5-7 Using Whole Vectors

The standard method of solving a problem involving Newton's laws is to break the forces into components. However, using whole vectors is an alternate approach. Let's see how whole vectors can be applied in a particular situation.

EXAMPLE 5.7 – Using whole vectors

(a) A box is placed on a frictionless ramp inclined at an angle θ with the horizontal. The box is then released from rest. Find an expression for the normal force acting on the box in this situation. What is the role of the normal force? What is the acceleration of the box?

(b) A box truck is traveling in a horizontal circle around a banked curve that is inclined at an angle θ with the horizontal. The curve is covered with ice and is effectively frictionless, so the truck can make it safely around the curve only if it travels at a particular constant speed (known as the *design speed* of the curve). Find an expression for the normal force acting on the truck in this situation. What is the role of the normal force? What is the design speed of the curve?

(c) Compare and contrast these two situations.

SOLUTION

(a) As usual, our first step is to draw a diagram, and then a free-body diagram showing the forces acting on the box, as in Figure 5.18. The two forces are the downward force of gravity, and the normal force applied by the ramp to the box. Because we are using whole vectors, we don't need to worry about splitting vectors into components. It is crucial, however, to think about the direction of the acceleration, which in this case is directed down the slope.

Let's apply Newton's second law, $\sum \vec{F} = m\vec{a}$, adding the forces as vectors. In this case, we get the right-angled triangle in Figure 5.18c. Each side of the triangle represents one vector in the equation $m\vec{g} + \vec{F}_N = m\vec{a}$. The vector $m\vec{a}$ (the net force) is parallel to the ramp, while the normal force is perpendicular to the ramp and the force of gravity is directed straight down. θ, the angle of the ramp, is the angle at the bottom of the triangle.

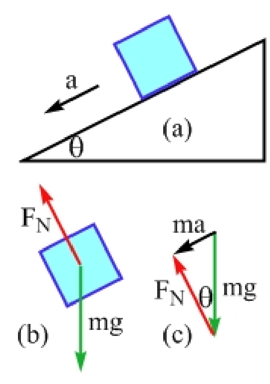

Figure 5.18: (a) A diagram, (b) free-body diagram, and (c) a right-angled triangle to show how the force of gravity and normal force combine to give the net force on the box.

Because the force of gravity is on the hypotenuse of the triangle, we get:

$$\cos\theta = \frac{F_N}{mg}, \qquad \text{so} \qquad F_N = mg\cos\theta .$$

The role of the normal force is simply to prevent the box from falling through the ramp.

We can find the acceleration from the geometry of the right-angled triangle:

$$\sin\theta = \frac{ma}{mg} = \frac{a}{g} .$$

This relationship gives $\vec{a} = g\sin\theta$ directed down the slope.

(b) The situation of the box truck traveling around the banked curve resembles the box on the incline. A diagram of the situation, showing the back of the truck, is illustrated in Figure 5.19.

The same forces, the force of gravity and the normal force, appear on this free-body diagram as in the free-body diagram for the box. The difference lies in the direction of the acceleration, which for the circular motion situation is directed horizontally to the left, toward the center of the circle.

Again we apply Newton's second law, $\sum \vec{F} = m\vec{a}$, adding the forces as vectors, where the magnitude of the acceleration has the special form $a_c = v^2 / r$. This gives:

$m\vec{g} + \vec{F}_N = \frac{mv^2}{r}$, directed toward the center of the circle.

Again, each force represents one side of a right-angled triangle. Now the normal force is on the hypotenuse, and clearly must be larger than it is in part (a).

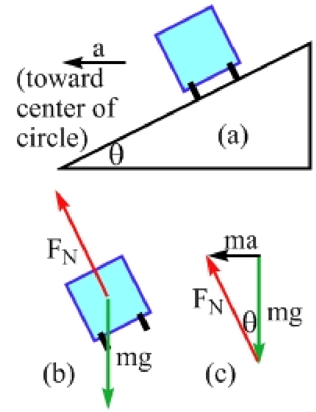

Figure 5.19: (a) A diagram, (b) free-body diagram, and (c) right-angled triangle to show how the force of gravity and normal force combine to give the net force on the truck.

Using what we know about the geometry of right-angled triangles, we get:

$\cos\theta = \frac{mg}{F_N}$, so now $F_N = \frac{mg}{\cos\theta}$.

If the angle of the ramp is larger than zero, then $\cos\theta$ is a number less than 1 and $F_N > mg$. In part (a) we had $F_N < mg$. The normal force for the box truck is larger because it has two roles. Not only does the normal force prevent the truck from falling through the incline, it must also provide the force directed toward the center to keep the truck moving around the circle.

To find the design speed of the curve, we can use the other side of the triangle:

$\sin\theta = \frac{ma}{F_N} = \frac{mv^2}{rF_N} = \frac{mv^2\cos\theta}{rmg} = \frac{v^2\cos\theta}{rg}$.

Re-arranging the preceding equation gives a design speed of $v = \sqrt{\frac{rg\sin\theta}{\cos\theta}} = \sqrt{rg\tan\theta}$.

This is an interesting result. First, there is a design speed, a safest speed to negotiate the curve. At the design speed, the vehicle needs no assistance from friction to travel around the circle. Going significantly faster is dangerous because the vehicle is only prevented from sliding toward the outside of the curve by the presence of friction - the faster you go, the larger the force of friction required. Second, the design speed does not depend on the vehicle mass, which is fortunate for the road designers. The same physics applies to a Mini Cooper as to a large truck.

(c) A key similarity is the free-body diagram: in both cases there is a downward force of gravity and a normal force perpendicular to the slope. The key difference is that the accelerations are in different directions, requiring a larger normal force in the circular motion situation.

Related End-of-Chapter Exercises: 25 – 27.

Essential Question 5.7: Consider again the situation of the truck on the banked curve. In icy conditions, is it safest to drive very slowly around the curve or to drive at the design speed?

Answer to Essential Question 5.7: Even in very low-friction conditions, it is safer to travel at the design speed than at a slower speed! If you go too slowly around a banked curve, there is a tendency for your vehicle to slide down the slope and run off the road on the inside of the curve.

5-8 Vertical Circular Motion

A common application of circular motion is an object moving in a vertical circle. Examples include roller coasters, cars on hilly roads, and a bucket of water on a string. The bucket and roller coaster turn completely upside down as they travel, so they differ a little from the situation of the car on the road, which (we hope) remains upright.

EXAMPLE 5.8A – Whirling a bucket of water

A bucket of water is being whirled in a vertical circle of constant radius *r* at a constant speed *v*. What is the minimum speed required for the water to remain in the bucket at the top of the circle?

SOLUTION

Let's apply the general method, starting with the diagram in Figure 5.20. We then draw a free-body diagram, although we have to decide whether to analyze the bucket or the water. If we consider the bucket, two forces act on it, the force of gravity and the tension in the string, both of which are directed down when the bucket is at the top of the circle. If we consider the water, there is a downward force of gravity, and a downward normal force from the bucket takes the place of the tension. The analysis is the same in both cases, so let's consider the water.

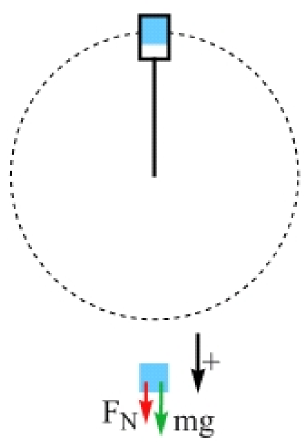

Figure 5.20: A bucket of water whirled in a vertical circle and a free-body diagram showing the forces acting on the water at the top of the loop.

Next, we choose an appropriate coordinate system. It is generally best to choose the positive direction as the direction of the acceleration, which points toward the center of the circle. When the bucket is at the top of the path, the acceleration, and the positive direction, points down. We don't need to split any forces into components. Let's apply Newton's second law, $\sum \vec{F} = m\vec{a}$.

Have a look at the free-body diagram to evaluate the left-hand side, and write the right-hand side in the usual circular-motion form. This gives:

$$+mg + F_N = +\frac{mv^2}{r} .$$

Solving for the normal force gives: $F_N = \frac{mv^2}{r} - mg$.

As long as the first term on the right exceeds the second term (in other words, as long as the normal force is positive), we're in no danger of having the water fall on us. Objects lose contact with one another (the water starts to fall out) when the normal force goes to zero. Setting the normal force to zero gives us the minimum safe speed of the bucket at the top of the circle:

$0 = \frac{mv_{\text{min}}^2}{r} - mg$,   which leads to $v_{\text{min}} = \sqrt{gr}$.

Related End-of-Chapter Exercises: 12, 61.

EXAMPLE 5.8B – Apparent weight on a roller coaster

You are riding on a roller coaster that is going around a vertical circular loop. What is the expression for the normal force on you at the bottom of the circle?

SOLUTION

Once again, we apply the general method, starting with a diagram and a free-body diagram in Figure 5.21. We then draw a free-body diagram, which shows an upward normal force and a downward force of gravity. The system can be you or the car – the analysis is the same. Here we choose a coordinate system with a positive direction up, in the direction of the acceleration (toward the center of the circle). There is no need to split forces into components, so we can go straight to step 5 of the general method and apply Newton's Second Law:

$$\sum \vec{F} = m\vec{a} .$$

Have a look at the free-body diagram to evaluate the left-hand side, and remember that the right-hand side can be written in the usual circular-motion form. This gives:

$$+F_N - mg = +\frac{mv^2}{r} .$$

Solving for the normal force at the bottom of the circle gives:

$$F_{N,bottom} = \frac{mv^2}{r} + mg .$$

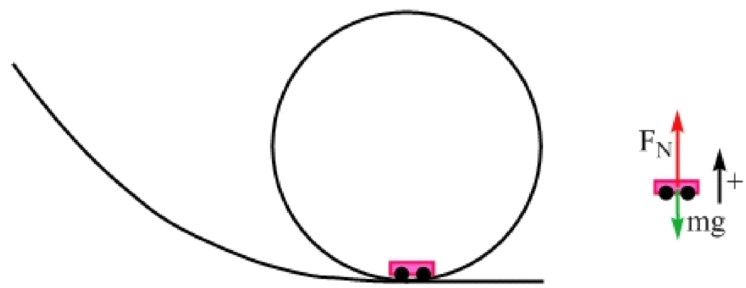

Figure 5.21: A car on a roller-coaster track (left), as well as (right) the free-body diagram when the car is at the bottom of the loop.

Let's compare our expression for the normal force on the car (or you) at the bottom of the loop to the expression for the normal force when the car (or you) is at the top. We can use the expression that we derived for the bucket at the top, in Example 5.8A, because the free-body diagram is the same in the two situations at the top of the loop.

$$F_{N,top} = \frac{mv^2}{r} - mg .$$

Note that the normal force at the bottom is larger than it is at the top. This difference is enhanced by the fact that the speed of the roller coaster at the bottom of the loop is larger than the speed at the top. Does this change in the normal force match the experience of a rider, who feels that she is lighter than usual at the top of the loop and heavier than usual at the bottom? Yes, because the normal force is the rider's apparent weight. Roller coasters are generally designed to have non-zero but fairly small normal forces at the top, so a rider feels almost weightless. At the bottom of the loop, the apparent weight can be considerably larger than *mg*, so a rider feels much heavier than usual.

Related End-of-Chapter Exercises: 20, 63.

Essential Question 5.8: You are on a roller coaster that reaches a top speed of 120 km/h at the bottom of a circular loop of radius 30 m. If you have a mass of 50 kg (and therefore a weight of 490 N), what is your apparent weight at the bottom of the loop? If the roller coaster's speed at the top of the loop has dropped to 80 km/h, what is your apparent weight at the top of the loop?