

$$L = rp\sin\theta$$

In this case, $L_i = Rmv\sin(90^\circ) = Rmv$.

The angular momentum is directed clockwise.

Jumping on a merry-go-round

Conserving angular momentum: $\vec{L}_i = \vec{L}_f$

Let's define counterclockwise to be positive.

$$+Rmv = +I_{total}\omega_f$$

$$+Rmv = +\left(cMR^2 + mR^2\right)\omega_f$$

We can treat Sarah as a point, a distance *R* from the center.

Solving for the final angular speed: $\omega_f = \dfrac{mv}{cMR + mR}$

Rotational Kinetic Energy

Energy associated with rotation is given by an equation analogous to that for straight-line motion.

For an object that is moving but not rotating: $K = \frac{1}{2}mv^2$

For an object that is rotating only: $K = \frac{1}{2}I\omega^2$

For an object that is translating and rotating simultaneously, such as a rolling object:

$$K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

A figure skater

When the figure skater moves her arms in closer to her body while she is spinning, what happens to the skater’s rotational kinetic energy?

1. It increases
2. It decreases
3. It must stay the same, because of conservation of energy

Kinetic energy

$$K_i = \frac{1}{2} I_i \omega_i^2 = \frac{1}{2}\left(I_i \omega_i\right) \times \omega_i$$

$$K_f = \frac{1}{2} I_f \omega_f^2 = \frac{1}{2}\left(I_f \omega_f\right) \times \omega_f$$

The terms in brackets are the same, so the final kinetic energy is larger than the initial kinetic energy, because $\omega_i < \omega_f$.

Where does the extra kinetic energy come from?

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Where does the extra kinetic energy come from?

The skater does work on her arms in bringing them closer to her body, and that work shows up as an increase in kinetic energy.

A race

We have three objects, a solid disk, a ring, and a solid sphere. If we release them from rest at the top of an incline, which object will win the race? Assume the objects roll down the ramp without slipping.

1. The sphere
2. The ring
3. The disk
4. It’s a three-way tie
5. Can't tell - it depends on mass and/or radius.

Racing shapes

Let's use conservation of energy to analyze the race between two objects that roll without slipping down the ramp.

Let's analyze a generic object with a mass *M*, radius *R*, and a rotational inertia of:

$$I = cMR^2$$

Start with the usual five-term energy conservation equation.

$$U_i + K_i + W_{nc} = U_f + K_f$$

Eliminate the terms that are zero:

Insert the expressions for the various terms.

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Insert the expressions for the various terms.

$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$

Racing shapes, continued

$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$

Because the object rolls without slipping, we can use $\omega = \frac{v}{R}$

We can also substitute the expression for rotational inertia. $I = cMR^2$

$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}(cMR^2)\frac{v^2}{R^2}$$

Both the mass and the radius cancel out!

$$gh = \frac{1}{2}v^2 + \frac{1}{2}cv^2$$

Solving for the speed at the bottom: $v = \sqrt{\frac{2gh}{1+c}}$

What does this tell us?

$$gh = \frac{1}{2}v^2 + \frac{1}{2}cv^2$$

$$v = \sqrt{\frac{2gh}{1+c}}$$

Mass and radius do not matter. All that matters is c. The larger the value of c, the slower the object is, because a larger fraction of the potential energy is directed toward the rotational kinetic energy, with less available for the translational kinetic energy.

Simulation

A race

If we take the winner of the rolling race (the sphere) and race it against a frictionless block, which object wins the race? Assume the sphere rolls without slipping.

1. The sphere
2. The block
3. It’s a tie
4. Can't tell