

6-1 Rewriting Newton's Second Law

In this chapter, we will begin by taking a look at two ideas that we are familiar with from previous chapters. Let's see what happens when we combine Newton's second law, $\vec{F}_{net} = m\vec{a}$, with the definition of acceleration, $\vec{a} = \Delta\vec{v} / \Delta t$. This gives $\vec{F}_{net} = m\Delta\vec{v} / \Delta t$. Let's be a bit creative and write this relationship in a form more as Newton himself originally did:

$$\vec{F}_{net} = \frac{\Delta(m\vec{v})}{\Delta t} \quad (\text{Equation 6.1: General form of Newton's Second Law})$$

Equation 6.1 is more general than Newton's second law stated in the form $\vec{F}_{net} = m\vec{a}$, because equation 6.1 allows us to work with systems (such as rockets) where the mass changes. $\vec{F}_{net} = m\vec{a}$ applies only to systems where the mass is constant, although so many systems have constant mass that we find this form of the equation to be very useful.

The general form of Newton's second law connects the net force on an object with the rate of change of the quantity $m\vec{v}$. This quantity has a name, which you may already be familiar with.

An object's **momentum** is the product of its mass and its velocity. Momentum is a vector, pointing in the direction of the velocity. The symbol we use to represent momentum is \vec{p} .

$$\vec{p} = m\vec{v} . \quad (\text{Equation 6.2: Momentum})$$

Equation 6.1 can be re-arranged to read: $\vec{F}_{net} \Delta t = \Delta(m\vec{v}) = \Delta\vec{p}$.

Thus, to change an object's momentum, all we have to do is to apply a net force for a particular time interval. To produce a larger change in momentum, we can apply a larger net force or apply the same net force over a longer time interval. The product of the net force and the time interval over which the force is applied is such an important quantity that we should name it, too.

The product of the net force and the time interval over which the force is applied is called **impulse**. An impulse produces a change in momentum. An impulse is a vector.

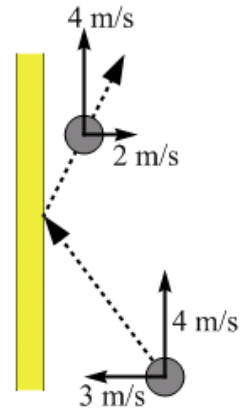
$$\vec{F}_{net} \Delta t = \Delta(m\vec{v}) = \Delta\vec{p} . \quad (\text{Equation 6.3: Impulse})$$

EXPLORATION 6.1 – Hitting the boards

Just before hitting the boards of an ice rink, a hockey puck is sliding along the ice at a constant velocity. The components of the velocity are 3.0 m/s in the direction perpendicular to the boards and 4.0 m/s parallel to the boards. After bouncing off the boards, the puck's velocity component perpendicular to the boards is 2.0 m/s and the component parallel to the boards is unchanged. The puck's mass is 160 g.

Step 1 - What is the impulse applied to the puck by the boards? Let's sketch a diagram (see Figure 6.1) to help visualize what is going on. Impulse is the product of the net force and the time--but we don't know the net force so we can't get impulse that way. Impulse is also equal to the change in momentum, as we can see from equation 6.3, so let's figure out that change.

Figure 6.1: The puck's velocity components before and after it collides with the boards.



Remember that momentum, impulse, and velocity are all vectors. Let's choose a coordinate system with the positive x -direction to the right and the positive y -direction up. It is important to notice that there has been no change in the puck's velocity in the y -direction, so there is no change in momentum, no impulse, and no net force in that direction. We can focus on the x -direction to answer the question.

The puck's mass is constant, so the puck's momentum changes because there is a change in velocity. What is the magnitude of the change in the puck's velocity in the x -direction, perpendicular to the boards? It's tempting to say 1.0 m/s, but it is actually 5.0 m/s. That result comes from:

$$\Delta \vec{v}_x = \vec{v}_x - \vec{v}_{ix} = +2.0 \text{ m/s} - (-3.0 \text{ m/s}) = +5.0 \text{ m/s} .$$

Knowing the change in velocity, we can find the impulse, which is in the $+x$ -direction:

$$\Delta \vec{p}_x = m \Delta \vec{v}_x = 0.16 \text{ kg} \times (+5.0 \text{ m/s}) = +0.80 \text{ kg m/s} .$$

Step 2 - If the puck is in contact with the boards for 0.050 s, what is the average force applied to the puck by the boards? The force varies over the 0.050 s the puck is in contact with the boards, but we can get the average force from:

$$\vec{F}_x = \frac{\Delta \vec{p}_x}{\Delta t} = \frac{+0.80 \text{ kg m/s}}{0.050 \text{ s}} = +16 \text{ N}$$

Key ideas for impulse and momentum: Analyzing situations from an impulse-momentum perspective can be very useful, as it allows us to directly connect force, velocity, and time. It is absolutely critical to account for the fact that momentum, impulse, force, and velocity are all vectors when carrying out such an analysis. **Related End-of-Chapter Exercises: 14, 15**

Let's now summarize a general method we can use to solve a problem involving impulse and momentum. We will apply this method in Exploration 6.2.

Solving a Problem Involving Impulse and Momentum

A typical impulse-and-momentum problem relates the net force acting on an object over a time interval to the object's change in momentum. A method for solving such a problem is:

1. Draw a diagram of the situation.
2. Add a coordinate system to the diagram, showing the positive direction(s). Keeping track of direction is important because force and momentum are vector quantities.
3. Organize what you know, and what you're looking for, such as by drawing one or more free-body diagrams, or drawing a graph of the net force as a function of time.
4. Apply equation 6.3 $(\vec{F}_{net} \Delta t = \Delta(m\vec{v}))$ to solve the problem.

Essential Question 6.1: At some time T after a ball is released from rest, the force of gravity has accelerated that ball to a velocity v directed straight down. Taking into account impulse and momentum, what is the ball's velocity at a time $2T$ after being released?

Answer to Essential Question 6.1: The force is constant, and Equation 6.3 tells us that the velocity increases linearly with time. Thus, at a time $2T$, the velocity will be $2v$ directed down.

6-2 Relating Momentum and Impulse

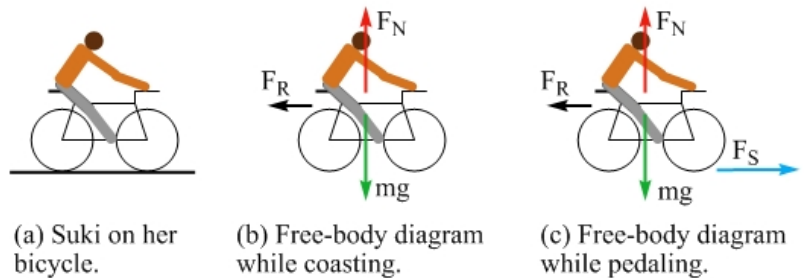
In this section, we will apply the general method from the end of Section 6-1 to solve a problem using the concepts of impulse and momentum.

EXPLORATION 6.2 – An impulsive bike ride

Suki is riding her bicycle, in a straight line, along a flat road. Suki and her bike have a combined mass of 50 kg. At $t = 0$, Suki is traveling at 8.0 m/s. Suki coasts for 10 seconds, but when she realizes she is slowing down, she pedals for the next 20 seconds. Suki pedals so that the static friction force exerted on the bike by the road increases linearly with time from 0 to 40 N, in the direction Suki is traveling, over that 20-second period. Assume there is constant 10 N resistive force, from air resistance and other factors, acting on her and the bicycle the entire time.

Step 1 - Sketch a diagram of the situation. The diagram is shown in Figure 6.2, along with the free-body diagram that applies for the first 10 s and the free-body diagram that applies for the 20-second period while Suki is pedaling.

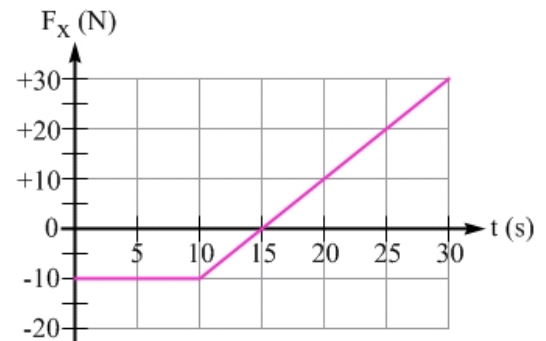
Figure 6.2: A diagram of (a) Suki on her bike, as well as free-body diagrams while she is (b) coasting and while she is (c) pedaling. Note that in free-body diagram (c), the static friction force \vec{F}_S gradually increases because of the way Suki pedals.



Step 2 - Sketch a graph of the net force acting on Suki and her bicycle as a function of time.

Take the positive direction to be the direction Suki is traveling. In the vertical direction, the normal force exactly balances the force of gravity, so we can focus on the horizontal forces. For the first 10 seconds, we have only the 10 N resistive force, which acts to oppose the motion and is thus in the negative direction. For the next 20 seconds, we have to account for the friction force that acts in the direction of motion and the resistive force. We can account for their combined effect by drawing a straight line that goes from -10 N at $t = 10$ s, to $+30$ N (40 N $- 10$ N) at $t = 30$ s. The result is shown in Figure 6.3.

Figure 6.3: A graph of the net force acting on Suki and her bicycle as a function of time.



Step 3 - What is Suki's speed at $t = 10$ s? Let's apply Equation 6.3, which we can write as:

$$\vec{F}_{net} \Delta t = \Delta(m\vec{v}) = m \Delta\vec{v} = m (\vec{v}_{10s} - \vec{v}_i).$$

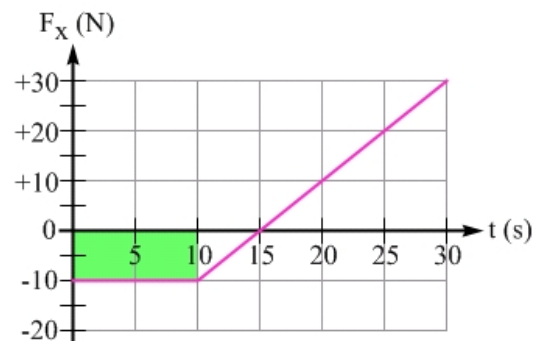
Solving for the velocity at $t = 10$ s gives:

$$\vec{v}_{10s} = \vec{v}_i + \frac{\vec{F}_{net} \Delta t}{m} = +8.0 \text{ m/s} + \frac{(-10 \text{ N})(10 \text{ s})}{50 \text{ kg}} = +8.0 \text{ m/s} - 2.0 \text{ m/s} = +6.0 \text{ m/s}.$$

Thus, Suki's speed at $t = 10$ s is 6.0 m/s. We can also obtain this result from the force-versus-time graph, by recognizing that the impulse, $\vec{F}_{net} \Delta t$, represents the area under this graph over some time interval Δt . Let's find the area under the graph, over the first 10 seconds, shown highlighted in green in Figure 6.4. The area is negative, because the net force is negative over that time interval. The area under the graph is the impulse:

$$\vec{F}_{net} \Delta t = -10 \text{ N} \times 10 \text{ s} = -100 \text{ N s} = -100 \text{ kg m/s}.$$

Figure 6.4: The green rectangle represents the area under the graph for the first 10 s. The area is negative, because the force is negative.



From Equation 6.3, we know the impulse is equal to the change in momentum. Suki's initial momentum is $m\vec{v}_i = 50 \text{ kg} \times (+8.0 \text{ m/s}) = +400 \text{ kg m/s}$. Her momentum at $t = 10$ s is therefore $+400 \text{ kg m/s} - 100 \text{ kg m/s} = +300 \text{ kg m/s}$. Dividing this by the mass to find the velocity at $t = 10$ s confirms what we found above:

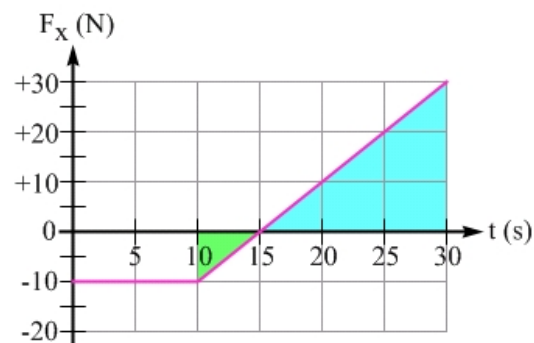
$$\vec{v}_{10s} = \frac{\vec{p}_{10s}}{m} = \frac{\vec{p}_i + \Delta\vec{p}}{m} = \frac{+400 \text{ kg m/s} - 100 \text{ kg m/s}}{50 \text{ kg}} = \frac{+300 \text{ kg m/s}}{50 \text{ kg}} = +6.0 \text{ m/s}.$$

Step 4 - What is Suki's speed at $t = 30$ s? Let's use the area under the force-versus-time graph, between $t = 10$ s and $t = 30$ s, to find Suki's change in momentum over that 20-second period. This area is highlighted in Figure 6.5, split into a negative area for the time between $t = 10$ s and $t = 15$ s, and a positive area between $t = 15$ s and $t = 30$ s. These regions are triangles, so we can use the equation for the area of a triangle, $0.5 \times \text{base} \times \text{height}$. The area under the curve, between 10 s and 15 s, is $0.5 \times (5.0 \text{ s}) \times (-10 \text{ N}) = -25 \text{ kg m/s}$. The area between 15 s and 30 s is $0.5 \times (15 \text{ s}) \times (30 \text{ N}) = +225 \text{ kg m/s}$. The total area (total change in momentum) is $+200 \text{ kg m/s}$.

Note that another approach is to multiply the average net force acting on Suki and the bicycle (+10 N) over this interval, by the time interval (20 s), for a $+200 \text{ kg m/s}$ change in momentum.

Figure 6.5: The shaded regions correspond to the area under the curve for the time interval from $t = 10$ s to $t = 30$ s.

In step 3, we determined that Suki's momentum at $t = 10$ s is $+300 \text{ kg m/s}$. With the additional 200 kg m/s , the net momentum at $t = 30$ s is $+500 \text{ kg m/s}$. Dividing by the 50 kg mass gives a velocity at $t = 30$ s of $+10 \text{ m/s}$.



Key idea for the graphical interpretation of impulse: The area under the net force versus time graph for a particular time interval is equal to the change in momentum during that time interval.
Related End-of-Chapter Exercises: 24, 27 – 30.

Essential Question 6.2: Return to the 30-second interval covered in Exploration 6.2. At what time during this period does Suki reach her minimum speed?

Answer to Essential Question 6.2: At $t = 15$ s. The graph in Figure 6.4 is helpful in determining when Suki reaches her minimum speed. As long as the net force is negative, Suki slows down (unless her velocity becomes negative, which never happens in this case). Suki continues to slow down until $t = 15$ s. After that time, the net force is positive, so Suki speeds up after $t = 15$ s.

6-3 Implication of Newton's Third Law: Momentum is Conserved

Figure 6.6: Two carts colliding.



EXPLORATION 6.3A – Two carts collide

Let's do an experiment in which two carts, cart 1 and cart 2, collide with one another on a horizontal track, as shown in Figure 6.6. How does the momentum of each cart change? What happens to the momentum of the two-cart system? The upward normal force applied by the track on each cart is balanced by the downward force of gravity, so the net force experienced by each cart during the collision is that applied by the other cart.

Let's use the subscripts i for the initial situation (before the collision), and f for the final situation (after the collision).

The collision changes the momentum of cart 1 from \vec{p}_{1i} to $\vec{p}_{1f} = \vec{p}_{1i} + \Delta\vec{p}_1$.

Similarly, the collision changes the momentum of cart 2 from \vec{p}_{2i} to $\vec{p}_{2f} = \vec{p}_{2i} + \Delta\vec{p}_2$.

The total momentum of the system beforehand is $\vec{p}_{1i} + \vec{p}_{2i}$.

The total momentum of the system afterwards is $\vec{p}_{1f} + \vec{p}_{2f} = \vec{p}_{1i} + \Delta\vec{p}_1 + \vec{p}_{2i} + \Delta\vec{p}_2$.

Consider $\Delta\vec{p}_1$, the change in momentum experienced by cart 1 in the collision. This change in momentum comes from the force applied to cart 1 by cart 2 during the collision. Similarly, $\Delta\vec{p}_2$, cart 2's change in momentum, comes from the force applied to cart 2 by cart 1 during the collision. Newton's third law tells us that, no matter what, the force applied to cart 1 by cart 2 is equal and opposite to that applied to cart 2 by cart 1. Keeping in mind that the change in momentum is directly proportional to the net force, and that we're talking about vectors, this means:

$$\Delta\vec{p}_2 = -\Delta\vec{p}_1.$$

Substituting this result into our expression for the total momentum of the system after the collision shows that momentum is conserved (momentum remains constant):

$$\vec{p}_{1f} + \vec{p}_{2f} = \vec{p}_{1i} + \vec{p}_{2i}.$$

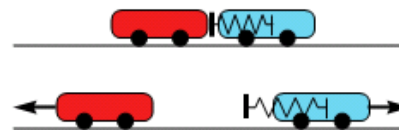
Key idea: The total momentum of the system after the collision equals the total momentum of the system before the collision. This **Law of Conservation of Momentum** applies to any system where there is no net external force. **Related End-of-Chapter Exercise: 4.**

We'll spend more time on the law of conservation of momentum in Chapter 7 but, for now, consider the following exploration.

EXPLORATION 6.3B – An explosive situation

Two carts are placed back-to-back on a horizontal track. One cart contains a spring-loaded piston. When the spring is released, the piston pushes against the other cart and the two carts move in opposite directions along the track, as shown in Figure 6.7. Assume the carts are initially at rest in the center of the track and that friction is negligible.

Figure 6.7: A diagram showing the initial situation (top), and the situation after the carts have moved apart (bottom).



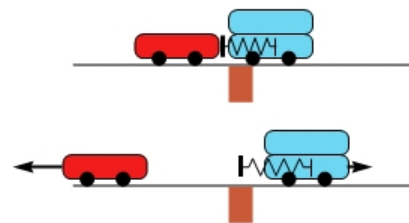
Step 1 - If the two carts have equal masses, is momentum conserved in this process? A good answer to this question is “it depends.” The momentum of each cart individually is not conserved, because each cart starts with no momentum and ends up with a non-zero momentum. This is because each cart experiences a net force (applied by the other cart), so its momentum changes according to the impulse equation (Equation 6.3).

On the other hand, the law of conservation of momentum tells us that *the momentum of the two-cart system is conserved because no net external force acts on this system*. The upward normal force, exerted by the track on this system, balances the downward force of gravity. Cart 1 acquires some momentum because of the force applied by cart 2, but cart 2 acquires an equal-and-opposite momentum because of the equal-and-opposite force applied to it by cart 1. The net momentum of the two-cart system is zero, even when the carts are in motion. Momentum is a vector, so the momentum of one cart is cancelled by the momentum of the other cart.

Step 2 - If we double the mass of one of the carts and repeat the experiment, is momentum conserved? Yes, the momentum of this system is conserved because no net external force acts on the system. Changing the mass of one cart will change the magnitude of the momentum acquired by each cart, but the momentum of the two-cart system is always zero, both before and after the spring is released. To conserve momentum, the force applied on cart 1 by cart 2 must be equal and opposite to the force applied on cart 2 by cart 1. Newton’s third law tells us that these forces are equal and opposite, no matter how the masses compare.

Step 3 - If we make the experiment more interesting, by balancing the track on a brick before releasing the spring, will the track tip over after the spring is released? If we tried this experiment when the masses are equal, what would happen? The track would remain balanced, even when the carts are in motion, because of the symmetry. The tendency of cart 1 to tip the track one way is balanced by the tendency of cart 2 to tip it the opposite way. We don’t have the same symmetry in step 2, but the track still remains balanced. The cart with half the mass of the other cart is always twice as far from the balance point. That maintains the balance, as shown in Figure 6.8.

Figure 6.8: As the carts move apart, the track remains balanced on the brick even if the carts have different masses.



Key idea for momentum conservation: Even if the momenta of individual parts of a system are not conserved, the momentum of the entire system is conserved (constant), as long as no net external force acts on the system. Conservation of momentum is a consequence of Newton’s third law. **Related End-of-Chapter Exercises: 44, 45.**

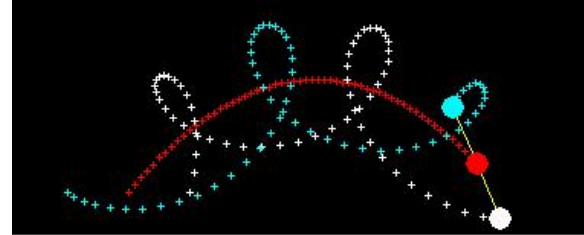
Essential Question 6.3: In Exploration 6.3B, the momentum of the system is always zero. Is there anything about the two-cart system that remains at rest and that shows clearly why the track doesn’t tip over when balanced on the brick?

Answer to Essential Question 6.3: No matter how the masses of the two carts in Exploration 6.3B compare, the center of mass of the system remains at rest at the balance point of the track.

6-4 Center of Mass

In the previous chapters, we treated everything as a particle, or, equivalently, as a ball. A ball, or particle, thrown through the air follows a parabolic path. What if you take a non-spherical object (a pen, for instance) and throw it so it spins? Most points on the object follow complicated paths, but the center of mass still follows a parabolic curve, as shown in Figure 6.9.

Figure 6.9: The motion of three balls on a stick. Only the red ball, located at the center of mass of the system, follows the parabolic path characteristic of free fall.



For a uniform object, the center of mass is located at the geometric center of the object. In general, the center of mass of an object, or a collection of objects, is given by Equation 6.4.

The **center of mass** is the point on an object that moves as though all the mass of the object is concentrated there. The x -coordinate of the center of mass is given by:

$$X_{CM} = \frac{x_1 m_1 + x_2 m_2 + x_3 m_3 + \dots}{m_1 + m_2 + m_3 + \dots} \quad (\text{Equation 6.4: Position of the center of mass})$$

where the m 's represent the masses of different objects in the system (or of various pieces of a single object) and the x 's represent the x -coordinates of those objects or pieces. Similar equations give us the y and z -coordinates of the center of mass.

EXAMPLE 6.4A – Three balls on a stick

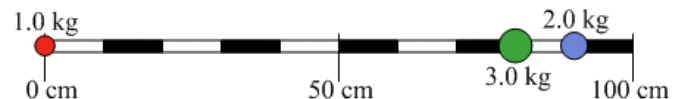
Three balls are placed on a meter stick. Ball 1, at the 0-cm mark, has a mass of 1.0 kg. Ball 2, at the 80-cm mark, has a mass of 3.0 kg. Ball 3, at the 90-cm mark, has a mass of 2.0 kg.

- If the meter stick has negligible mass, where is the system's center of mass?
- If the meter stick has a mass of 2.0 kg, where is the system's center of mass?

SOLUTION

(a) As usual, let's begin with a diagram of the situation (see Figure 6.10).

Figure 6.10: A diagram showing the position of the three balls on the meter stick.



To find the center of mass, we can substitute the given values into Equation 6.4:

$$X_{CM} = \frac{x_1 m_1 + x_2 m_2 + x_3 m_3}{m_1 + m_2 + m_3} = \frac{0 \times 1.0 \text{ kg} + (80 \text{ cm})(3.0 \text{ kg}) + (90 \text{ cm})(2.0 \text{ kg})}{1.0 \text{ kg} + 2.0 \text{ kg} + 3.0 \text{ kg}} = \frac{420 \text{ kg cm}}{6.0 \text{ kg}} = 70 \text{ cm}.$$

(b) If the stick's mass is uniformly distributed, we can treat the stick as a fourth ball, with a mass of 2.0 kg, located at the 50-cm mark. Making use of the result from part (a), which says that the first three balls are equivalent to a single 6.0-kg ball located at the 70-cm mark, we get:

$$X'_{CM} = \frac{(70 \text{ cm})(6.0 \text{ kg}) + (50 \text{ cm})(2.0 \text{ kg})}{6.0 \text{ kg} + 2.0 \text{ kg}} = \frac{520 \text{ kg cm}}{8.0 \text{ kg}} = 65 \text{ cm}.$$

Related End-of-Chapter Exercises: 31, 32.

The center of mass is particularly useful in systems experiencing no net external force. In such systems, the motion of the system's center of mass is unchanged, even if the motion of different parts of the system changes. This is a consequence of Newton's Second Law. Without a net external force acting, the acceleration of the center of mass of the system is zero. Different parts of the system can accelerate, but the forces associated with these accelerations cancel because the net force on the system is zero. Let's now consider an example of such a system.

EXAMPLE 6.4B – Canoe move the center of mass?

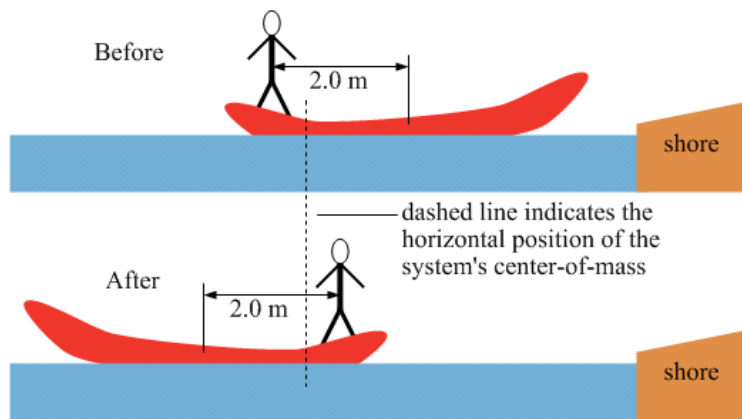
A man, with a mass of 90 kg, stands 2.0 m from the center of a 30 kg canoe that is floating on the calm water of a lake. Both the man and the canoe are initially at rest.

(a) If the man then moves to the point 2.0 m on the opposite side of the center of the canoe from where he starts, how far does the canoe move?

(b) How far does the man actually move relative to a fixed point on the shore?

SOLUTION

Figure 6.11: The position of the man and the canoe before and after the man moves from one end of the canoe to the other.



(a) We could solve this problem formally, but let's solve it conceptually by looking at Before and After pictures in Figure 6.11. First, let's determine the position of the center of mass of the system in the Before picture, before the man changes position.

Define the man's initial position as the origin (you can pick a different origin if you want), and assume the canoe's center of mass to be the middle of the canoe. We get:

$$X_{CM} = \frac{x_1 m_1 + x_2 m_2}{m_1 + m_2} = \frac{0 \times 90 \text{ kg} + (2.0 \text{ m})(30 \text{ kg})}{90 \text{ kg} + 30 \text{ kg}} = \frac{60 \text{ kg} \cdot \text{m}}{120 \text{ kg}} = 0.50 \text{ m}$$

In the Before picture, the man is 50 cm to the left, and the canoe's center of mass is 1.5 m to the right, of the system's center of mass. The canoe's center of mass is three times farther from the system's center of mass than the man is because the canoe's mass is 1/3 of the man's mass. Because no net external force acts on the canoe-man system, when the man moves to the right, the canoe moves left in such a way that the system's center of mass remains at rest. The man moves to a position that is a mirror image of his initial position, so the After picture is a mirror image of the Before picture (placing the mirror at the system's center of mass). The canoe's center of mass moves from 1.5 m to the right of the system's center of mass to 1.5 m to the left of the system's center of mass, for a net displacement of 3.0 m to the left.

(b) Applying a similar analysis to the man, the man moves from 0.5 m to the left of the system's center of mass to 0.5 m to the right, a net displacement of 1.0 m right. Equivalently, the man moves 4.0 m to the right relative to the canoe while the canoe moves 3.0 m to the left with respect to the shore, so the man ends up moving just 1.0 m right relative to the shore.

Related End-of-Chapter Exercise: 33.

Essential Question 6.4: In Example 6.4B, what force makes the canoe move when the man starts to move? What force stops the canoe when the man stops?

Answer to Essential Question 6.4: In each case, the force of friction (static friction if there is no slipping) between the man's shoes and the canoe causes the changes in the canoe's motion.

6-5 Playing with a Constant Acceleration Equation

Once again, let's start with a familiar relationship and look at in a new way to come up with a powerful idea. Return to one of our constant acceleration equations: $v_x^2 = v_{ix}^2 + 2\bar{a}_x\Delta\bar{x}$. If we re-arrange this equation to solve for the acceleration, we get: $\bar{a}_x = \frac{v_x^2 - v_{ix}^2}{2\Delta\bar{x}}$.

Substituting this into Newton's second law, $\bar{F}_{net,x} = m\bar{a}_x$, gives, after some re-arranging:

$$\frac{1}{2}mv_x^2 - \frac{1}{2}mv_{ix}^2 = \bar{F}_{net,x}\Delta\bar{x}.$$

We can do the same thing in the y -direction. Adding the x and y equations gives:

$$\left(\frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2\right) - \left(\frac{1}{2}mv_{ix}^2 + \frac{1}{2}mv_{iy}^2\right) = \bar{F}_{net,x}\Delta\bar{x} + \bar{F}_{net,y}\Delta\bar{y}.$$

Recognizing that $v_x^2 + v_y^2 = v^2$, the left side of the equation can be simplified:

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_i^2 = \bar{F}_{net,x}\Delta\bar{x} + \bar{F}_{net,y}\Delta\bar{y}.$$

The right side can also be simplified, because its form matches a dot product:

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_i^2 = \bar{F}_{net} \bullet \Delta\bar{r} = F_{net}\Delta r \cos\theta, \quad (\text{Equation 6.5})$$

where θ is the angle between the net force \bar{F}_{net} and the displacement $\Delta\bar{r}$.

So, we've now come up with two more useful concepts, which we name and define here.

Kinetic energy is energy associated with motion:

$$K = \frac{1}{2}mv^2 \quad ; \quad (\text{Equation 6.6: Kinetic energy})$$

Work relates force and displacement $W = \bar{F}_{net} \bullet \Delta\bar{r} = F_{net}\Delta r \cos\theta$. (Eq. 6.7: Work)

Both work and kinetic energy have units of joules (J), and they are both scalars.

Equation 6.5, when written in the form below, is known as the **work-kinetic energy theorem**. In this case the work is the work done by the net force.

$$\Delta K = W_{net} = F_{net}\Delta r \cos\theta, \quad (\text{Eq. 6.8: The work-kinetic energy theorem})$$

where θ is the angle between the net force \bar{F}_{net} and the displacement $\Delta\bar{r}$.

In general, when a force is perpendicular to the displacement, the force does no work. If the force has a component parallel to the displacement, the force does positive work. If the force has a component in the direction opposite to the displacement, the force does negative work.

Compare Exploration 6.5 to Exploration 6.2, in which Suki was riding her bike.

EXPLORATION 6.5 – A hard-working cyclist

Peter is riding his bicycle in a straight line on a flat road. Peter and his bike have a total mass of 60 kg and, at $t = 0$, he is traveling at 8.0 m/s. For the first 70 meters, he coasts. When Peter realizes he is slowing down, he pedals so that the static friction force exerted on the bike by the road increases linearly with distance from 0 to 40 N, in the direction Peter is traveling, over the next 140 meters. A constant 10 N resistive force acts on Peter and the bicycle the entire time.

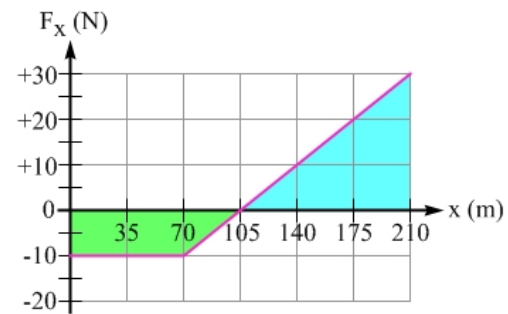
Step 1 - Sketch a graph of the net force acting on Peter and his bicycle as a function of position. Take the positive direction to be the direction Peter is traveling. In the vertical direction, the normal force balances the force of gravity, so we can focus on the horizontal forces. For the first 70 m, we have only the 10 N resistive force, which opposes the motion and is thus in the negative direction. For the next 140 m, we have to account for the friction force, which acts in the direction of motion, and the resistive force. We can account for their combined effect by drawing a straight line, as in Figure 6.12, that goes from -10 N at $x = 70$ m to $+30$ N (40 N $-$ 10 N) at $x = 210$ m.

Figure 6.12: A graph of the net force acting on Peter and his bicycle, as a function of position.



Step 2 - What is Peter's speed at $x = 210$ m? Let's use the area under the F_{net} versus position graph, between $x = 0$ and $x = 210$ m, to find the net work over that distance. This area is shown in Figure 6.13, split into a negative area for the region $x = 0$ to $x = 105$ m, and a positive area between $x = 105$ m and $x = 210$ m. Each box on the graph has an area of 10 N \times 35 m $=$ 350 N \cdot m. The negative area covers two-and-a-half boxes on the graph, while the positive area covers four-and-a-half boxes, for a net positive area of 2 boxes, or 700 N \cdot m.

Figure 6.13: The area within the shaded regions represents the area under the curve for the region from $x = 0$ to $x = 210$ m.



The net area under the curve in Figure 6.13 is the net work done on Peter and the bicycle, which is the change in kinetic energy ($W_{net} = \Delta K = K_f - K_i$). Thus, the final kinetic energy is:

$$K_f = K_i + W_{net} = \frac{1}{2}mv_i^2 + W_{net} = \frac{1}{2}(60 \text{ kg})(8.0 \text{ m/s})^2 + 700 \text{ N} \cdot \text{m} = 1920 \text{ J} + 700 \text{ J} = 2620 \text{ J}.$$

$$\text{Solving for the final speed from } K_f = (1/2)mv_f^2 \text{ gives: } v_f = \sqrt{\frac{2K_f}{m}} = \sqrt{\frac{2 \times 2620 \text{ J}}{60 \text{ kg}}} = 9.3 \text{ m/s}.$$

Key idea: The area under the net force-versus-position graph for a particular region is the work, and the change in kinetic energy, over that region. **Related End-of-Chapter Exercises: 48, 49.**

Essential Question 6.5: Initially, objects A and B are at rest. B 's mass is four times larger than A 's mass. Identical net forces are applied to the objects, as shown in Figure 6.14. Each force is removed once the object it is applied to has accelerated through a distance d . After the forces are removed, which object has more (a) kinetic energy? (b) momentum?

Figure 6.14: An overhead view of two objects, A and B , experiencing the same net force F as they move from rest through a distance d .

