

11-2 A General Method, and Rolling without Slipping

Let's begin by summarizing a general method for analyzing situations involving Newton's Second Law for Rotation, such as the situation in Exploration 1.1. We will then explore rolling. We will tie together the two themes of this section in sections 11-3 and 11-4.

A General Method for Solving a Newton's Second Law for Rotation Problem

These problems generally involve both forces and torques.

1. Draw a diagram of the situation.
2. Draw a free-body diagram showing all the forces acting on the object.
3. Choose a rotational coordinate system. Pick an appropriate axis to take torques about, and then apply Newton's Second Law for Rotation ($\sum \vec{\tau} = I\vec{\alpha}$) to obtain a torque equation.
4. Choose an appropriate x-y coordinate system for forces. Apply Newton's Second Law ($\sum \vec{F} = m\vec{a}$) to obtain one or more force equations. The positive directions for the rotational and x-y coordinate systems should be consistent with one another.
5. Combine the resulting equations to solve the problem.

Rolling without Slipping

Let's now examine a rolling wheel, which could be a bicycle wheel or a wheel on a car, truck, or bus. We will focus on a special kind of rolling, called **rolling without slipping**, in which the object rolls across a surface without slipping on that surface. This is actually what most rolling situations are, although our analysis would not apply to situations such as you spinning your car wheels on an icy road. Let's consider various aspects of rolling without slipping.

When we dealt with projectile motion in Chapter 4, we generally split the motion into two components, which were usually horizontal and vertical. To help understand rolling, we will follow a similar process. Rolling can be viewed as a combination, or superposition, of purely translational motion (moving a wheel from one place to another with no rotation) and purely rotational motion (only rotation with no movement of the center of the wheel). In the special case of rolling without slipping, there is a special connection between the translational component of the motion and the rotational component. Let's explore that connection.

EXPLORATION 11.2 – Rolling, rolling, rolling

We have a wheel of radius R that we will roll across a horizontal floor so that the wheel makes exactly one revolution. The wheel rolls without slipping on the floor.

Step 1 – *Consider the rotational part of the motion only (focus on the fact that the wheel spins around exactly once). What distance does a point on the outer edge of the wheel travel because of this spinning motion?*

Because we're ignoring the rotational motion, the distance traveled by a point on the outer edge of the wheel because of the spin is equal to the circumference of the wheel itself. This is a distance of $2\pi R$. See the top diagram in Figure 11.4.

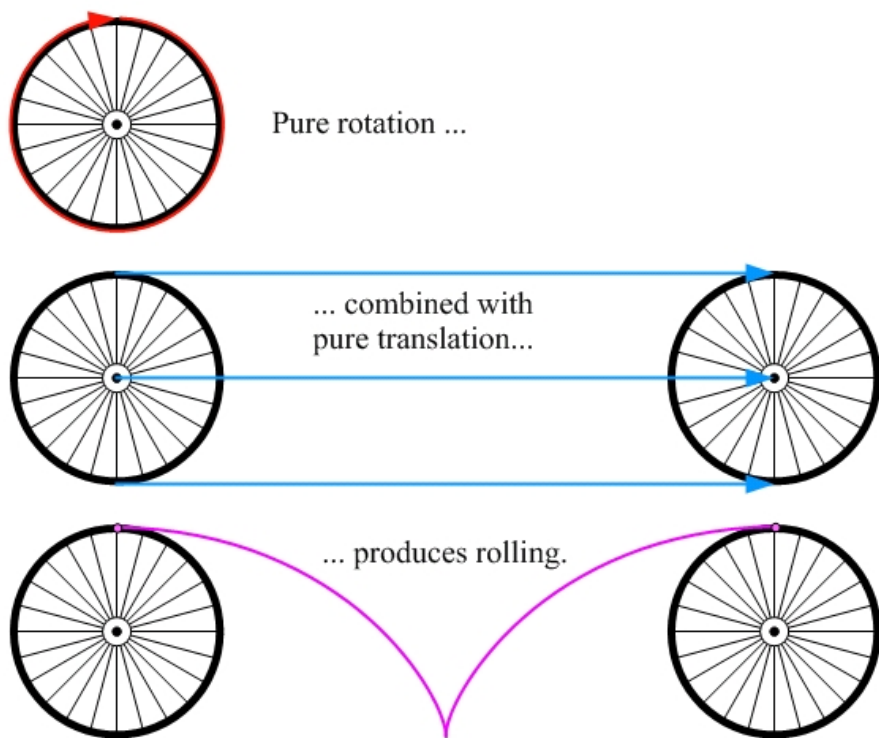


Figure 11.4: A pictorial representation of how the rotational component and the translational component of the motion combine to produce the interesting shape of the path traced out by a point on the outer edge of the wheel that is rolling without slipping. This shape is known as a cycloid.

Step 2 – Now consider the translational part of the motion only (i.e., ignore the fact that the wheel is spinning, and imagine that we simply drag the wheel a particular distance without allowing the wheel to rotate). What

is the distance that any point on the wheel moves if we drag the wheel a distance equal to that it would move if we rolled it so it rolled through exactly one revolution? To determine what this distance is, imagine that we placed some double-sided tape around the wheel before we rolled it, and that the tape sticks to the floor. This is shown in the middle diagram in Figure 11.4. Rolling the wheel through one revolution lays down all the tape on the floor, covering a distance that is again equal to the circumference of the wheel. Thus, focusing on the translational distance only, the translational distance moved by every point on the wheel as the wheel rolls through one revolution is $2\pi R$.

Step 3 – Assuming the rolling is done at constant speed, compare the speed of a point on the outer rim, associated only with the wheel's rotation, to the translational speed of the wheel's center of mass. We can find these speeds by dividing the appropriate distances by the time during which the motion takes place. Because the distances associated with the two components of the motion are equal, and the time of the motion is the same for the two components, these two speeds are equal.

Key ideas for rolling: Rolling can be considered to be a superposition of a pure translational motion and a pure rotational motion. In the special case of rolling without slipping, the distance moved by a point on the outer edge of a wheel associated with the rotational component is equal to the translational distance of the wheel. The speed of a point on the outer edge because of the rotational component is also equal to the translational speed of the wheel.

Related End-of-Chapter Exercises: 4, 17.

Essential Question 11.2: Different points on a wheel that is rolling without slipping have different speeds. Considering one particular instant, which point on the wheel is moving slowest? Which point is moving the fastest?

Answer to Essential Question 11.2: As we will investigate in more detail in section 11-3, when a wheel rolls without slipping, the point at the bottom of the wheel has the smallest speed (the speed there is zero, in fact), while the point at the top of the wheel is moving fastest.

11-3 Further Investigations of Rolling

Let's continue our analysis of rolling, starting by thinking about the velocity of various points on a wheel that rolls without slipping. We will then go on to investigate rolling spools.

EXPLORATION 11.3 – Determining velocity

Let's turn now from thinking about speeds to thinking about velocities. Consider a wheel rolling without slipping with a constant translational velocity \vec{v} across a level surface. For each point below, determine the point's net velocity by combining, as vectors, the point's translational velocity (the velocity associated with the translational component of the motion) with its velocity because of the rotational component of the motion.

Step 1 – Find the net velocity of the center of the wheel.

Considering the rotational motion to take place about an axis through the center of the wheel, the center of the wheel therefore has no rotational velocity (because $v_{rot} = r\omega$, and $r = 0$). Thus the net velocity of the center of the wheel is its translational velocity, \vec{v} , as shown in Figure 11.5.

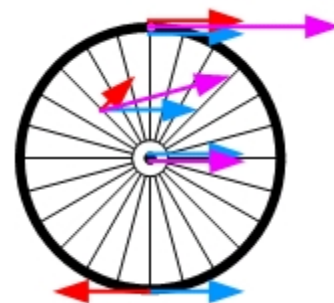


Figure 11.5: The translational (red), rotational (blue), and net velocities (purple) of various points on the wheel. The net velocity at a point is a vector sum of the translational and rotational velocities.

Step 2 – Find the net velocity of the point at the very top of the wheel.

Here we use the fact that the rotational speed is equal to the translational speed, so we are adding two velocities of equal magnitude. At the top of the wheel the velocities also point in the same direction, so the net velocity is $2\vec{v}$, as shown in Figure 11.5.

Step 3 – Find the net velocity of the point at the very bottom of the wheel.

As shown in Figure 11.5, the rotational velocity exactly cancels the translational velocity, since the vectors point in opposite directions and have equal magnitudes. The net velocity of that point is zero – the point is instantaneously at rest! This is a special condition that is characteristic of rolling without slipping. No slipping implies no relative motion between the surfaces in contact, which means the point at the bottom of the wheel that is in contact with the road surface is at rest.

Key ideas for rolling: The net velocity of a point on a rolling wheel can be found by adding, as vectors, the point's translational velocity and its rotational velocity. In the special case of a wheel rolling without slipping with a translational velocity \vec{v} , the net velocity of the center of the wheel is \vec{v} ; while that of the point at the top of the wheel is $2\vec{v}$. A point on the outer edge of the wheel actually comes instantaneously to rest when it reaches the bottom of the wheel.

Related End-of-Chapter Exercises: 5, 6.

EXAMPLE 11.3 – Unrolling a ribbon from a spool

A long ribbon is wrapped around the outer edge of a spool. You pull horizontally on the end of the ribbon so the ribbon starts to unwind from the spool as the spool rolls without slipping across a level surface.

- (a) When you have moved the end of the ribbon through a horizontal distance L , how far has the spool moved?
- (b) Does your answer change if the ribbon is instead wrapped around the spool's axle, which has a radius equal to half the radius of the spool? If so, how does the answer change?

SOLUTION

(a) A diagram of the situation is shown in Figure 11.6. Once again, we can think of the spool's rolling motion as a combination of its translational motion and its rotational motion. We can thus say that the end of the ribbon moves because (a) the spool has a translational motion, and (b) the spool is rotating. The speed of the ribbon matches the speed of the top of the spool, because there is no slipping between the ribbon and the spool. Recalling the result from Exploration 11.3, the top of the spool has a velocity twice that of the center of the spool. Putting these facts together means that the center of the spool has a velocity half that of the end of the ribbon at any instant, and so the spool covers a distance of $L/2$, half the distance covered by the end of the ribbon.

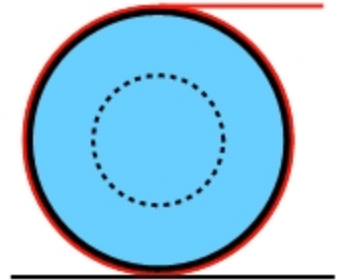


Figure 11.6: A spool is rolling without slipping to the right because you are pulling, to the right, on the red ribbon that is wrapped around the spool.

(b) What if the ribbon is wrapped around the spool's axle and you move the end of the ribbon through a distance L ? The answer changes because the rotational contribution to the net velocity changes. As shown in Figure 11.7, the ribbon now comes off the axle at the top of the axle, at a point halfway between the edge and the center of the spool. The net velocity at that point on the spool is 1.5 times the velocity of the center of the spool: the translational velocity is equal to the velocity of the center, while the rotational velocity is half that of the center, because at a radius of $R/2$ we have:

$$v_{rot} = \frac{R}{2} \omega = \frac{1}{2} R\omega = \frac{1}{2} v_{trans} .$$

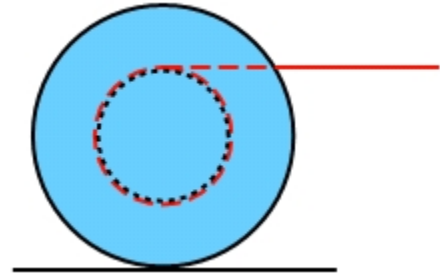


Figure 11.7: The ribbon is wrapped around the axle of the spool, which has a radius half that of the spool. The ribbon comes off the axle at the top.

Putting it another way, the velocity of the center of the spool is now two-thirds of the velocity of the end of the ribbon. If the end of the ribbon travels a distance L , the spool translates through a distance of $2L/3$.

Related End-of-Chapter Exercises: 18, 19.

Essential Question 11.3: In a situation similar to that in Figure 11.7, you pull to the right on a ribbon wrapped around the axle of a spool. This time, however, the ribbon is wound so it comes away from the spool underneath the axle, as shown in Figure 11.8. When you pull to the right on the ribbon, the spool rolls without slipping. In which direction does it roll? Sketch a free-body diagram of the spool to help you think about this.

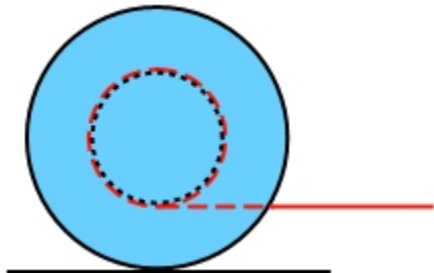


Figure 11.8: A ribbon is wrapped around the axle of the spool so the ribbon comes off the axle below the axle.

Answer to Essential Question 11.3: Many people focus on the counterclockwise torque, relative to an axis perpendicular to the page that passes through the center of the spool, exerted by the force of tension and conclude that the spool rolls to the left. Before jumping to conclusions, however, draw the free-body diagram (after drawing your own, see Figure 11.9). As usual there is a downward force of gravity and an upward normal force. Horizontally there is a force of tension, directed right, exerted by the ribbon. With no friction, the force of tension would cause the spool to move right and spin counterclockwise, so the bottom of the spool would move right with respect to the horizontal surface. Friction must therefore be directed left to oppose this, and, because we know the spool rolls without slipping, the force of friction must be static friction.

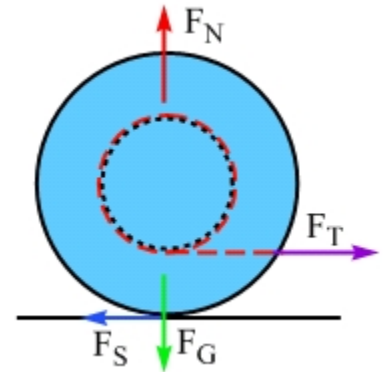


Figure 11.9: The free-body diagram of the spool.

Now we have the complete free-body diagram, we can see that the answer to the question is not obvious. There is one force left and one force right – which is larger? Relative to an axis through the center, there is one torque clockwise and one counterclockwise – which is larger? A quick way to get the answer is to consider an axis perpendicular to the page, passing through the point where the spool makes contact with the horizontal surface. Relative to this axis, three of the four forces give no torque, and the torque from the tension in the string is in a clockwise direction. Clockwise rotation of the spool, relative to the point where the spool touches the surface, is consistent with the spool rolling without slipping to the right. This is opposite to what you would conclude by focusing only on the torque about the center from the force of tension. The spool rolls to the right.

11-4 Combining Rolling and Newton's Second Law for Rotation

Let's now look at how we can combine torque ideas with rolling-without-slipping concepts.

EXPLORATION 11.4 – A vertical force but a horizontal motion

A spool of mass M has a string wrapped around its axle. The radius of the axle is half that of the spool. An upward force of magnitude F_T is exerted on the end of the string, as shown in Figure 11.10. This causes the spool, which is initially at rest, to roll without slipping as it accelerates across the level surface.

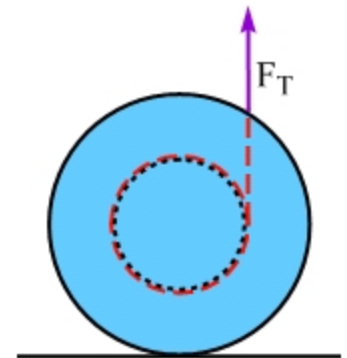
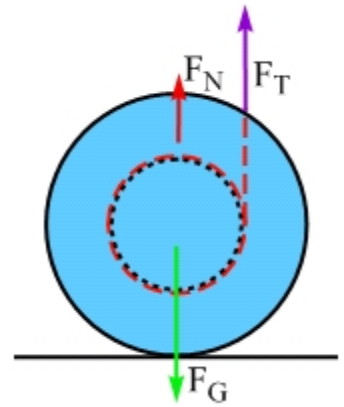


Figure 11.10: An upward force is exerted on the string wrapped around the axle of the spool.

In which direction does the spool roll? Which horizontal force is responsible for the spool's horizontal acceleration? Let's begin by drawing a free-body diagram of the spool. Figure 11.11 shows a partial free-body diagram, showing only the vertical forces acting on the spool. There is a downward force of gravity acting on the spool, and an upward force of tension applied by the string (note that F_T must be less than or equal to Mg , so the spool has no vertical acceleration). There is also an upward normal force, required to balance the vertical forces.

Figure 11.11: A partial free-body diagram of the spool, showing the vertical forces acting on it.

Is there a horizontal force? If there is, what could it be? Let's go back and think about what is interacting with the spool. The force of gravity accounts for the interaction between the Earth and the spool, and the force of tension accounts for the interaction between the string and the spool. The only interaction left is the interaction between the surface and the spool. The surface exerts a contact force on the spool. Remember that we generally split the contact force into components, the normal force (which we have accounted for) and the force of friction (which we have not).



If there is a horizontal force acting, it can only be a force of friction. Do we need friction in this situation? Consider what would happen if the free-body diagram shown in Figure 11.11 was complete, and there was no friction. Taking an axis perpendicular to the page through the center of the spool, the tension force would give rise to a counterclockwise torque. Because the net force acting on the spool would be zero, however, the spool would simply spin counterclockwise without moving. This is inconsistent with the rolling-without-slipping motion we are told is occurring. There must be a force of friction acting on the spool to cause the horizontal motion.

Note that, without friction, the bottom of the spool rotates to the right relative to the surface. The force of friction must therefore be directed to the left, acting to oppose the relative motion that would occur without friction. Because the force of friction is the only horizontal force acting on the spool, the spool accelerates to the left.

Is the force of friction kinetic friction or static friction? Because the spool is rolling without slipping, and the bottom of the spool is instantaneously at rest relative to the surface it is in contact with, the force of friction is the static force of friction. This may sound counter-intuitive, since there is relative motion between the spool as a whole and the surface, but it is very similar to the walking (without slipping) situation that we thought about in Chapter 5. The complete free-body diagram for the rolling-without-slipping situation is shown in Figure 11.12.

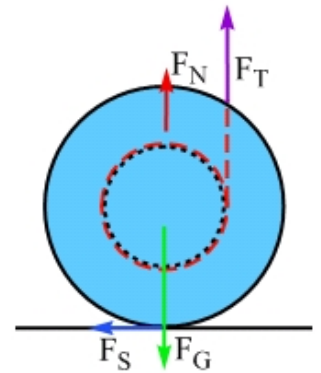
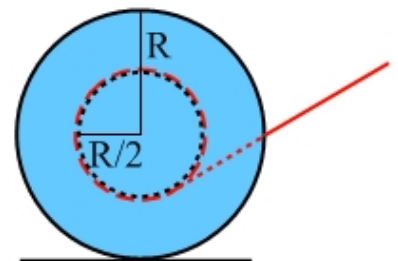


Figure 11.12: The complete free-body diagram of the spool.

Key ideas for rolling without slipping: Rolling without slipping often involves a force of friction, which must be a static force of friction. The static force of friction is often (although not always) in the direction of motion. **Related End-of-Chapter Exercises: 3, 50.**

Essential Question 11.4: In the situation shown in Figure 11.13, you pull on the end of a ribbon wrapped around the axle of a spool. Your force is exerted in the direction shown. If the spool rolls without slipping, in which direction does the spool roll?

Figure 11.13: A ribbon is wrapped around the axle of the spool so the ribbon comes off the axle in the direction shown.



Answer to Essential Question 11.4: Once again, it is simplest to take torques about an axis perpendicular to the page, passing through the point at which the spool touches the ground. The only force giving rise to a torque about this point is the tension in the ribbon, which gives a clockwise torque. If the spool rotates clockwise with respect to its bottom point, the motion of the spool is to the right.

11-5 Analyzing the Motion of a Spool

EXPLORATION 11.5 – Continuing the analysis of the rolling spool

Let's return to the situation described in Exploration 11.4, and focus in particular on the free-body diagram in Figure 11.12. Our goal is to determine the magnitude of the spool's acceleration in terms of F_T and M . The spool consists of two disks, each of mass $M/3$ and radius R , connected by an axle of mass $M/3$ and radius $R/2$.

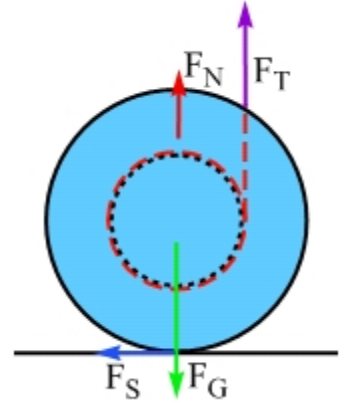


Figure 11.12: The complete free-body diagram of the spool.

Step 1 – Apply Newton's Second Law for the horizontal forces.

Because the spool accelerates left, take left to be the positive direction.

$$\Sigma \vec{F}_x = M \vec{a}_x.$$

Because the acceleration is entirely in the x-direction, we can replace \vec{a}_x by \vec{a} . Evaluating the left-hand side of this expression with the aid of Figure 11.13 gives:

$$+F_s = +M a.$$

Step 2 – Find the expression for the spool's rotational inertia about an axis perpendicular to the page passing through the center of the spool.

Why are we doing this? Well, we'll need to apply Newton's Second Law for Rotation to solve this problem, and that involves the spool's rotational inertia. To find the spool's rotational inertia, we can use the expression for the rotational inertia of a solid disk or cylinder ($I = \frac{1}{2} mr^2$) about the center. Let's apply this equation to the three pieces of the spool and add them together to find the net rotational inertia.

Each of the two disks contributes $\frac{1}{2} \left(\frac{M}{3} \right) R^2 = \frac{1}{6} MR^2$ to the rotational inertia.

The axle contributes $\frac{1}{2} \left(\frac{M}{3} \right) \left(\frac{R}{2} \right)^2 = \frac{1}{24} MR^2$.

The total rotational inertia is $I = \frac{1}{6} MR^2 + \frac{1}{6} MR^2 + \frac{1}{24} MR^2 = \frac{9}{24} MR^2 = \frac{3}{8} MR^2$.

Step 3 – Apply Newton's Second Law for Rotation to obtain a connection between the force of friction and the upward force \vec{F}_T applied to the string.

Taking torques about an axis perpendicular to the page and passing through the center of the spool is a good way to do this, since the force of gravity and the normal force pass through this axis and therefore give no torque about that axis. Since the spool has a counterclockwise angular acceleration let's take counterclockwise to be positive for torques. Applying Newton's Second Law for Rotation gives:

$$\Sigma \bar{\tau} = I\bar{\alpha} .$$

Referring to Figure 11.13, and using the equation $\bar{\tau} = r F \sin \theta$, we have:

$$+\frac{R}{2}F_T \sin(90^\circ) - RF_S \sin(90^\circ) = +I\alpha .$$

Recognizing that $\sin(90^\circ) = 1$, and substituting the expression for the spool's rotational inertia we found above, gives:

$$+\frac{R}{2}F_T - RF_S = +\frac{3}{8}MR^2\alpha .$$

Canceling a factor of R gives: $+\frac{1}{2}F_T - F_S = +\frac{3}{8}MR\alpha$.

Step 4 – What is the connection between the spool's acceleration and its angular acceleration?

For rolling without slipping, the connection between the acceleration and the angular acceleration is $a = R\alpha$, although it is always a good idea to check whether the positive direction for the straight-line motion is consistent with the positive direction for rotation. In our case they are consistent, since we chose them based on the motion. If we had reversed one of the positive directions, however, we would have had a negative sign in the equation.

Step 5 – Combine the results above to determine the spool's acceleration in terms of F_T and M .

Let's first substitute $a = R\alpha$ into our final expression from step 3, to get:

$$+\frac{1}{2}F_T - F_S = +\frac{3}{8}Ma .$$

In step 1, we determined that $F_S = M a$, so we get:

$$+\frac{1}{2}F_T - Ma = +\frac{3}{8}Ma ;$$

$$+\frac{1}{2}F_T = +\frac{11}{8}Ma ;$$

$$\bar{a} = \frac{4F_T}{11M} , \text{ directed to the left.}$$

Key idea: Solving a rolling-without-slipping problem often involves analyzing the rotational motion, analyzing the one-dimensional motion, and combining the analyses.

Related End-of-Chapter Exercises: 51, 53.

Essential Question 11.5: Consider a hard ball that is rolling without slipping across a smooth level surface. If the ball maintains a constant velocity, in what direction is the static force of friction acting on the ball? Consider the three-possible free-body diagrams for the ball in Figure 11.14 below, and state which free-body diagram is appropriate for this situation.

Figure 11.14: Possible free-body diagrams for a ball rolling, without slipping, at constant velocity to the right across a horizontal surface.

