11-6 Angular Momentum

By now, we have looked at enough analogies between straight-line motion and rotational motion that we can simply take a straight-line motion equation, replace the straight-line motion variables by their rotational counterparts, and write down the equivalent rotational equation. We could also derive the rotational equations following a derivation parallel to the one we used for the straight-line motion equation, but the end result would be the same.

Let's try this for angular momentum. In Chapter 6, we used the following expression for the linear momentum, $\vec{p}$, of an object of mass *m* moving with velocity $\vec{v}$: $\vec{p} = m\vec{v}$.

Using the symbol $\vec{L}$ to represent angular momentum, we can come up with the equivalent expression for angular momentum by replacing mass *m* by its rotational equivalent, rotational inertia *I*, and velocity $\vec{v}$ by its rotational equivalent $\vec{\omega}$:

$\vec{L} = I\,\vec{\omega}$. (Equation 11.1: **Angular momentum**)

We made a number of statements about momentum in Chapter 6. Equivalent statements apply to angular momentum, including:

- Angular momentum is a vector, pointing in the direction of angular velocity.
- The angular momentum of a system can be changed by applying a net torque.
- If no net torque acts on a system, its angular momentum is conserved.

Let's explore this idea of angular momentum conservation.

EXPLORATION 11.6 – Jumping on the merry-go-round

A little red-haired girl named Sarah, with mass *m*, runs toward a playground merry-go-round, which is initially at rest, and jumps on at its edge. Sarah's velocity $\vec{v}$ is tangent to the circular merry-go-round. Sarah and the merry-go-round then spin together with a constant angular velocity $\vec{\omega}_f$. The merry-go-round has a mass *M*, a radius *R*, has the form of a uniform solid disk. Assume that Sarah's "radius" is small compared to *R*. The goal of this Exploration is to determine an expression for $\vec{\omega}_f$. We can treat this as a collision.

Step 1 – ***Sketch two diagrams, one showing Sarah running toward the merry-go-round and the other showing Sarah and the merry-go-round rotating together after Sarah has jumped on. Imagine that you're looking down on the situation from above.*** These two diagrams are shown in Figure 11.15.

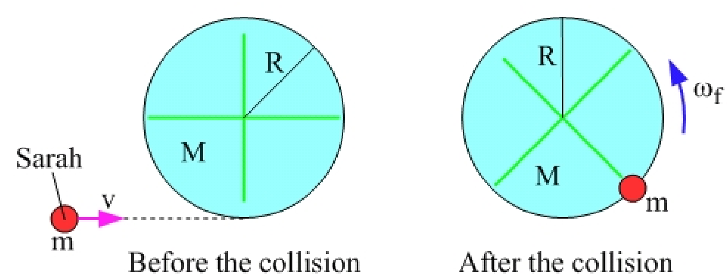

Figure 11.15: On the left is the situation before the collision, as Sarah runs toward the merry-go-round, while on the right is the situation after the collision, with Sarah and the merry-go-round rotating together with a constant angular velocity.

Step 2 – ***What kind of momentum does the Sarah/merry-go-round system have, if any, before Sarah jumps on the merry-go-round? What about after Sarah jumps on?*** After the collision, when the system is rotating, the system clearly has a non-zero angular momentum. Before the collision, however, it is not obvious that the system has any angular momentum, because nothing is rotating. Sarah certainly has a linear momentum, however, because she has a non-zero velocity.

Step 3 – ***Convert Sarah's linear momentum before the collision to an angular momentum, using a method modeled on the way we convert a force to a torque.*** Although there is no rotation before the collision, we can say that the system has an angular momentum with respect to an axis perpendicular to the page that passes through the center of the merry-go-round. Consider how we get torque from force, where the magnitude of the torque is given by $\tau = r\,F\sin\phi$. Angular momentum is found from linear momentum in a similar fashion, with its magnitude given by:

$L = r\,p\sin\phi = r\,(mv)\sin\phi$, (Eq. 11.2: **Connecting angular momentum to linear momentum**)

where ϕ is the angle between the line we measure distance along and the line of the linear momentum.

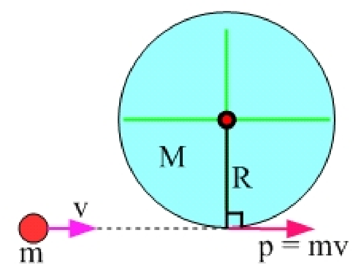

Figure 11.16: The lever-arm method to determine Sarah's angular momentum, with respect to an axis passing through the center of the system. merry-go-round.

Relative to the axis through the center of the merry-go-round, the angular momentum is: $\vec{L}_i = R\,mv\sin(90°) = R\,mv$, in a counterclockwise direction.

Step 4 – ***Apply angular momentum conservation to express $\vec{\omega}_f$, the angular velocity of the system after the collision, in terms of the variables above.*** Angular momentum is conserved because there are no external torques acting on the Sarah/merry-go-round system, relative to a vertical axis passing through the center of the turntable. We will justify this further in section 11-7. Thus, we can say: Angular momentum before the collision = angular momentum afterwards.

The angular momentum afterwards is $\vec{L}_f = I\,\vec{\omega}_f$. The system's rotational inertia after the collision is the sum of the rotational inertias of Sarah, and the ½ MR^2 of the merry-go-round. Sarah's "radius" is small compared to *R,* so we treat Sarah as a point, assuming that all her mass is the same distance, R, from the center of the turntable. Sarah's rotational inertia is thus mR^2.

Thus, the rotational inertia of the system after the collision is $I = \frac{1}{2}MR^2 + mR^2$.

Taking counterclockwise to be positive, angular momentum conservation gives: $\vec{L}_i = \vec{L}_f$.

$$+R\,mv = I\vec{\omega}_f = \left(\frac{1}{2}MR^2 + mR^2\right)\vec{\omega}_f .$$

Solving for the final angular velocity of the system gives:

$$\vec{\omega}_f = +\frac{mv}{\frac{1}{2}MR + mR} \quad \text{or,} \quad \vec{\omega}_f = \frac{mv}{\frac{1}{2}MR + mR}$$ directed counterclockwise.

Key ideas: Linear momentum converts to angular momentum in the same way force converts to torque. Also, we apply momentum conservation ideas to rotational collisions in the same way we analyze collisions in one and two dimensions. **Related End-of-Chapter Exercises: 32, 34, 59.**

Essential Question 11.6: Is it possible for Sarah, with the same initial speed, to jump onto the merry-go-round at the same point, but not make it spin? If so, how could she do this?

Answer to Essential Question 11.6: One way for Sarah to jump onto the merry-go-round, without causing the merry-go-round to spin, is for Sarah to direct her velocity at the center of the merry-go-round, instead of tangent to it. If Sarah ran directly toward the center of the merry-go-round she would have no angular momentum before the collision and there would be no reason for the system to spin after the collision.

11-7 Considering Conservation, and Rotational Kinetic Energy

In step 4 of Exploration 11.6, we stated that the angular momentum of the system consisting of Sarah and the merry-go-round was conserved, because no external torques were acting on the system. Let's justify that statement. We do not have to concern ourselves with vertical forces, such as the force of gravity or the normal force applied to the merry-go-round by the ground, because vertical forces give no torque about a vertical axis of rotation. We also do not have to concern ourselves with the force that Sarah exerts on the merry-go-round, or the equal-and-opposite force the merry-go-round exerts on Sarah, because the system we're considering consists of the combination of Sarah and the merry-go-round, so those are internal forces and cancel one another. Still, let's examine those forces a little.

Individual free-body diagrams for Sarah and the merry-go-round when Sarah first jumps on the merry-go-round are shown in Figure 11.17. Through some combination of friction between her shoes and the merry-go-round, and a contact force between her hands and any handholds on the merry-go-round, there is a force component that acts to the left on Sarah from the merry-go-round (this reduces her speed), and an equal-and-opposite force component that acts to the right on the turntable by Sarah (providing the torque that gives the merry-go-round an angular acceleration). However, the turntable does not accelerate to the right. This is because there is a horizontal force applied on the turntable by whatever the turntable's axis is connected to, which we can consider to be the Earth. As shown in Figure 11.17, the Sarah/merry-go-round system has a net external force acting on it at this point, which is why the *linear* momentum of the system is not conserved. However, this net external force gives rise to no torque about an axis through the center of the merry-go-round, because the force passes through that axis. Because there is no net external torque acting on the system, the system's angular momentum is conserved.

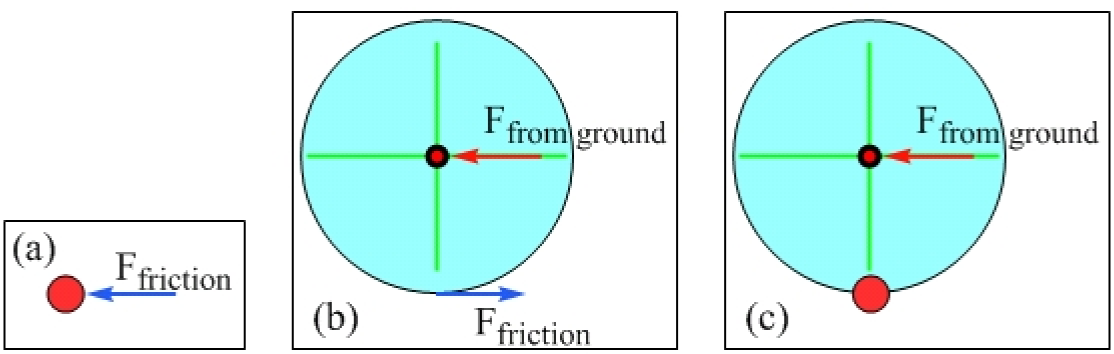

Figure 11.17: Free-body diagrams for Sarah, the merry-go-round, and the system consisting of Sarah and the merry-go-round together, when Sarah initially makes contact with the merry-go-round. Vertical forces are ignored in this overhead view.

Rotational Kinetic Energy

Let's now move from the rotational equivalent of linear momentum to the rotational equivalent of translational kinetic energy. The equation we used previously for kinetic energy is $K = \frac{1}{2} mv^2$. We can find the equivalent expression for kinetic energy in a rotational setting by replacing mass *m* by rotational inertia *I*, and speed *v* by angular speed ω. The kinetic energy of a purely rotating object is thus given by:

$$K = \frac{1}{2} I\omega^2 . \qquad \text{(Equation 11.3: \textbf{Rotational kinetic energy})}$$

Let's make sure our substituting-the-equivalent-rotational-variables method of arriving at rotational equations makes sense. Consider, for instance, a uniform rod that can rotate about an axis through one end. If we hold the rod horizontal and then release it from rest, the rod swings down. What is the rod's kinetic energy at a particular instant, say at the instant shown in Figure 11.18 (a)? One thing we could do is, as shown in Figure 11.18 (b), break the rod into small pieces of mass m_i, determine the speed v_i of each piece, find the kinetic energy $\frac{1}{2} m_i v_i^2$ of each piece, and then add up all these kinetic energies to find the total kinetic energy:

$$K = \sum \frac{1}{2} m_i v_i^2 .$$

Because the speed of each piece is different, while the angular speed of each piece is the same, let's write the sum in terms of the rod's angular speed instead:

$$K = \sum \frac{1}{2} m_i \left(r_i \omega\right)^2 .$$

If we bring the constants of ½ and ω^2 out in front of the sum, our expression becomes $K = \frac{1}{2}\omega^2 \sum m_i r_i^2$, which we can write as $K = \frac{1}{2} I\omega^2$, because the definition of rotational inertia is $I = \sum m_i r_i^2$. This expression for the kinetic energy agrees with what we came up with above (and it works for any rotating object, not just a rod!).

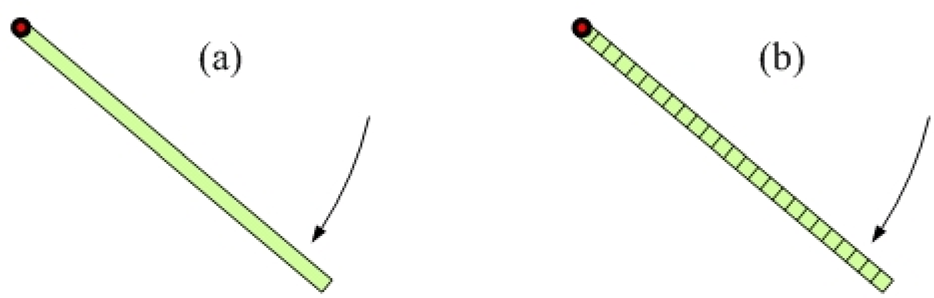

Figure 11.18: (a) A rod that has been released from rest when it was horizontal is now moving. We can find its kinetic energy by breaking the rod into small pieces, as shown in (b), finding the kinetic energy of each piece, and adding these kinetic energies together to find the net kinetic energy.

Essential Question 11.7: In Chapter 7 we used names such as "elastic collision" and "inelastic collision" to classify various collisions. Under what category would the Sarah/merry-go-round collision described in the previous Exploration fall?

Answer to Essential Question 11.7: Because Sarah and the merry-go-round stick together and move as one after the collision, the collision is completely inelastic.

11-8 Racing Shapes

Let's make use of the expression for rotational kinetic energy we derived in section 11-7, and apply it to analyze the motion of an object that rolls without slipping down a slope. The analysis can be done in terms of energy conservation (as we will do), or in terms of thinking about forces and torques and applying Newton's Second Law and Newton's Second Law for Rotation. The analysis in those terms can be found on the accompanying web site.

EXPLORATION 11.8 – Racing shapes

You have various shapes, including a few different solid spheres, a few rings, and a few uniform disks and cylinders. The objects have various masses and radii. When you race the objects by releasing them from rest two at a time, they roll without slipping down an incline of constant angle. Our goal is to determine which object reaches the bottom of the incline in the shortest time. Let's analyze this for a generic object of mass *M*, radius *R*, and rotational inertia, about an axis through the center of mass, of cMR^2 .

Step 1 – ***Sketch a free-body diagram for the object as it rolls without slipping down the ramp.*** A diagram and a free-body diagram is shown in Figure 11.19. The Earth applies a downward force of gravity to the object, while the incline applies a contact force. We split the contact force into two forces, a normal force perpendicular to the incline and a force of friction directed up the slope. This is a static force of friction, because the object does not slip as it rolls. The force of static friction is directed up the slope, not because the motion of the object is down the slope, but because the object has a clockwise angular acceleration (its angular velocity is clockwise and increasing as it rolls down). Taking an axis through the center of the object, the static force of friction is the only force that can provide the torque associated with this angular acceleration – the other two forces pass through the center of the object and thus give no torque about that axis.

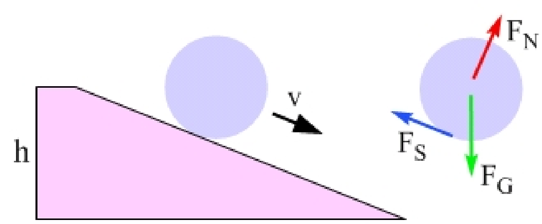

Figure 11.19: The diagram and free-body diagram of an object as it rolls without slipping down a ramp. A force of friction directed up the ramp provides the clockwise torque associated with the object's clockwise angular acceleration. The force of friction is static because the object does not slip as it rolls.

Step 2 – ***Let's analyze this in terms of energy conservation, using the same conservation of energy equation we used in previous chapters. Start by eliminating the terms that are zero in the equation.*** Recall that the energy conservation equation is: $K_i + U_i + W_{nc} = K_f + U_f$ .The object is released from rest, so the initial kinetic energy K_i is zero. We can also define the bottom of the incline to be the zero level for gravitational potential energy, so the final potential energy is $U_f = 0$. We also have no work being done by non-conservative forces. This may seem somewhat counter-intuitive at first, because static friction acts on each object as it rolls down the hill, but it is kinetic friction that is associated with a loss of mechanical energy. Static friction, because it involves no relative motion (and therefore no displacement to use in the work equation), does not produce a loss of mechanical energy.

The conservation of energy equation can thus be written: $U_i = K_f$.

Let's say that each object starts from a height h above the bottom of the incline. Because the zero for potential energy is at the bottom, the initial gravitational potential energy can be written as: $U_i = Mgh$. Our energy conservation term can thus be written $Mgh = K_f$.

Step 3 – *Split the kinetic energy term into two pieces, one representing the translational kinetic energy and one representing the rotational kinetic energy. Express the rotational kinetic energy in terms of M and v_f (the speed at the bottom of the incline) and solve for v_f*. First, let's think about why considering two types of kinetic energy is appropriate. When an object's center-of-mass is moving, the object has translational kinetic energy $KE_{trans} = \frac{1}{2}Mv^2$. When an object is only rotating, it has a rotational kinetic energy $KE_{rot} = \frac{1}{2}I\omega^2$. A rolling object, however, is both translating as well as rotating, and thus it has both these forms of kinetic energy.
Our energy equation now becomes: $Mgh = \frac{1}{2}Mv_f^2 + \frac{1}{2}I\omega_f^2$.

Let's make two substitutions to rewrite the rotational kinetic energy term. First, we can use our expression for rotational inertia, $I = c\,MR^2$. Then, we use the relationship between speed and angular speed that applies to rolling without slipping,: $\omega = v/R$. Our energy equation is now:

$$Mgh = \frac{1}{2}Mv_f^2 + \frac{1}{2}c\,MR^2\,\frac{v_f^2}{R^2}.$$

Note that all factors of mass M and radius R cancel, leaving: $gh = \frac{1}{2}v_f^2 + \frac{1}{2}c\,v_f^2$.

Solving for v_f, the object's speed at the bottom of the incline, gives: $v_f = \sqrt{\frac{2gh}{1+c}}$.

This result is consistent with the $v_f = \sqrt{2gh}$ result we obtained in previous chapters (for the speed of a ball dropped from rest through a height h, for instance), giving us some confidence that the answer is correct.

So, which object wins the race? The winner is the object with the highest speed at the bottom, which requires the smallest value of c. Recall that c is the numerical factor in the moment of inertia, $I = c\,MR^2$. For the various shapes we were racing we have c = 2/5 for solid spheres; c = 1/2 for uniform disks and cylinders; and $c = 1$ for rings. Thus, in the rolling races, a solid sphere beats any disk (or cylinder) and any ring, while any disk or cylinder beats any ring.

Key ideas: We can apply energy conservation in an analysis of rotating, or rolling, objects, just as we did in previous situations. Our energy conservation equation from Chapter 7 needs no modification. All we have to do is to use the expression for the kinetic energy of rotating objects: $KE_{rot} = \frac{1}{2}I\omega^2$. **Related End-of-Chapter Exercises: 7, 8, 10.**

Essential Question 11.8: In Exploration 11.8, we determined that, in the races of rolling objects, a solid sphere would beat a disk or cylinder, which would beat a ring. What if we raced two of the same kind of object against one another (such as a sphere versus a sphere)? Which object would win? The object with the larger mass, smaller mass, larger radius, or smaller radius?

Answer to Essential Question 11.8: Review the analysis in step 3 of Exploration 11.8. Both the mass and radius cancel out of the energy conservation equation. This tells us, surprisingly, that the mass and radius are irrelevant. In other words, all uniform solid spheres roll the same, all uniform solid disks (or cylinders) roll the same, and all rings roll the same – all the races involving two of the same kind of object end in a tie.

11-9 Rotational Impulse and Rotational Work

Let's continue our method of determining rotational equations from their straight-line motion counterparts by writing down expressions for rotational impulse and rotational work. In Chapter 6, the impulse relationship we came up with was: $\Delta\vec{p} = \vec{F}_{net}\,\Delta t$. In words, this equation tells us that the change in momentum an object experiences is equal to the product of the net force applied to the object multiplied by the time interval over which it is applied. Transforming this to a rotational setting, an object's change in angular momentum is equal to the net torque it experiences multiplied by the time interval over which that net torque is applied:

$\Delta\vec{L} = \vec{\tau}_{net}\,\Delta t$. (Equation 11.4: **Rotational impulse**)

Similarly, we can consider the concept of work in a rotational setting. For straight-line motion, if we meld the work equation with the work-energy theorem we get:

$\Delta K = W_{net} = \vec{F}_{net} \bullet \Delta\vec{r} = F_{net}\,\Delta r \cos\phi$. (Equation 6.8: **Work-kinetic energy theorem**)

In chapter 6, we used the variable θ to represent the angle between the net force $\vec{F}_{net}$ and the displacement $\Delta\vec{r}$. We'll use ϕ here instead because in this chapter we're using θ to represent the angular position of a rotating object.

To find the expression for work in a rotational setting, start with equation 6.8. Replace force $\vec{F}$ by its rotational equivalent, $\vec{\tau}$, and replace displacement $\Delta\vec{r}$ by its rotational equivalent $\Delta\vec{\theta}$. This gives:

$\Delta K = W_{net} = \vec{\tau}_{net} \bullet \Delta\vec{\theta} = \tau_{net}\,\Delta\theta\,\cos\phi$. (Equation 11.5: **Rotational work**)

If the dot product notation confuses you, feel free to ignore it! Because we'll deal only with rotation about one axis (rotation in one dimension), we can make Equation 11.5 simpler:

$\Delta K = W_{net} = \pm\tau_{net}\,\Delta\theta$. (Eq. 11.6: **Work-kinetic energy theorem, for rotation**)

We use the plus sign when the torque is in the same direction as the angular displacement, and the minus sign when the torque is opposite to the direction of the angular displacement.

EXAMPLE 11.9 – Comparing the motions

Note – compare this example to Example 6.3. The methods of analysis in that example and this one are virtually identical. Two objects, *A* and *B*, are initially at rest. The objects have the same mass and radius. Object *A* is a uniform solid disk, while object *B* is a bicycle wheel that can, for this purpose, be considered to be a ring. Each object rotates with no friction about an axis through its center, perpendicular to the plane of the disk/wheel. Identical net torques are then applied to the objects by pulling on strings wrapped around their outer rims. Each net torque is removed once the object it is applied to has accelerated through one complete rotation.

(a) After the net torques are removed which object has more kinetic energy?
(b) After the net torques are removed which object has more speed?
(c) After the net torques are removed, which object has more momentum?

SOLUTION

(a) A diagram of this situation is shown in Figure 11.20. Because the objects start from rest, the angular displacement of each is in the same direction as the net torque (counterclockwise, in the case shown in Figure 11.20). Because the objects experience equal torques and equal angular displacements the work done on the objects is the same, by Equation 11.6. This means the change in kinetic energy is the same for each, and since they both start with no kinetic energy their final kinetic energies are equal.

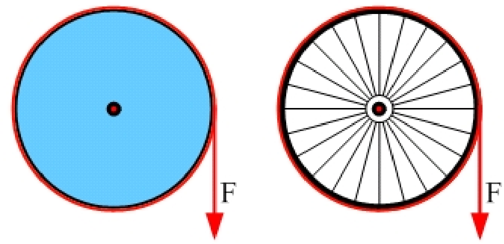

Figure 11.20: Diagrams of the disk and wheel. Each object starts from rest and rotates about an axis perpendicular to the page passing through the center of the object. The force exerted on the string wrapped around the object is removed once the object has accelerated through exactly one revolution.

(b) Unlike Example 6.3, in which the objects had different masses, these objects have the same mass M and the same radius R. This is a rotational situation, however, so what matters is how their rotational inertias compare. Object A, a uniform solid disk rotating about an axis through its center, has a rotational inertia of $I_A = \frac{1}{2}MR^2$. Object B, which we are treating as a ring, has a rotational inertia of $I_B = MR^2$. Thus the relationship between the rotational inertias is $I_A = \frac{1}{2}I_B$. If the objects have the same kinetic energy but B has a larger rotational inertia then A must have a larger angular speed. Setting the final kinetic energies equal, $K_A = K_B$, gives:

$$\frac{1}{2}I_A\,\omega_A^2 = \frac{1}{2}I_B\,\omega_B^2 .$$

Canceling factors of $\frac{1}{2}$ gives: $I_A\,\omega_A^2 = I_B\,\omega_B^2$

Bringing in the relationship between the rotational inertias gives: $\frac{1}{2}I_B\,\omega_A^2 = I_B\,\omega_B^2$.

This gives $\omega_A = \sqrt{2}\,\omega_B$, so object A has a larger angular speed than object B.

(c) One way to find the angular momenta is as follows:

$$\vec{L}_A = I_A\,\vec{\omega}_A = \frac{1}{2}I_B\,\vec{\omega}_A = \frac{1}{2}I_B\left(\sqrt{2}\,\vec{\omega}_B\right) = \frac{1}{\sqrt{2}}I_B\,\vec{\omega}_B = \frac{1}{\sqrt{2}}\vec{L}_B .$$

Thus, object B, the wheel, has a larger angular momentum than object A, the disk. As in Example 6.3, we can understand this result conceptually. The change in angular momentum is the net torque multiplied by the time over which the net torque acts. Both objects experience identical torques, but because B has a larger rotational inertia, B takes more time to spin through one revolution than A does. Because the torque is applied to B for a longer time, B's change in angular momentum, and final angular momentum, has a larger magnitude than A's.

Related End-of-Chapter Exercises: 22, 23.

Essential Question 11.9: Return to the situation described in Example 11.9, but now object B is replaced by object C, a bicycle wheel of the same mass as object A but with a different radius. Once again, we can treat the bicycle wheel as a ring. The situation described in Example 11.9 is repeated, but this time objects A and C end up with the same rotational kinetic energy and the same angular momentum. How is this possible? Be as quantitative about your answer as you can.