10-4 Torque

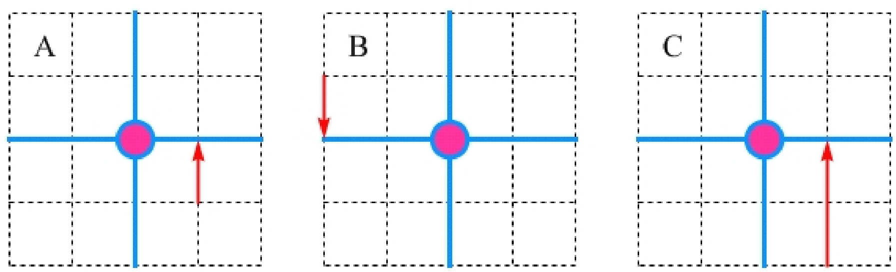
If an object is at rest, how can we get it to rotate? If an object is already rotating, how can we change its rotational motion? We answered equivalent questions about straight-line motion by saying “Apply a net force!” Let’s now consider the rotational equivalent of force.

EXPLORATION 10.4 – Turning a revolving door

From an overhead view, a revolving door looks like a + sign mounted on a vertical axle. The door can spin freely, clockwise or counterclockwise, about its center.

Step 1 – ***Consider the three cases illustrated in Figure 10.9, in which a force (the red arrow) is applied to a revolving door. In each case, determine the direction the door will start to rotate, assuming it starts from rest.***

Figure 10.9: Three cases of forces (shown in red) applied to a revolving door, shown from an overhead perspective.



Although the direction of the force in case B is opposite to that in cases A and C, in each case the door will rotate counterclockwise. If you are ever confused about the direction an object will tend to rotate, place your pen or pencil on the diagram and hold it at the axis of the object, in this case at the center. Then push on the object in the direction, and at the location, of the applied force and see which way the object spins. Knowing the direction of a force applied to an object is not enough to determine the direction of rotation; we also need to know where the force is applied in relation to the axis of rotation.

Step 2 – ***Rank the three cases based on how quickly the revolving door spins, from largest to smallest, assuming the door is initially at rest.*** In case C the door will rotate more quickly than in case A, because the applied force in C is twice as large as that in A while everything else (the point at which the force is applied, and the direction of the force) is equal. The door in case B also rotates faster than that in A because, even though the force has the same magnitude, in case B the force is applied further from the axis of rotation. Applying a force farther from the axis of rotation generally has a larger effect on the rotation of an object, which you have probably experienced. If you have ever come to a door where it was not obvious which side was connected to the hinges, and given the door a push on the edge where the hinges were, you most likely came close to running straight into the door as it opened very slowly in response to your push. Applying the same force at the edge of the door furthest from the hinges, however, is far more effective at opening the door.

The comparison that is hardest to rank is that between B and C. In case C the applied force is twice as large as that in B, but the force in B is applied twice as far from the axis of rotation as that in C. Which effect is more important? It turns out that these effects are equally important, so cases B and C are equivalent. The overall ranking is B=C>A.

The point of this discussion is that the angular acceleration of the door is proportional to both the applied force and the distance of the applied force from the axis of rotation. Let's now consider whether the direction at which the force is applied makes any difference.

Step 3 – ***Consider the three cases shown in Figure 10.10. Rank these three cases based on the revolving door's angular acceleration, from largest to smallest.***

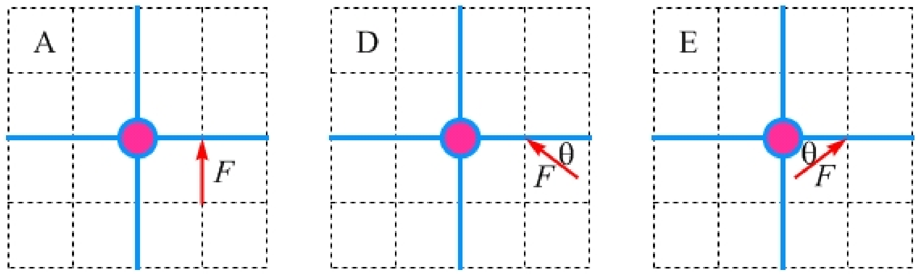


Figure 10.10: Three cases involving the same magnitude force applied at the same point on a revolving door, but applied in different directions.

Let's split the forces in cases D and E into components, as shown in Figure 10.11. How do the components of the force influence the door in each case? If you've ever tried to open a door by exerting a force parallel to the door itself, you'll know that this is completely ineffective. Similarly, the parallel components in cases D and E do absolutely nothing to affect the door's rotation. Only the perpendicular components, which have a magnitude of $F \sin\theta$, affect the rotation. Because these components are smaller than *F*, the magnitude of the perpendicular force in case A, ranking the three cases gives A>D=E.

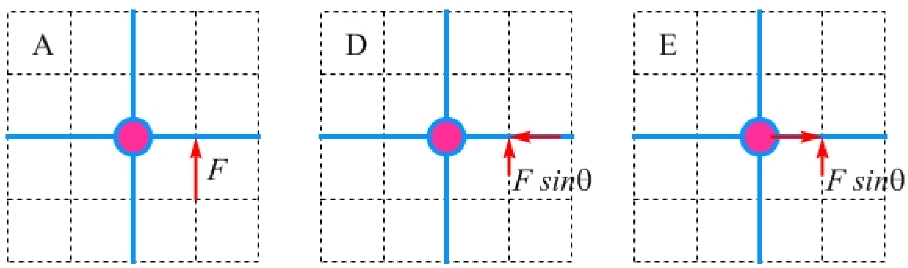


Figure 10.11: Splitting the force in case D, and case E, into components parallel to the door and perpendicular to the door.

Key ideas: The angular acceleration of a door depends on three factors: the magnitude of the applied force; the distance from the axis of rotation to where the force is applied; and the direction of the applied force. **Related End-of-Chapter Exercises: 48, 49.**

In Exploration 10.4, we learned about the rotational equivalent of force, which is torque.

The name for the rotational equivalent of force is **torque**, which we symbolize with the Greek letter tau (τ). Whereas a force is a push or a pull, a torque is a twist. A torque can result from applying a force. The torque resulting from applying a force *F* at a distance *r* from an axis of rotation is:

$$\tau = r F \sin\theta .$$ (Equation 10.9: **Magnitude of the torque**)

The angle θ represents the angle between the line of the force and the line the distance *r* is measured along.

Essential Question 10.4: Make a list of common household items or tools that exploit principles of torque.

Answer to Essential Question 10.4: Quite a number of tools and gadgets exploit torque, in the sense that they enable you to apply a small force at a relatively large distance from an axis, and the tool converts that into a large force acting at a relatively small distance from an axis. Examples include scissors, bottle openers, can openers, nutcrackers, screwdrivers, crowbars, wrenches, wheelbarrows, and bicycles.

10-5 Three Equivalent Methods of Finding Torque

EXPLORATION 10.5 – Three ways to find torque

A rod of length L is attached to a wall by a hinge. The rod is held in a horizontal position by a string that is tied to the wall and attached to the end of the rod, as shown in Figure 10.12.

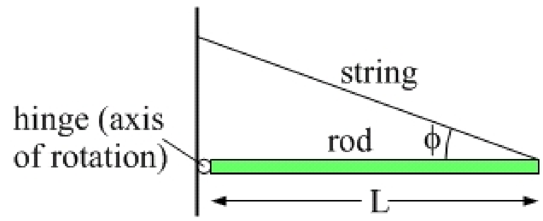


Figure 10.12: A rod attached to a wall at one end by a hinge, and held horizontal by a string.

Step 1 – ***In what direction is the torque applied by the string to the rod, about an axis that passes through the hinge and is perpendicular to the page?*** As we did in previous chapters, it's a good idea to draw a free-body diagram of the rod (or at least part of a free-body diagram, as in Figure 10.13) to help visualize what is happening. For now the only force we'll include on the free-body diagram is the force of tension applied by the string (we'll go on to look at all the forces applied to the rod in Exploration 10.8). Try placing your pen over the picture of the rod. Hold the pen where the hinge is and push on the pen, at the point where the string is tied to the rod, in the direction of the force of tension. You should see the pen rotate counterclockwise. Thus, we can say that the torque applied by the string, about the axis through the hinge, is in a counterclockwise direction.

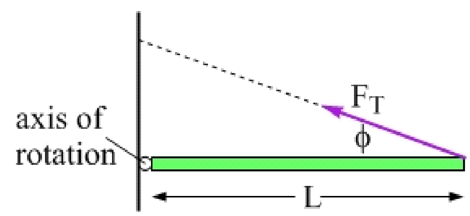


Figure 10.13: A partial free-body diagram for the rod, showing the force of tension applied to the rod by the string.

Note that we are dealing with direction for torque much as we did for angular velocity. The true direction of the torque can be found by curling your fingers on your right hand counterclockwise and placing your hand, little finger down, on the page. When you stick out your thumb it points up, out of the page. This is the true direction of the torque, but for simplicity we can state directions as either clockwise or, as in this case, counterclockwise.

Now we know the direction of the torque, relative to an axis through the hinge, applied by the string, let's focus on determining its magnitude.

Step 2 – ***Measuring the distance r in Equation 10.9 along the bar, apply Equation 10.9 to find the magnitude of the torque applied by the string on the rod, with respect to the axis passing through the hinge perpendicular to the page.*** Finding the magnitude of the torque means identifying the three variables, r, F, and θ, in Equation 10.9. In this case we can see from Figure 10.13 that the distance r is the length of the rod, L; the force $\vec{F}$ is the force of tension, $\vec{F}_T$; and the angle θ is the angle between the line of the force (i.e., the string) and the line the distance r is measured along (the rod), so θ is the angle ϕ in Figure 10.13. In this case, then, applying Equation 10.9 tells us that the magnitude of the torque is $\tau = L\,F_T \sin\phi$.

Step 3 – ***Now, determine the torque, about the axis through the hinge that is perpendicular to the page, by first splitting the force of tension into components, and then applying Equation 10.9.*** Which set of axes should we use when splitting the force into components? The most sensible coordinate system is one aligned parallel to the rod and perpendicular to the rod, giving the two components shown in Figure 10.14. Because the force component that is parallel to the rod is directed at the hinge, where the axis goes through, that component gives a torque of zero (it's like trying to open a door by pushing on the door with a force directed at the line passing through the hinges). Another way to prove that the force is zero is to apply Equation 10.9 with an angle of 180˚, which means multiplying by a factor of sin(180˚), which is zero.

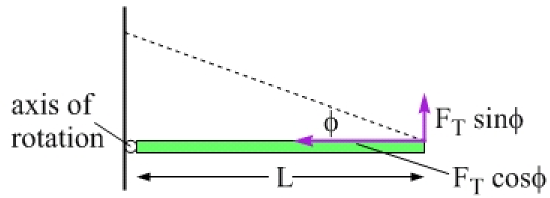


Figure 10.14: Splitting the force of tension into a component parallel to the rod, and a component perpendicular to the rod.

The torque from the force of tension is associated entirely with the perpendicular component of the force of tension. Now, identifying the three pieces of Equation 10.9 gives a force magnitude of $F = F_T \sin\phi$; a distance measured along the rod of $r = L$, and an angle of $\theta = 90^\circ$ between the line of the perpendicular force component and the line we measured *r* along. Because $\sin(90^\circ) = 1$, applying Equation 10.9 tells us that the magnitude of the torque from the tension, with respect to our axis through the hinge, is $\tau = L\,(F_T \sin\phi)\sin(90^\circ) = L\,F_T \sin\phi$. This agrees with our calculation in Step 2.

Step 4 – ***Instead of measuring r along the rod, draw a line from the hinge that meets the string (the line of the force of tension) at a 90˚ angle. Apply Equation 10.9 to find the magnitude of the torque applied by the string on the rod, with respect to the axis passing through the hinge, by measuring r along this line.***

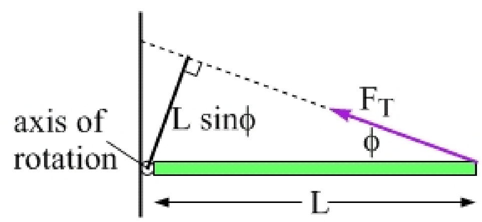


Figure 10.15: A diagram showing the lever arm, in which the distance used to find torque is measured from the axis along a line perpendicular to the line of the force.

As we can see from Figure 10.15, the *r* in this case is not *L*, the length of the rod, but is instead $L\sin\phi$. This result comes from applying the geometry of right-angled triangles. The magnitude of the force, F , is F_T , the magnitude of the full force of tension, and the angle between the line we measure *r* along and the line of the force is 90˚. This is known as the **lever-arm method** of calculating torque, where the lever-arm is the perpendicular distance from the axis of rotation to the force. Applying Equation 10.9 gives the magnitude of the torque as $\tau = (L\sin\phi)\,F_T \sin(90^\circ) = L\,F_T \sin\phi$, agreeing with the other two methods discussed above.

Key idea for torque: We can find torque in three equivalent ways. It can be found using the whole force and the most obvious distance; after splitting the force into components; or by using the lever-arm method in which the distance from the axis is measured along the line perpendicular to the force. Use whichever method is most convenient in a particular situation. **Related End-of-Chapter Exercises: 7, 23, 50.**

Essential Question 10.5: Torque can be calculated with respect to any axis. In Exploration 10.5, what is the torque, due to the force of tension, with respect to an axis passing through the point where the string is tied to the wall? In each case, assume the axis is perpendicular to the page.

Answer to Essential Question 10.5: The torque, from the tension, is zero with respect to any axis that passes through the string, because the line of the force (the string, in this case) passes through an axis that lies on the string. It is important to remember that the torque (both its direction and magnitude) associated with a force depends on the particular axis of rotation the torque is being measured with respect to.

10-6 Rotational Inertia

In Chapter 3, we found that an object's acceleration is proportional to the net force acting on the object:

$$\vec{a} = \frac{\sum \vec{F}}{m} . \qquad \text{(Equation 3.1: \textbf{Connecting acceleration to net force})}$$

A similar relationship connects the angular acceleration of an object to the net torque acting on it:

$$\vec{\alpha} = \frac{\sum \vec{\tau}}{I} . \qquad \text{(Eq. 10.10: \textbf{Connecting angular acceleration to net torque})}$$

Thus, the angular acceleration of an object is proportional to the net torque acting on the object. The I in the denominator of Equation 10.10 is known as the rotational inertia, which is the rotational equivalent of mass.

We have already looked at how the angular acceleration $\vec{\alpha}$ is the rotational equivalent of the acceleration $\vec{a}$, and how torque, $\vec{\tau}$, is the rotational equivalent of force, $\vec{F}$. The I in the denominator of Equation 10.10 must therefore be the rotational equivalent of the mass, m. I is known as the **rotational inertia**, or the **moment of inertia**. In the same way that mass is a measure of an object's tendency to maintain its state of straight-line motion, an object's rotational inertia is a measure of the object's tendency to maintain its rotational motion. Something with a large mass is hard to get moving, and it is also hard to stop if it is already moving. Similarly, if an object has a large rotational inertia it is difficult to start it rotating, and difficult to stop if it is already rotating.

One question to consider is, are rotational inertia and mass the same thing? In other words, does an object's mass, by itself, determine the rotational inertia? Let's check the units of rotational inertia. Re-arranging Equation 10.10, we find that rotational inertia has units of torque units (N m) divided by angular acceleration units (rad/s^2). Remembering that the newton is equivalent to kg m/s^2, and that we can treat the radian as being dimensionless, we find that rotational inertia has units of kg m^2. Rotational inertia depends on more than just mass, it depends on both mass and, somehow, length squared. Let's investigate this further.

EXPLORATION 10.7 – Rotational inertia

Consider a ball of mass M mounted at the end of a stick that has a negligible mass, and a length L (which is large compared to the ball's radius). The other end of the stick is pinned so the stick can rotate freely about the pin.

Step 1 – ***If the ball and stick are held horizontal and then released from rest, what is the ball's initial acceleration?*** The ball's initial acceleration is $\vec{g}$, the acceleration due to gravity. The force of the stick acting on the ball only becomes non-zero after the ball starts moving. We should also draw a diagram to help analyze the situation. The diagram is shown in Figure 10.16.

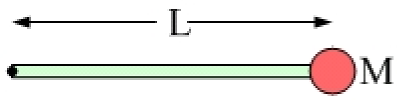


Figure 10.16: The initial position of the ball and stick. The system can rotate about an axis passing through the left end of the stick.

Step 2 – ***What is the ball's initial angular acceleration?***

The angular acceleration can be found from the equation $a = r\alpha$. Here $r = L$, the length of the stick, so we have $\vec{\alpha} = g / L$, directed clockwise.

Step 3 – ***What is the torque acting on the ball at the instant it is released?***

Here we can draw a free-body diagram of the ball, shown in Figure 10.17. Initially the only force acting on the ball is the force of gravity, Mg directed down. Considering an axis perpendicular to the page and passing through the pin, the torque is $\vec{\tau} = L\,Mg$, directed clockwise.

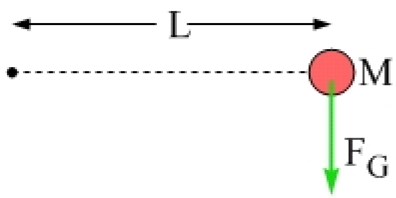


Figure 10.17: The free-body diagram of the ball immediately after the system is released from rest.

Step 4 – ***Using Equation 10.10, and the results from steps 2 and 3, determine the rotational inertia of the ball relative to the axis passing through the pin.***

Re-arranging Equation 10.10 to solve for the rotational inertia gives:

$$I = \frac{\sum \vec{\tau}}{\vec{\alpha}}.$$

The torque and the angular acceleration are both clockwise, allowing us to divide the magnitude of the torque by the magnitude of the angular acceleration to determine the ball's rotational inertia about an axis through the pin.

$$I = \frac{L\,Mg}{g / L} = M\,L^2 .$$

Thus the rotational inertia of an object of mass M in which all the mass is at a particular distance L from the axis of rotation is $I = M\,L^2$.

Key ideas for rotational inertia: An object's rotational inertia is determined by three factors: the object's mass; how the object's mass is distributed; and the axis the object is rotating around. **Related End-of-Chapter Exercises: 10, 27.**

Essential Question 10.6: Consider the three cases shown in Figure 10.18. In each case, a ball of a particular mass is placed on a light rod of a particular length. Each rod can rotate without friction about an axis through the left end. Rank the cases based on their rotational inertias, from largest to smallest.

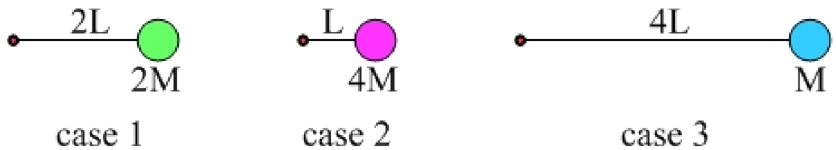


Figure 10.18: Three cases, each involving a ball on the end of a rod that can rotate about its left end.

Answer to Essential Question 10.6: The correct ranking is 3>1>2. In the rotational inertia equation, the distance from the axis to the ball (the length of the rod) is squared, while the mass is not. Thus, changing the length by a factor of 2 changes the rotational inertia by a factor of 4, whereas changing the mass by a factor of 2 changes the rotational inertia by only a factor of 2.

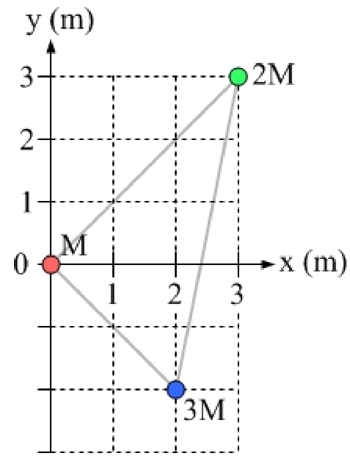
10-7 An Example Problem Involving Rotational Inertia

Our measure of inertia for rotational motion is somewhat more complicated than inertia for straight-line motion, which is just mass. Consider the following example.

EXAMPLE 10.7 – Spinning the system.

Three balls are connected by light rods. The mass and location of each ball are:
Ball 1 has a mass M and is located at $x = 0$, $y = 0$.
Ball 2 has a mass of $2M$ and is located at $x = +3.0$ m, $y = +3.0$ m.
Ball 3 has a mass of $3M$ and is located at $x = +2.0$ m, $y = -2.0$ m.
Assume the radius of each ball is much smaller than 1 meter.

(a) Find the location of the system's center-of-mass.
(b) Find the system's rotational inertia about an axis perpendicular to the page that passes through the system's center-of-mass.
(c) Find the system's rotational inertia about an axis parallel to, and 2.0 m from, the axis through the center-of-mass.



SOLUTION

Let's begin, as usual, by drawing a diagram of the situation. The diagram is shown in Figure 10.19.

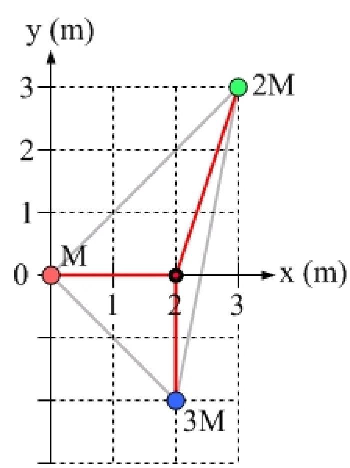
Figure 10.19: A diagram showing the location of the balls in the system described in Example 10.7.

(a) To find the location of the system's center-of-mass, let's apply Equation 6.3. To find the x-coordinate of the system's center-of-mass:

$$X_{CM} = \frac{x_1 m_1 + x_2 m_2 + x_3 m_3}{m_1 + m_2 + m_3} = \frac{(0)M + (+3.0\,\text{m})(2M) + (+2.0\,\text{m})(3M)}{M + 2M + 3M} = \frac{(+12.0\,\text{m})M}{6M} = +2.0\,\text{m}$$

The y-coordinate of the system's center-of-mass is given by:

$$Y_{CM} = \frac{y_1 m_1 + y_2 m_2 + y_3 m_3}{m_1 + m_2 + m_3} = \frac{(0)M + (+3.0\,\text{m})(2M) + (-2.0\,\text{m})(3M)}{M + 2M + 3M} = \frac{(0)M}{6M} = 0 \,.$$



(b) To find the system's rotational inertia about an axis through the center-of-mass we can find the rotational inertia for each ball separately, using $I = M\,L^2$, and then simply add them to find the total rotational inertia. Figure 10.20 is helpful for seeing where the different L values come from.

Figure 10.20: The center-of-mass of the system is marked at (+2 m, 0). The axis of rotation passes through that point. The red lines show how far each ball is from the axis of rotation.

For ball 1, $L^2 = (2.0\,\text{m})^2 = 4.0\,\text{m}^2$ so $I_1 = M\,L^2 = \left(4.0\,\text{m}^2\right)M$.

For ball 2, $L^2 = 10\,\text{m}^2$ so $I_2 = 2M\,L^2 = \left(20\,\text{m}^2\right)M$.

For ball 3, $L^2 = 4.0\,\text{m}^2$ so $I_3 = 3M\,L^2 = \left(12\,\text{m}^2\right)M$.

The total rotational inertia is the sum of these three values, $(36\,\text{m}^2)M$.

(c) To find the rotational inertia through an axis parallel to the first axis and 2.0 m away from it, let's choose a point for this second axis to pass through. A convenient point is the origin, $x = 0$, $y = 0$. Figure 10.21 shows where the L values come from in this case.

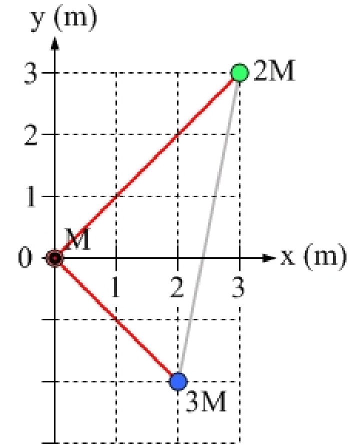


Figure 10.21: The axis of rotation now passes through the ball of mass M at the origin. The red lines show how far the other two balls are from the axis of rotation.

Repeating the process we followed in part (c) gives:

For ball 1, $L^2 = 0$ so $I_1' = 0$.

For ball 2, $L^2 = 18\,\text{m}^2$ so $I_2' = 2M\,L^2 = \left(36\,\text{m}^2\right)M$.

For ball 3, $L^2 = 8.0\,\text{m}^2$ so $I_3' = 3M\,L^2 = \left(24\,\text{m}^2\right)M$.

The total rotational inertia is the sum of these three values, $(60\,\text{m}^2)M$.

Related End-of-Chapter Exercises: 29, 31.

Does it matter which point the second axis passes through? What if we had used a different point, such as x = +2.0 m, y = -2.0 m, or any other point 2.0 m from the center-of-mass? Amazingly, it turns out that it doesn't matter. Any axis parallel to the axis through the center-of-mass and 2.0 m from it gives a rotational inertia of $(60\,\text{m}^2)M$. It turns out that the rotational inertia of a system is minimized when the axis goes through the center-of-mass, and the rotational inertia of the system about any parallel axis a distance h from the axis through the center-of-mass can be found from

$I = I_{CM} + mh^2$, (Equation 10.11: **The parallel-axis theorem**)

where m is the total mass of the system.

Let's check the parallel-axis theorem using our results from (b) and (c). In part (b) we found that the rotational inertia about the axis through the center-of-mass is $I_{CM} = (36\,\text{m}^2)M$. The mass of the system is $m = 6M$ and the second axis is h = 2.0 m from the axis through the center-of-mass. This gives $I = (36\,\text{m}^2)M + 6M(2.0\,\text{m})^2 = (60\,\text{m}^2)M$, as we found above.

Essential Question 10.7: To find the total mass of a system of objects, we simply add up the masses of the individual objects. To find the total rotational inertia of a system of objects, can we follow a similar process, adding up the rotational inertias of the individual objects.

Answer to Essential Question 10.7: Yes, the rotational inertia of a system of objects can be found be adding up the rotational inertias of the various objects making up the system. This is precisely the process we followed in Example 10.7.

10-8 A Table of Rotational Inertias

Consider now what happens if we take an object that has its mass distributed over a length, area, or volume, rather then being concentrated in one place. Generally, the rotational inertia in such a case is calculated by breaking up an object into tiny pieces, finding the rotational inertia of each piece, and adding up the individual rotational inertias to determine the total rotational inertia.

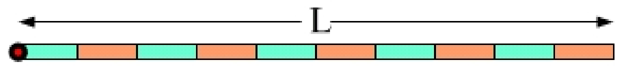


Figure 10.22: A uniform rod of length *L* and mass *M*, divided into 10 equal pieces. The axis of rotation passes through the left end of the rod and is perpendicular to the page.

We can get a feel for the process by considering how we would find the rotational inertia of a uniform rod of length *L* and mass *M*, rotating about an axis through the end of the rod that is perpendicular to the rod itself. If all the mass were concentrated at the far end of the rod, a distance *L* from the axis, then the rotational inertia would be ML^2. Because most of the mass is closer than *L* to the axis of rotation, the rod's rotational inertia turns out to be less than ML^2. If we broke up the rod into ten equal pieces, with centers at 5%, 15%, 25%, 35%,…,95% of the length of the rod (see Figure 10.22), we would calculate a rotational inertia of 0.3325 ML^2. This is very close to the value we would get by doing the integration, $I_{rod,end} = ML^2/3$. The rotational inertia's of various shapes, and for various axes of rotation, are shown in Figure 10.23.

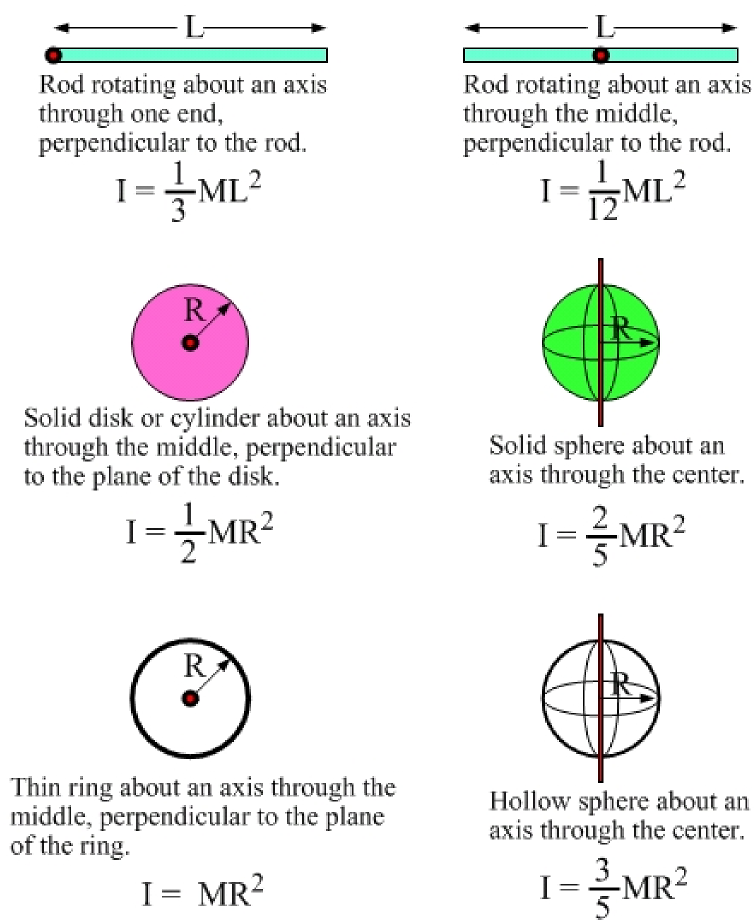


Figure 10.23: Expressions for the rotational inertia of various objects about a particular axis. In each case, the object has a mass *M*.

Essential Question 10.8: In Figure 10.23, all the values for rotational inertia are of the form $I = cMR^2$, or $I = cML^2$, where *c* is generally less than 1. The exception is the rotational inertia of a ring rotating about an axis through the center of the ring and perpendicular to the plane of the ring, where *c* = 1. Why do we expect to get $I = MR^2$ for the ring rotating about that central axis?

Answer to Essential Question 10.8: The expression for the rotational inertia of the ring has no factor less than 1 in front of the MR^2 because every bit of mass in the ring is a distance R from the center of the ring. In all the other cases shown in Figure 10.23, most of the mass of the given object is at a distance less than R (or less than L) from the axis in question.

10-9 Newton's Laws for Rotation

In Chapter 3 we considered Newton's three laws of motion. The first two of these laws have analogous statements for rotational motion.

Newton's First Law for Rotation: an object at rest tends to remain at rest, and an object that is spinning tends to spin with a constant angular velocity, unless it is acted on by a nonzero net torque ***or there is a change in the way the object's mass is distributed***.

Recall that the net torque is the sum of all the forces acting on an object. Always remember to add torques as vectors. The net torque can be symbolized by $\sum \vec{\tau}$.

The first part of the statement of Newton's first law for rotation parallel's Newton's first law for straight-line motion, but the phrase about how spinning motion can be affected by a change in mass distribution is something that only applies to rotation.

Newton's second law for rotation, on the other hand, is completely analogous to Newton's second law for straight-line motion, $\sum \vec{F} = m\,\vec{a}$. Replacing force by torque, mass by rotational inertia, and acceleration by angular acceleration, we get:

$\sum \vec{\tau} = I\vec{\alpha}$. (Equation 10.12: **Newton's Second Law for Rotation**)

We'll spend the rest of this chapter, and a good part of the next chapter, looking at how to apply Newton's second law in various situations. In Chapter 11, we will deal with rotational dynamics, involving motion and acceleration. For the remainder of this chapter, however, we will focus on situations involving static equilibrium.

Conditions for static equilibrium

An object is in static equilibrium when it remains at rest. Two conditions apply to objects in static equilibrium. These are:

$$\sum \vec{F} = 0 \qquad \text{and} \qquad \sum \vec{\tau} = 0 .$$

Expressed in words, an object in static equilibrium experiences no net force and no net torque. Using these conditions, we will be able to analyze a variety of situations. Many excellent examples of static equilibrium involve the human body, such as when you hold your arm out; when you bend over; and when you stand on your toes. In each case, forces associated with muscles, bones, and tendons maintain the equilibrium situation.

Essential Question 10.9: Newton's first law for rotation includes a phrase that says spinning motion can be affected by a change in the way an object's mass is distributed. Can you think of a real-life example of this?