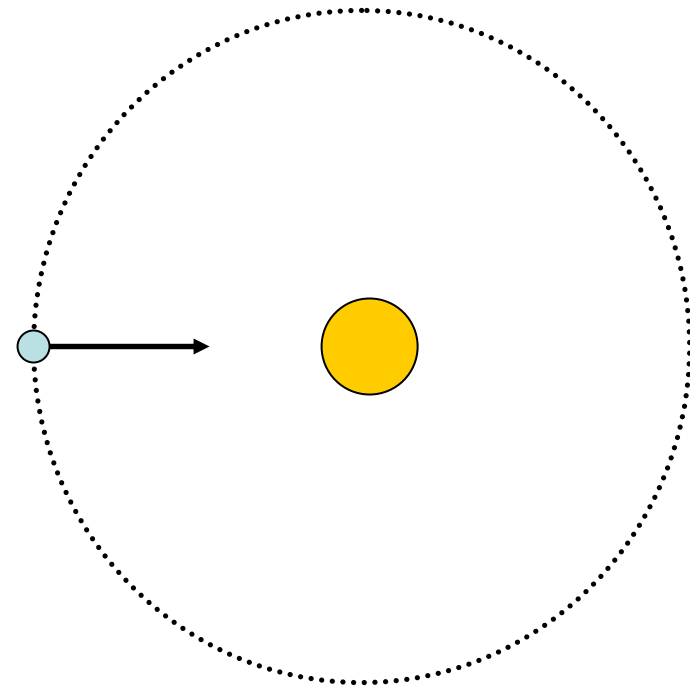# Kepler's Third Law

Consider an object (mass *m*) in a circular orbit (radius *r*) around a much larger object (mass *M*).

$$\sum \vec{F} = ma$$

$$\frac{GmM}{r^2} = \frac{mv^2}{r}$$

$$\frac{GM}{r} = v^2$$

We can also use $v = \dfrac{2\pi r}{T}$ to get: $T^2 = \dfrac{4\pi^2}{GM} r^3$

Rotational motion

How do we describe the motion of rotating objects, or objects that travel along circular paths?

Let's first look at the parallels between straight-line motion and rotational motion.

Compare the motion of:

• a ball, thrown straight up into the air

• a disk, that has a counter-clockwise angular acceleration, and a clockwise initial angular velocity.

Which motion do the graphs in the simulation correspond to?

A spinning wheel

Choose a point (but not the center) on a wheel that is spinning. At a particular instant in time, are there any other points on the wheel that have the same speed as that of the first point? At that same instant, are there any other points on the wheel that have the same velocity as that of the first point?

1. There are plenty of other points with the same speed and plenty of points with the same velocity.

2. There are plenty of other points with the same speed, but no other points with the same velocity.

3. There are no other points with the same speed, but plenty of points with the same velocity.

4. There are no other points with the same speed, and no other points with the same velocity.

A spinning wheel

Choose a point (but not the center) on a wheel that is spinning.

Is there any other point on the wheel that has the same speed as the speed of the first point?

Is there any other point on the wheel that has the same velocity as the velocity of the first point?

A spinning wheel

Choose a point (but not the center) on a wheel that is spinning.

Is there any other point on the wheel that has the same speed as the speed of the first point?

Yes – all points at the same distance r from the center.

Is there any other point on the wheel that has the same velocity as the velocity of the first point?
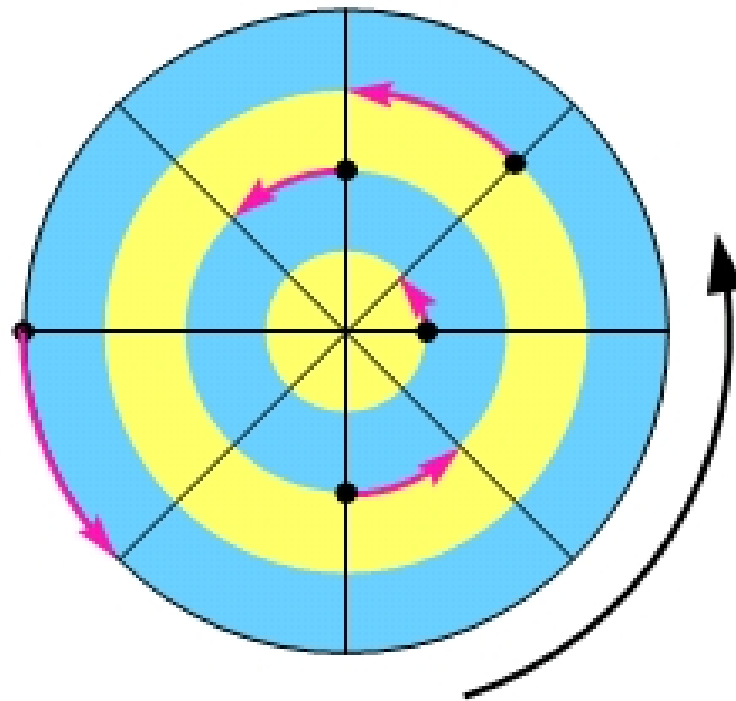
A spinning wheel

Is there any other point on the wheel that has the same velocity as the velocity of the first point?

No! Each point has a unique velocity.
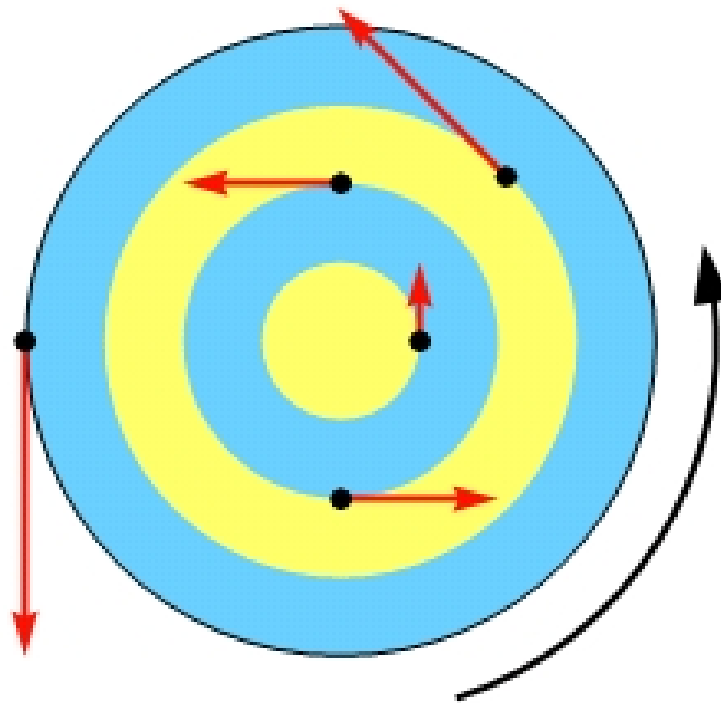
A spinning wheel

However, every point on the wheel has the same angular velocity (the rate at which a point sweeps out an angle).

Rotational variables

For rotational motion, we define a new set of variables that naturally fit the motion.

Angular position: θ , in units of radians. (π rad = 180°)

Angular displacement: $\Delta \vec{\theta}$

Angular velocity: $\vec{\omega} = \dfrac{\Delta \vec{\theta}}{\Delta t}$, in units of rad/s.
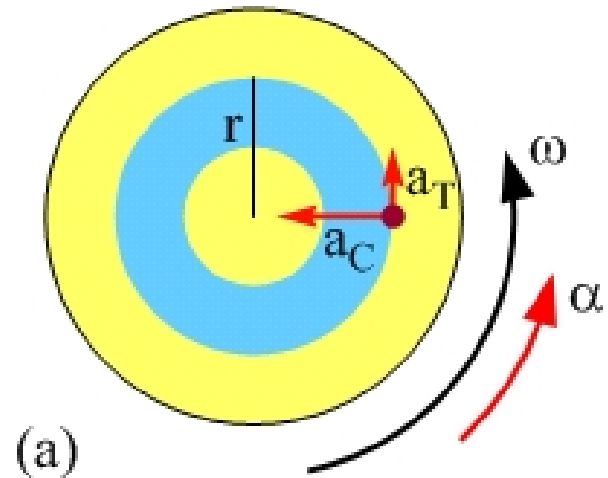
For a direction, we often use clockwise or counterclockwise, but the direction is actually given by the right-hand rule.

Angular acceleration: $\vec{\alpha} = \dfrac{\Delta \vec{\omega}}{\Delta t}$, in units of rad/s².
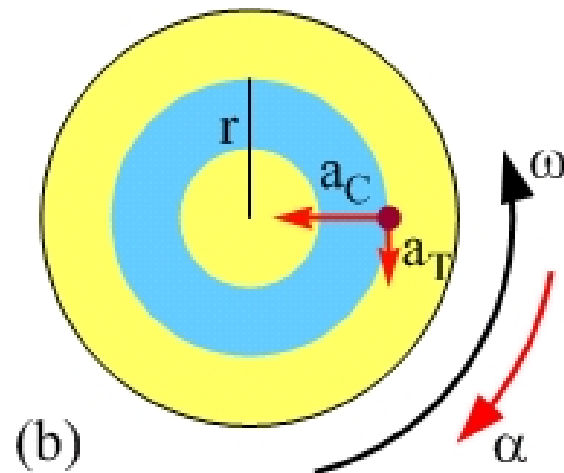
Speeding up vs. slowing down

In (a), the disk is speeding up. A point on the disk has both a centripetal acceleration and a tangential acceleration.

In (b), the disk is slowing down. Both the angular acceleration and the tangential acceleration are in the opposite direction to what they are in (a).

Analogies, part 1

We'll look at several analogies between straight-line motion variables and rotational motion variables.

| Variable	Straight-line motion	Rotational motion	Connection
Position	x	θ	$\theta = \dfrac{x}{r}$
Velocity	v	ω	$\omega = \dfrac{v_t}{r}$
Acceleration	a	α	$\alpha = \dfrac{a_t}{r}$

The t subscript stands for tangential.

Worksheet, part 1

A large block is tied to a string wrapped around the outside of a large pulley that has a radius of 2.0 m. When the system is released from rest, the block falls with a constant acceleration of 0.5 m/s^2, directed down.

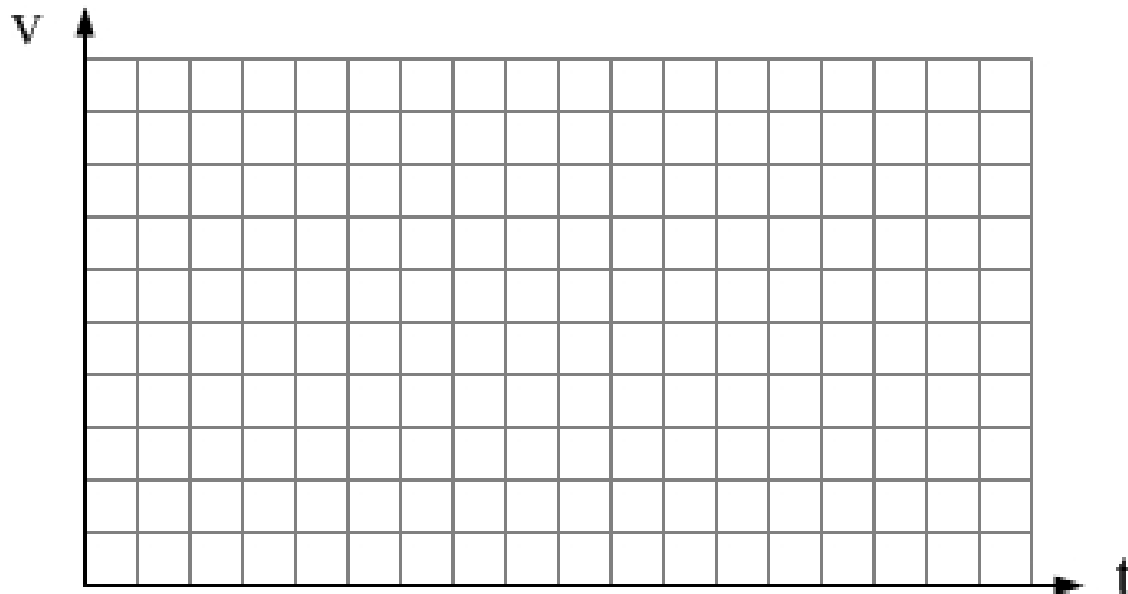What is the speed of the block after 4.0 s?

How far does the block travel in 4.0 s?
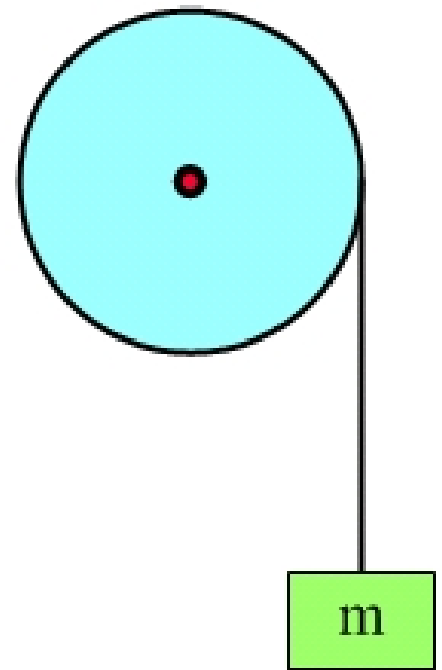
Worksheet, part 1

Plot a graph of the speed of the block as a function of time, up until 4.0 s.

On the same set of axes, plot the speed of a point on the pulley that is on the outer edge of the pulley, 2.0 m from the center, and the speed of a point 1.0 m from the center.
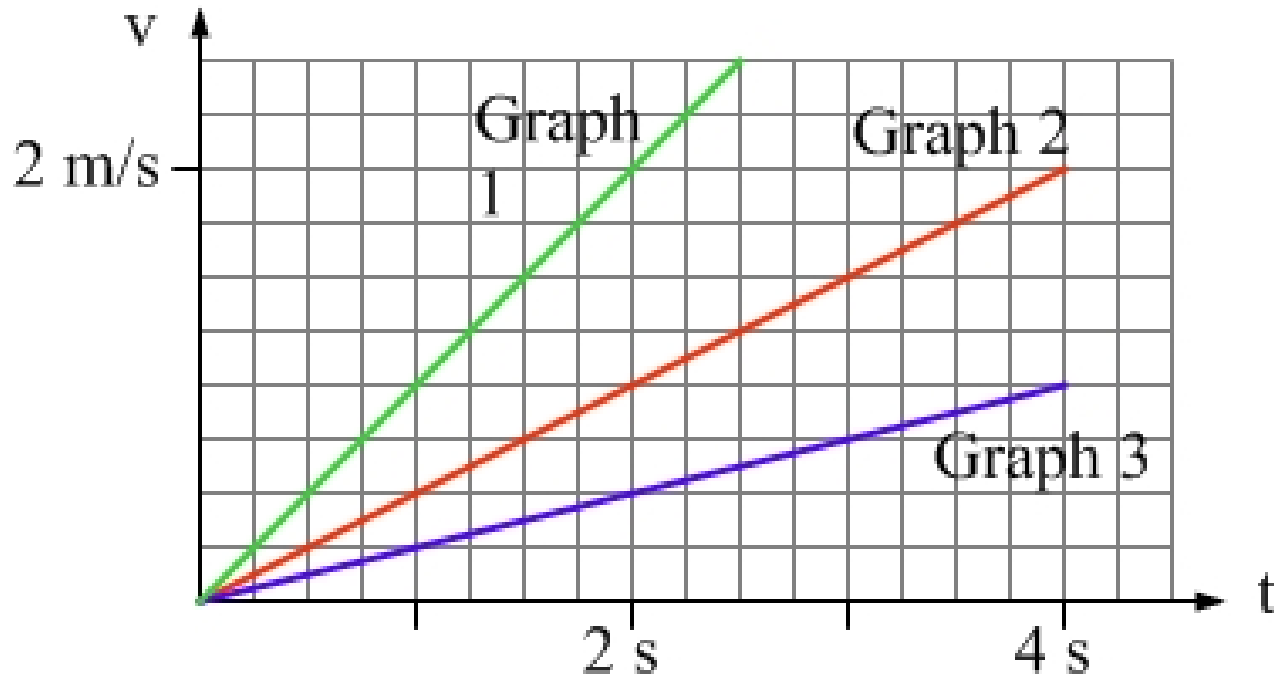
Worksheet

If the speed of the block follows graph 2…

Which graph represents the speed of a point on the outer edge of the pulley?

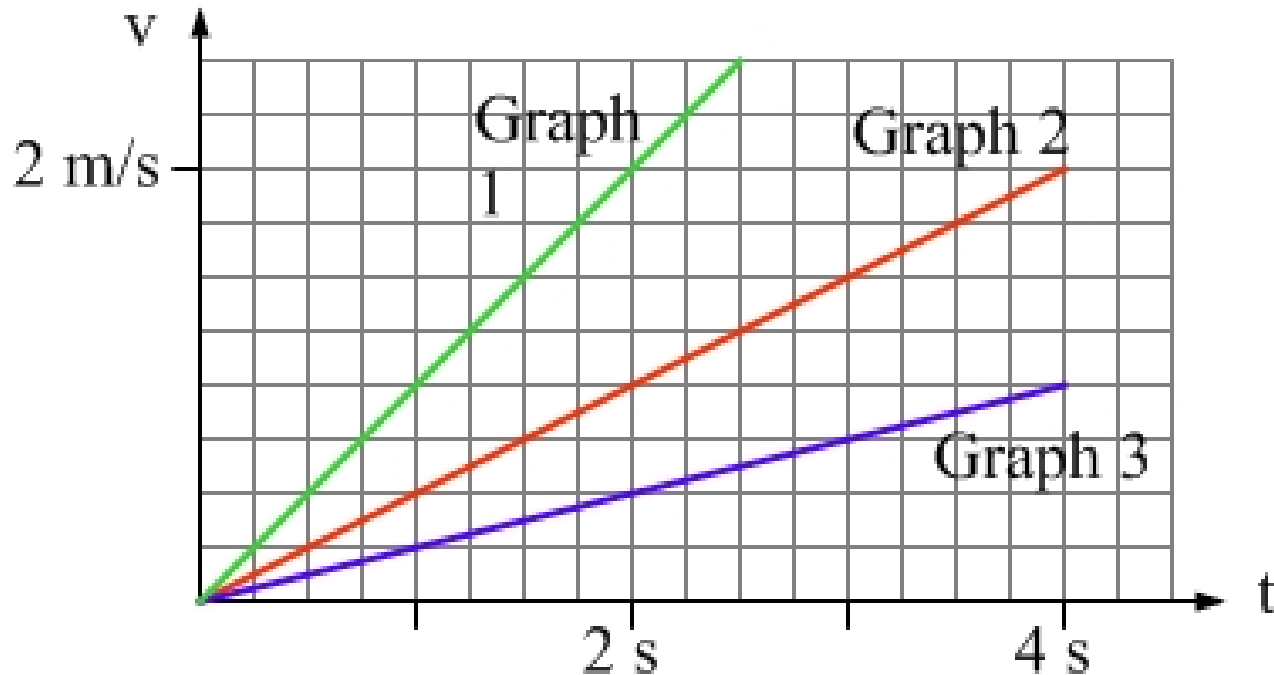Which graph represents the speed of a point 1.0 m from the center of the pulley?

Worksheet

If the speed of the block follows graph 2…

Which graph represents the speed of a point on the outer edge of the pulley? Graph 2

Which graph represents the speed of a point 1.0 m from the center of the pulley?

Worksheet

If the speed of the block follows graph 2…

Which graph represents the speed of a point on the outer edge of the pulley? Graph 2

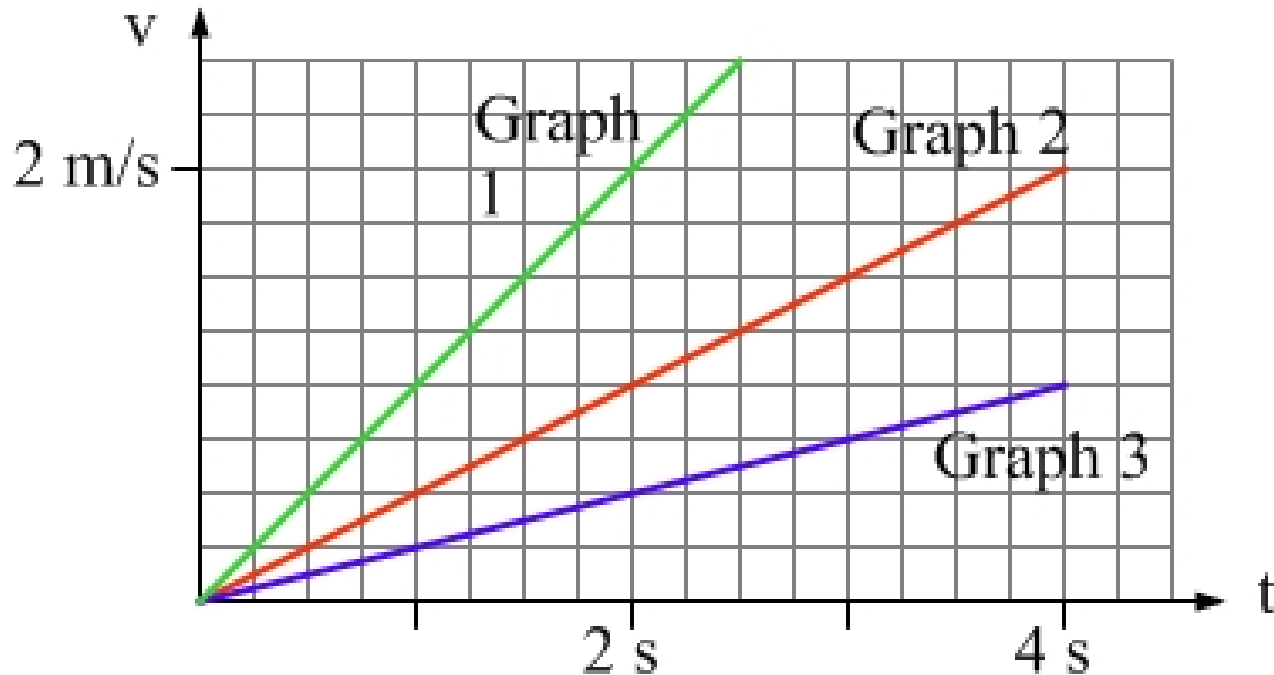Which graph represents the speed of a point 1.0 m from the center of the pulley? Graph 3

Angular vs. linear variables

Consider the simulation of two objects on a turntable that rotates at a constant angular velocity. The red object is farther from the center than the blue one.

Simulation

Angular vs. linear variables

The red object is farther from the center. Choose the best statement from the list below, about the two objects.

1. They have the same speed, but different velocities.
2. They have the same speed and the same velocity.
3. The have the same angular speed, but different angular velocities.
4. They have the same angular speed and the same angular velocity.
5. Both 1 and 3
6. Both 1 and 4
7. Both 2 and 3
8. Both 2 and 4
9. None of the above. .

Angular vs. linear variables

The red object is farther from the center. Choose the best statement from the list below, about the two objects.

1. Their accelerations have the same magnitude, but the red one has a larger angular acceleration.

2. Their accelerations have the same magnitude, but the blue one has a larger angular acceleration.

3. They have the same angular accelerations, but the red one has a larger magnitude acceleration.

4. They have the same angular accelerations, but the blue one has a larger magnitude acceleration.

5. Both the acceleration and the angular acceleration equal zero for both objects.

6. None of the above.

Rotational kinematics problems

When the angular acceleration is constant we can use the basic method we used for one-dimensional motion situations with constant acceleration.

1. Draw a diagram.

2. Choose an origin.

3. Choose a positive direction (generally clockwise or counter-clockwise).

4. Make a table summarizing everything you know.

5. Only then, assuming the angular acceleration is constant, should you turn to the equations.

Constant acceleration equations

Straight-line motion equation	Rotational motion equation
$v = v_i + at$	$\omega = \omega_i + \alpha t$
$x = x_i + v_i t + \dfrac{1}{2}at^2$	$\theta = \theta_i + \omega_i t + \dfrac{1}{2}\alpha t^2$
$v^2 = v_i^2 + 2a(\Delta x)$	$\omega^2 = \omega_i^2 + 2\alpha(\Delta\theta)$

Don't forget to use the appropriate + and - signs!

Example problem

You are on a ferris wheel that is rotating at the rate of 1 revolution every 8 seconds. The operator of the ferris wheel decides to bring it to a stop, and puts on the brake. The brake produces a constant acceleration of -0.11 radians/s^2.

(a) If your seat on the ferris wheel is 4.2 m from the center of the wheel, what is your speed when the wheel is turning at a constant rate, before the brake is applied?

(b) How long does it take before the ferris wheel comes to a stop?

(c) How many revolutions does the wheel make while it is slowing down?

(d) How far do you travel while the wheel is slowing down?

Simulation

Get organized

Origin: your initial position.
Positive direction: counterclockwise (the direction of motion).

Use a consistent set of units.
1 revolution every 8 s is 0.125 rev/s.

$$\omega_i = \frac{0.125 \text{ rev}}{1 \text{ s}} \times \frac{2\pi \text{ rad}}{1 \text{ rev}} = 0.785 \text{ rad/s}$$

θ_i	0
ω_i	$+0.785$ rad/s
ω	0
α	-0.11 rad/s^2

Part (a)

If your seat on the ferris wheel is 4.2 m from the center of the wheel, what is your speed when the wheel is turning at a constant rate, before the brake is applied?

θ_i	0
ω_i	+0.785 rad/s
ω	0
α	−0.11 rad/s^2

$$v_i = r\omega_i = 4.2 \text{ m} \times 0.785 \text{ rad/s} = 3.3 \text{ m/s}$$

Note that the radian unit can be added or removed whenever we find it convenient to do so.

Part (b)

How long does it take before the ferris wheel comes to a stop?

θ_i	0
ω_i	+0.785 rad/s
ω	0
α	−0.11 rad/s^2

$$\omega = \omega_i + \alpha t$$

$$t = \frac{\omega - \omega_i}{\alpha} = \frac{0 - 0.785 \text{ rad/s}}{-0.11 \text{ rad/s}^2} = 7.1 \text{ s}$$

Part (c)

How many revolutions does the wheel make while it is slowing down?

θ_i	0
ω_i	+0.785 rad/s
ω	0
α	–0.11 rad/s^2

$$\omega^2 = \omega_i^2 + 2\alpha(\Delta\theta)$$

$$\Delta\theta = \frac{\omega^2 - \omega_i^2}{2\alpha} = \frac{0 - (0.785 \text{ rad/s})^2}{2 \times (-0.11 \text{ rad/s}^2)} = 2.80 \text{ rad}$$

$$2.80 \text{ rad} \times \frac{1 \text{ rev}}{2\pi \text{ rad}} = 0.45 \text{ rev}$$

Part (d)

How far do you travel while the wheel is slowing down?

$\Delta\theta = 2.80$ rad

We're looking for the distance you travel along the circular arc. The arc length is usually given the symbol *s*.

$s = r(\Delta\theta) = 4.2 \text{ m} \times 2.80 \text{ rad} = 11.8 \text{ m}$

Worksheet, part 2

While fixing the chain on your bike, you have the bike upside down. Your friend comes along and gives the front wheel, which has a radius of 30 cm, a spin. You observe that the wheel has an initial angular velocity of 2.0 rad/s, and that the wheel comes to rest after 50 s.

Assume that the wheel has a constant angular acceleration.

Determine how many revolutions the wheel makes.

Worksheet, part 2

One method is to use the average angular velocity of 1.0 rad/s.

With a time of 50 s, the wheel has an angular displacement of 1.0 rad/s multiplied by 50 s, or 50 rad.

$$50 \text{ rad} \times \frac{1 \text{ rev}}{2\pi \text{ rad}} = \frac{25}{\pi} \text{ rev}$$

This is very close to 8 revolutions, just slightly less.

Torque

Torque is the rotational equivalent of force.

A torque is a twist applied to an object.

A net torque acting on an object at rest will cause it to rotate.

If you have ever opened a door, you have a working knowledge of torque.
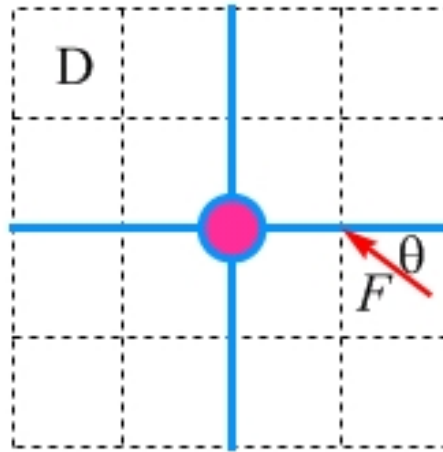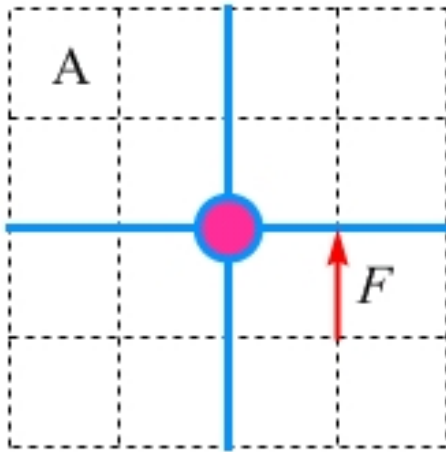
A revolving door

A force is applied to a revolving door that rotates about its center:
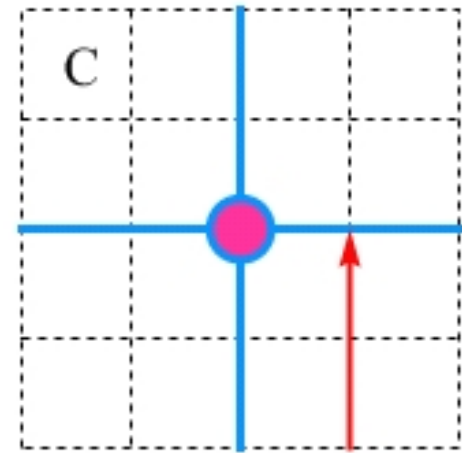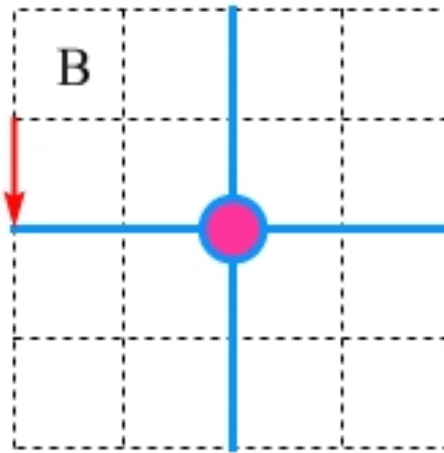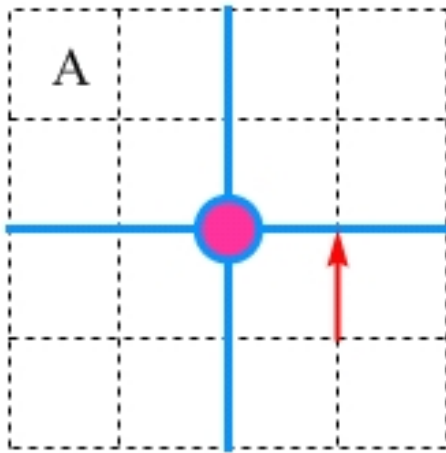
Rank these situations based on the magnitude of the torque experienced by the door, from largest to smallest.
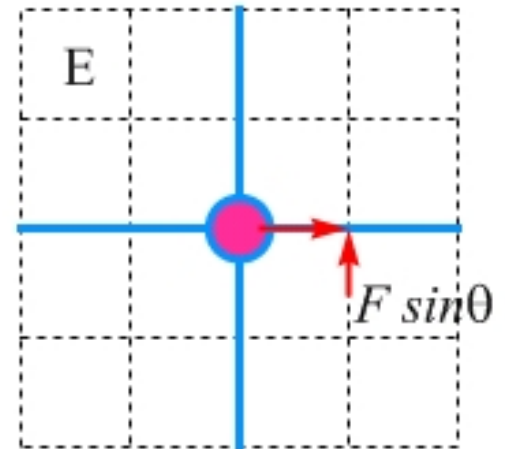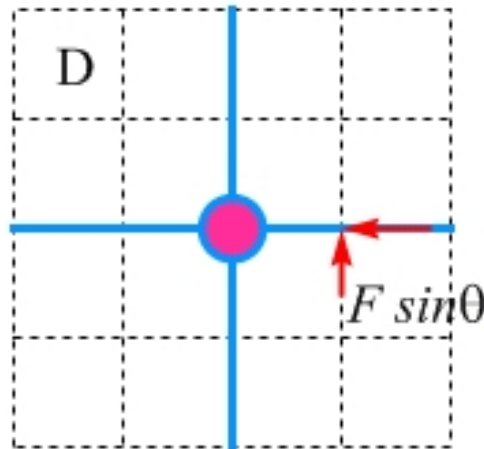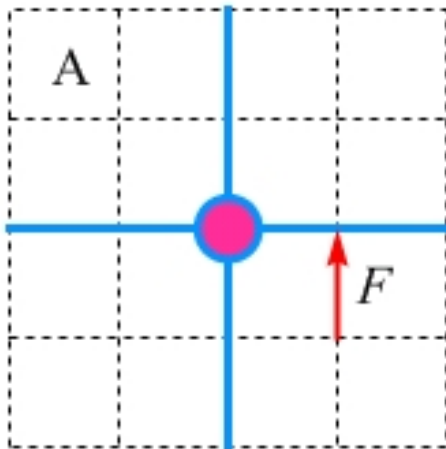
1. C>A>B
2. C>B>A
3. C>A=B

4. B>C>A
5. B>A>C
6. B>A=C
7. None of the above

Simulation

Revolving door simulation

A revolving door

A force is applied to a revolving door that rotates about its center:
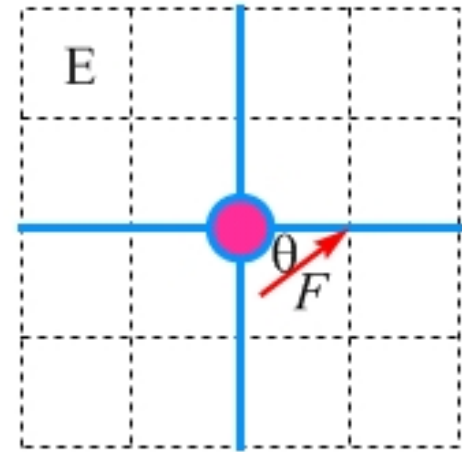
Rank these situations based on the magnitude of the torque experienced by the door, from largest to smallest.
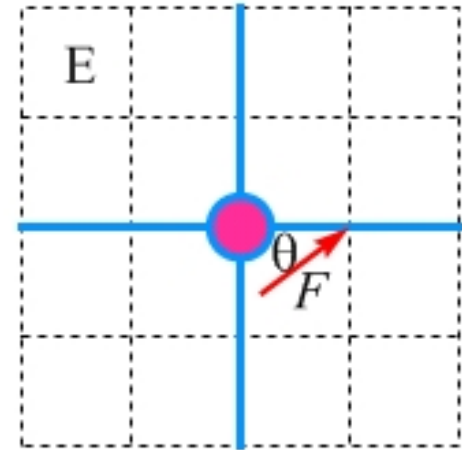
1. E>A>D
2. E>D>A
3. E>A=D
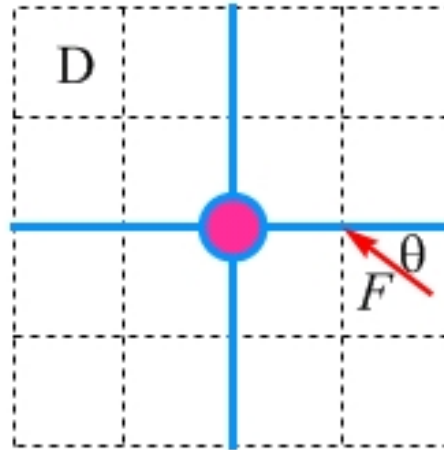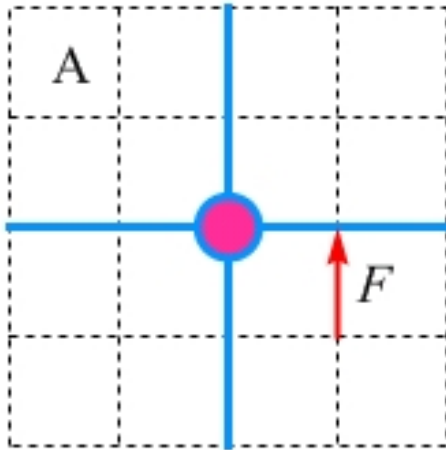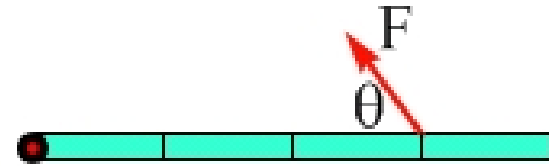4. A>E>D
5. A>D>E
6. A>D=E
7. None of the above

Use components

The force components directed toward, or away from, the axis of rotation do nothing, as far as getting the door to rotate.

Torque

Forces can produce torques. The magnitude of a torque depends on the force, the direction of the
force, and where the force is applied.

The magnitude of the torque is $\tau = r\,F\sin\theta$.
$\vec{r}$ is measured from the axis of rotation to the line of the force, and θ is the angle between $\vec{r}$ and $\vec{F}$.

To find the direction of a torque from a force, pin the object at the axis of rotation and push on it with the force. We can say that the torque from that force is whichever direction the object spins (counterclockwise, in the picture above).

Torque is zero when $\vec{r}$ and $\vec{F}$ are along the same line.
Torque is maximum when $\vec{r}$ and $\vec{F}$ are perpendicular.

Three ways to find torque

Find the torque applied by the string on the rod .
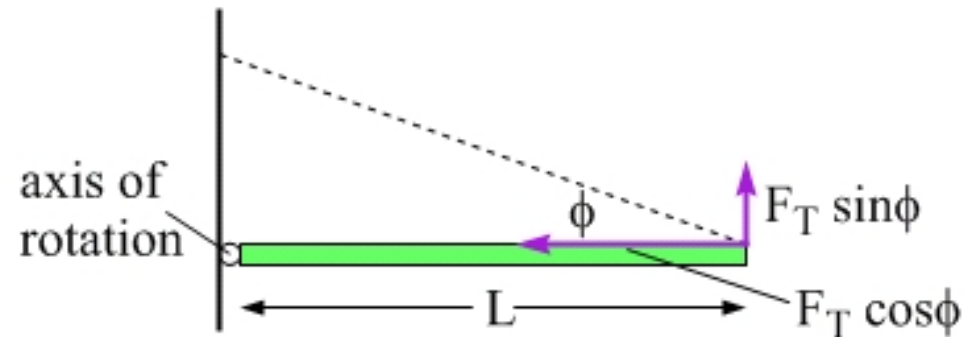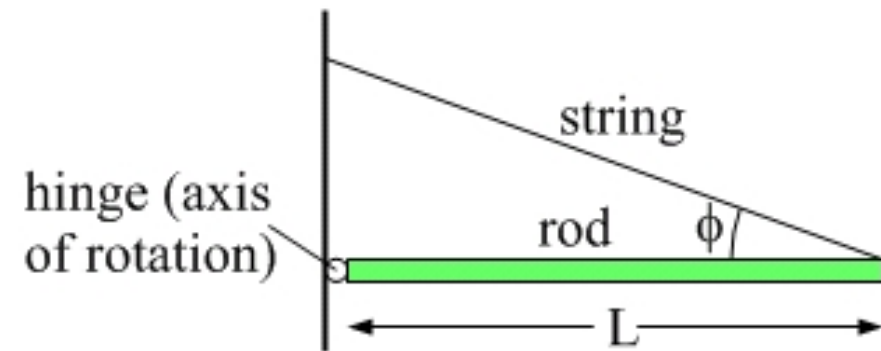
1. Just apply the equation $\tau = r\,F\sin\theta$
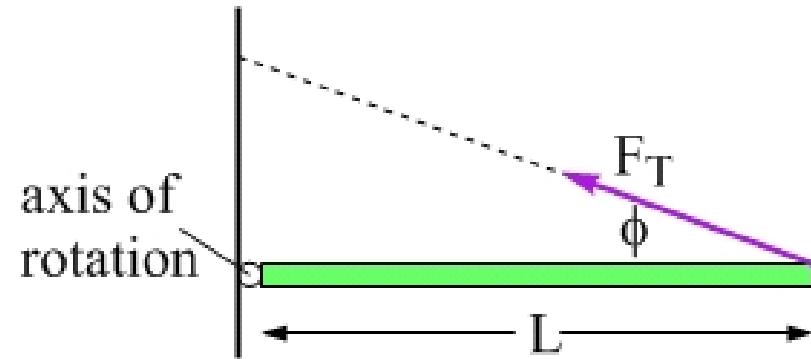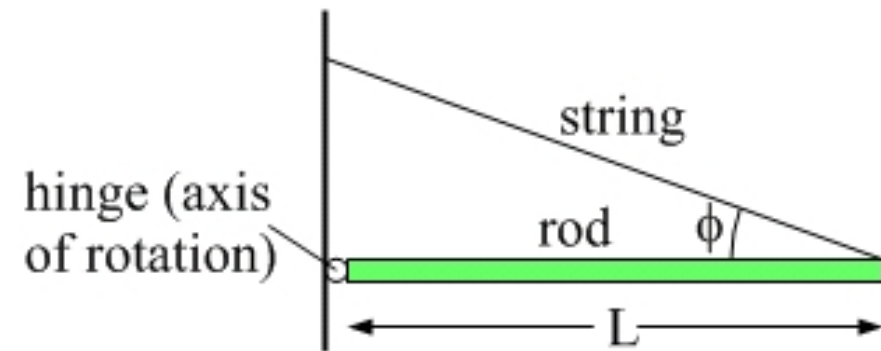
$$\tau = L \times F_T \times \sin\phi$$

Three ways to find torque

Find the torque applied by the string on the rod .

2. Break the force into components first, then use $\tau = r\,F\sin\theta$.
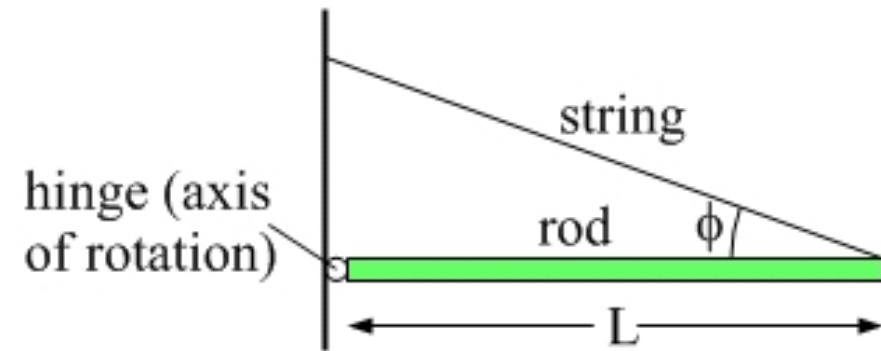

The force component along the rod gives no torque.

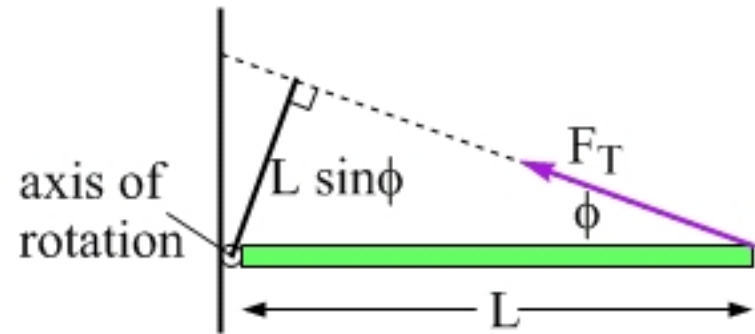$$\tau = L \times \left(F_T \sin\phi \right) \times \sin 90°$$

Three ways to find torque

Find the torque applied by the string on the rod .

3. Use the lever-arm method: measure *r* along the line that meets the line of the force at a 90° angle.

$$\tau = r\,F\sin\theta$$

$$\tau = \left(L\sin\phi\right) \times F_T \times \sin 90°$$

Worksheet, part 2

Try drawing a free-body diagram for a horizontal rod that is hinged at one end. The rod is held horizontal by an upward force applied by a spring scale at the far end of the rod.

How does the weight of the rod compare to the reading on the spring scale?

An equilibrium example

This is a model of our lower arm,

with the elbow being the hinge.

Summing the torques

To solve for the unknown force, we can't use forces, because we get one equation with two unknowns (the force of gravity and the hinge force).

Use torques instead. We can take torques about any axis we want, but if we take torques about an axis through the hinge, we eliminate the unknown hinge force.

Define clockwise as positive, and say the rod has a length L.

$$\sum \vec{\tau} = 0$$

$$+\frac{L}{2} \times mg - L \times F_S = 0 \qquad \Rightarrow \qquad mg = 2F_S$$

Equilibrium

For an object to remain in equilibrium, two conditions must be met.

The object must have no net force: $\sum \vec{F} = 0$

and no net torque: $\sum \vec{\tau} = 0$

Moving the spring scale

What, if anything, happens when the spring scale is moved a little closer to the hinge? To maintain equilibrium:

1. The magnitude of the spring-scale force increases.
2. The magnitude of the spring-scale force decreases.
3. The magnitude of the upward hinge force increases.
4. The magnitude of the upward hinge force decreases.
5. Both 1 and 3
6. Both 1 and 4
7. Both 2 and 3
8. Both 2 and 4
9. None of the above.

Worksheet, part 2

Try drawing a free-body diagram for a horizontal rod that is hinged at one end. The rod is held horizontal by an upward

force applied by a spring scale ¼ of the way along the rod.

How does the weight of the rod compare to the reading on the spring scale?

An equilibrium example

This is a model of our lower arm,

with the elbow being the hinge.

Summing the torques

To solve for the unknown force, we can't use forces, because we get one equation with two unknowns (the force of gravity and the hinge force).

Use torques instead. We can take torques about any axis we want, but if we take torques about an axis through the hinge we eliminate the unknown hinge force.

Define clockwise as positive, and say the rod has a length *L*.

$$\sum \vec{\tau} = 0$$

$$+\frac{L}{2} \times mg - \frac{L}{4} \times F_S = 0 \qquad \Rightarrow \qquad mg = \frac{F_S}{2}$$

Red and blue rods

Two rods of the same shape are held at their centers and rotated back and forth. The red one is much easier to rotate than the blue one. What is the best possible explanation for this?

1. The red one has more mass.

2. The blue one has more mass.

3. The red one has its mass concentrated more toward the center; the blue one has its mass concentrated more toward the ends.

4. The blue one has its mass concentrated more toward the center; the red one has its mass concentrated more toward the ends.

5. Either 1 or 3   6. Either 1 or 4

7. Either 2 or 3   8. Either 2 or 4

9. Due to the nature of light, red objects are just inherently easier to spin than blue objects are.

Newton's First Law for Rotation

An object at rest tends to remain at rest, and an object that is spinning tends to spin with a constant angular velocity, unless it is acted on by a nonzero net torque **or there is a change in the way the object's mass is distributed**.

The net torque is the vector sum of all the torques acting on an object.

The tendency of an object to maintain its state of motion is known as inertia. For straight-line motion mass is the measure of inertia, but mass by itself is not enough to define rotational inertia.

Rotational Inertia

How hard it is to get something to spin, or to change an object's rate of spin, depends on the mass, and on how the mass is distributed relative to the axis of rotation. Rotational inertia, or moment of inertia, accounts for all these factors.

The moment of inertia, I, is the rotational equivalent of mass.

For an object like a ball on a string, where all the mass is the same distance away from the axis of rotation:
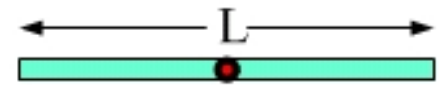
$I = mr^2$

If the mass is distributed at different distances from the rotation axis, the moment of inertia can be hard to calculate. It's much easier to look up expressions for I from the table on page 291 in the book (page 10-15 in *Essential Physics*).

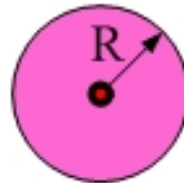# A table of rotational inertias
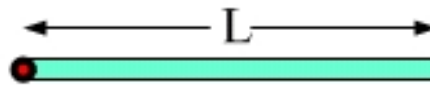
Rod rotating about an axis through one end, perpendicular to the rod.

$$I = \frac{1}{3}ML^2$$

Rod rotating about an axis through the middle, perpendicular to the rod.
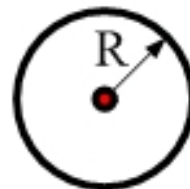
$$I = \frac{1}{12}ML^2$$

Solid disk or cylinder about an axis through the middle, perpendicular to the plane of the disk.

$$I = \frac{1}{2}MR^2$$

Solid sphere about an axis through the center.

$$I = \frac{2}{5}MR^2$$

Thin ring about an axis through the middle, perpendicular to the plane of the ring.

$$I = MR^2$$

Hollow sphere about an axis through the center.

$$I = \frac{3}{5}MR^2$$

The parallel axis theorem

If you know the rotational inertia of an object of mass *m* when it rotates about an axis that passes through its center of mass, the object's rotational inertia when it rotates about a parallel axis a distance *h* away is:

$$I = I_{CM} + mh^2$$