11-1 Applying Newton's Second Law for Rotation

Let's learn how to apply Newton's second law for rotation to systems in which the angular acceleration $\vec{\alpha}$ is non-zero. The analysis of such systems is known as **rotational dynamics**.

EXPLORATION 11.1 – A mass and a pulley

A pulley with a mass M and a radius R is mounted on a frictionless horizontal axle passing through the center of the pulley. A block with a mass M hangs down from a string that is wrapped around the outside of the pulley. Assume that the pulley is a uniform solid disk. The goal of this Exploration is to determine the acceleration of the block when the system is released from rest. Before we do anything else, we should draw a diagram of the situation based on the description above. The diagram is shown in Figure 11.1.
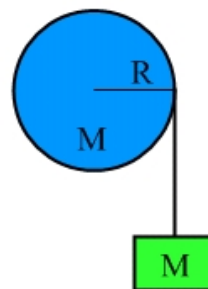
Figure 11.1: A diagram of the pulley and block system. The block hangs down from a string wrapped around the outside of the pulley.

Step 1 – *Draw a free-body diagram of the block after the system is released from rest.* The block accelerates down when the system is released, which means that the block must have a net force acting on it that is directed down. There are only two forces acting on the block, an upward force of tension and a downward force of gravity. Thus, for the net force to be directed down, the force of tension must have a smaller magnitude than the force of gravity. Note that a common mistake is to assume that the force of tension is equal to the force of gravity acting on the block. Thinking about Newton's second law and how it applies to the block helps us to avoid making this error.
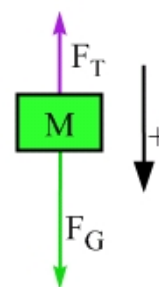
Figure 11.2: The free-body diagram of the block.

Step 2 – *Draw a free-body diagram of the pulley.* A complete free-body diagram of the pulley, shown in Figure 11.3 (a), reflects that fact that the center-of-mass of the pulley remains at rest, so the net force must be zero. There is still a non-zero net torque, about an axis through the center of the pulley and perpendicular to the page, that gives rise to an angular acceleration. Generally, when we sum torques about an axis through the center, we draw a rotational free-body diagram, as in Figure 11.3 (b), including only forces that produce a torque. In this case, the only force producing a torque about the center of the pulley is the force of tension. As we discussed in Step 1, above, the force of tension is not equal to the weight of the block when the block has a non-zero acceleration.

Step 3 – *Apply Newton's Second Law to the block.* The block accelerates down, so let's define down to be the positive direction. Newton's second law is $\sum \vec{F} = M\vec{a}$. Using the free-body diagram in Figure 11.2 to evaluate the left-hand side gives: $+Mg - F_T = +Ma$.

Figure 11.3: The full free-body diagram of the pulley, in (a), and the rotational free-body diagram in (b), showing the only force acting on the pulley that produces a torque about an axis perpendicular to the page through the center of the pulley.
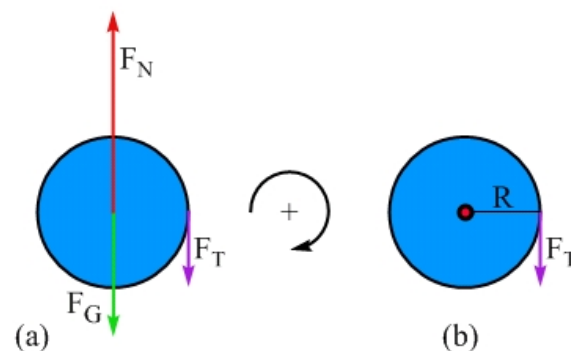
Step 4 – *Apply Newton's second law for rotation to the pulley.* Here, let's define clockwise to be positive for rotation, both because this is the direction of the angular acceleration and because the pulley going clockwise is consistent with the block moving down, in the direction we defined as positive for the block's motion.

Newton's second law for rotation is $\sum \vec{\tau} = I\vec{\alpha}$. Using the free-body diagram in 11.4(b) to evaluate the left-hand side gives: $+R\,F_T \sin(90°) = +I\alpha$.

In the description of the problem we were told, "the pulley is a uniform solid disk." This tells us what to use for the rotational inertia, *I*, on the right-hand side of the equation. Looking up the expression for the rotational inertia of a solid disk in Figure 10.22, we get $I = MR^2/2$. Inserting this into the equation above, and using the fact that $\sin(90°) = 1$, gives:

$$+R\,F_T = +\frac{1}{2}MR^2\,\alpha.$$

Canceling a factor of *R* gives: $+F_T = +\frac{1}{2}MR\alpha$.

Step 5 – *Put the resulting equations together to solve for the block's acceleration.* Looking at the force equation, $+Mg - F_T = +Ma$, and the equation we obtained from summing torques, $+F_T = +MR\alpha/2$, we have only two equations but three unknowns, F_T, *a*, and α. We could put the two equations together to eliminate the force of tension but then we'd be stuck.

Fortunately, we have another connection we can exploit, which is $\alpha = a/R$. The justification here is as follows. As the block accelerates down, every point on the string moves with the same magnitude acceleration as the block. We assume the string does not slip on the pulley, so the outer edge of the pulley (the part in contact with the string), moves with the string. Thus, the tangential acceleration of a point on the outer edge of the pulley is equal in magnitude to the acceleration of the sting, which equals the magnitude of the block's acceleration. Finally, we connect the magnitude of the tangential acceleration of the outer edge of the pulley to the magnitude of the pulley's angular acceleration using $\alpha = a_T/R$. Putting everything together boils down to $\alpha = a/R$, which we can substitute into the equation that came from summing torques:

$$+F_T = +\frac{1}{2}MR\alpha = +\frac{1}{2}MR\frac{a}{R} = +\frac{1}{2}Ma.$$

Using this result in the force equation gives: $+Mg - F_T = +Ma$.

Substituting in for the force of tension gives: $+Mg - \frac{1}{2}Ma = +Ma$.

Thus, $a = 2g/3$, with the acceleration being directed down.

Key ideas: Applying Newton's second law for rotation helps us analyze situations that are purely rotational. Problems that involve both rotation and straight-line motion, as is the case in Exploration 11.1, can be analyzed by combining a torque analysis with a force analysis.
Related End-of-Chapter Exercises: 1, 13.

Essential Question 11.1: If the pulley in Exploration 11.1 is changed from a uniform solid disk to a uniform solid sphere of the same mass and radius as the disk, how does that affect the block's acceleration? How does that affect the tension in the string?