4-4 Projectile Motion

Projectile motion is, in general, two-dimensional motion that results from an object with an initial velocity in one direction experiencing a constant force in a different direction. A good example is a ball you throw to a friend. You give the ball an initial velocity when you throw it, and then the force of gravity acts on the ball as it travels to your friend. In this section, we will learn how to analyze this kind of situation.

EXPLORATION 4.4 – A race

You release one ball (ball *A*) from rest at the same time you throw another ball (ball *B*), which you release with an initial velocity that is directed entirely horizontally. You release both balls simultaneously from the same height *h* above level ground. Neglect air resistance.

Step 1 - ***Which ball travels a greater distance before hitting the ground?*** Ball *A* takes the shortest path to the ground, so ball *B* travels farther.

Step 2 - ***Which ball reaches the ground first? Why?*** We can construct a motion diagram (see Figure 4.9) by, for instance, analyzing a video of the balls as they fall. Many people think that because ball *B* travels farther it takes longer to reach the ground; however, ball *B* also has a higher speed. The reality is that both balls reach the ground at the same time. The reason is that the motion of ball *B* can be viewed as a combination of its horizontal motion and its vertical motion. The horizontal motion has no effect whatsoever on the vertical motion, so what happens vertically for ball *B* is exactly the same as what happens vertically for ball *A*.

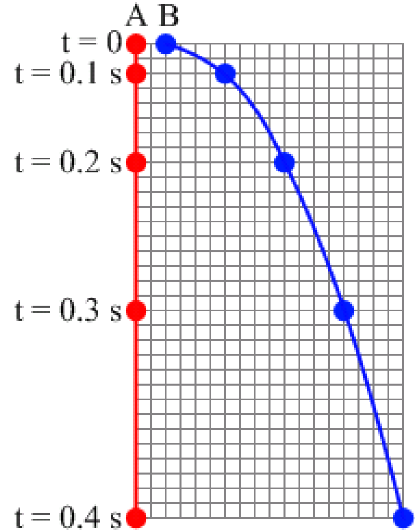


Figure 4.9: A motion diagram can be constructed from experimental evidence, such as by analyzing a video of the balls as they fall.

Key idea for projectile motion: The key idea of this chapter is the independence of *x* and *y*. The basic idea is that the motion that happens in one direction (*x*) is independent of the motion that happens in a perpendicular direction (*y*), and vice versa, as long as the force is constant.
Related End-of-Chapter Exercises: 9, 10.

The *x*-direction and *y*-direction motions are independent in the sense that each of the one-dimensional motions occurs as if the other motion is not happening. These motions are connected, though. The object's motion generally stops after a particular time, so the time is the same for the *x*-direction motion and the *y*-direction motion.

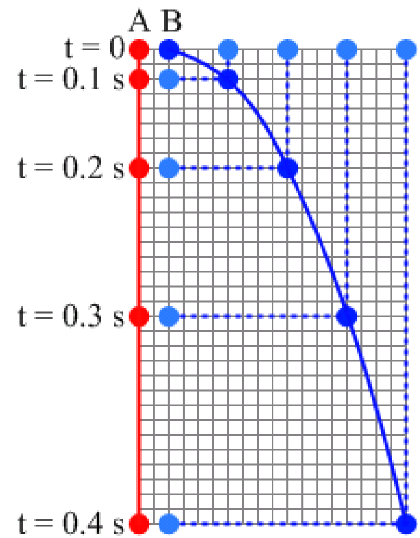
This powerful concept allows us to treat a two-dimensional projectile motion problem as two separate one-dimensional problems. We already have a good deal of experience with one-dimensional motion, so we can build on what we learned in Chapter 2. For the most part, we will deal with situations where the acceleration is constant, so all our experience with constant-acceleration situations in one dimension will be directly relevant here.

Solving a Two-Dimensional Constant-Acceleration Problem
Our general method for analyzing a typical projectile-motion problem builds on the method we used for analyzing one-dimensional constant-acceleration motion in Chapter 2. The basic idea is to split the two-dimensional problem into two one-dimensional subproblems, which we can call the x subproblem and the y subproblem.

1. Draw a diagram of the situation.
2. Draw a free-body diagram of the object showing all the forces acting on the object while it is in motion. A free-body diagram helps in determining the acceleration of the object.
3. Choose an origin.
4. Choose an x-y coordinate system, showing which way is positive for each coordinate axis.
5. Organize your data, keeping the information for the x subproblem separate from the information for the y subproblem.
6. Only then should you turn to the constant-acceleration equations. Make sure the acceleration is constant so the equations apply! We use the same three equations that we used in Chapter 2, but we customize them for the x and y subproblems, as follows:

Equation from Chapter 2	x-direction equations	y-direction equations
$v = v_i + at$ (2.7)	$v_x = v_{ix} + a_x t$ (4.2x)	$v_y = v_{iy} + a_y t$ (4.2y)
$x = x_i + v_i t + \frac{1}{2}at^2$ (2.9)	$x = x_i + v_{ix} t + \frac{1}{2}a_x t^2$ (4.3x)	$y = y_i + v_{iy} t + \frac{1}{2}a_y t^2$ (4.3y)
$v^2 = v_i^2 + 2a\Delta x$ (2.10)	$v_x^2 = v_{ix}^2 + 2a_x \Delta x$ (4.4x)	$v_y^2 = v_{iy}^2 + 2a_y \Delta y$ (4.4y)

Table 4.4: Constant acceleration equations for two-dimensional projectile motion. The equation numbers are shown in parentheses after each equation.



Let's apply the method above to analyze the race from Exploration 4.4. We begin by sketching a motion diagram and a free-body diagram for each ball, and continue the analysis in the next section. On the motion diagram for ball B, show the separate x (horizontal) and y (vertical) motions.

The motion diagrams are shown in Figure 4.10, while the free-body diagram of each ball is shown in Figure 4.11. Let's consider the motion from just after the balls are released until just before the balls make contact with the ground. Because the only force acting on either ball is the force of gravity, the same free-body diagram applies to both objects.

Figure 4.10: Motion diagram for balls A and B. For ball B, the vertical and horizontal motions are shown separately. These two independent motions combine to give the parabolic path followed by ball B.

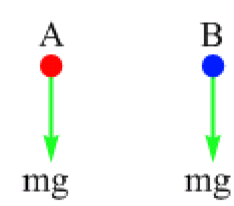


Figure 4.11: Free-body diagrams for balls A and B. From the instant just after you release the balls until the instant just before the balls hit the ground, the only force acting on either ball is the force of gravity, so the balls have identical free-body diagrams. The balls travel along different paths only because their initial velocities are different.

Essential Question 4.4 The free-body diagrams in Figure 4.11 imply that the balls have the same mass. What would happen if the balls had different masses?

Answer to Essential Question 4.4 Assuming that we can neglect air resistance, the relative mass of the balls is completely irrelevant. If *B*'s mass was double *A*'s mass, for instance, the force of gravity on *B* would be twice that on *A*, but both balls would still have an acceleration of $\vec{g}$, and the two balls would still hit the ground simultaneously.

4-5 The Independence of x and y

A key to understanding projectile motion is the independence of *x* and *y*, the fact that the horizontal (*x*-direction) motion is completely independent of the vertical (*y*-direction) motion. Let's exploit this concept to continue our analysis of the race from Exploration 4.4.

EXPLORATION 4.5 – Analyzing the race

Step 1 – ***Find the acceleration of each ball.*** The free-body diagram in Figure 4.11, combined with Newton's second law, tells us that the acceleration of each object is simply the acceleration due to gravity, $\vec{g}$. This comes from:

$$\vec{a} = \frac{\Sigma \vec{F}}{m} = \frac{m\vec{g}}{m} = \vec{g} \, .$$

Step 2 – ***Use the general method to find the time it takes the balls to reach the ground.*** Because the motion is directed right and down, let's choose positive directions as +*x* to the right and +*y* down, and set the origin for each ball to be the point from which it is released. Choosing up as positive, with an origin at ground level, would also work well.

Let's say both balls fall through a vertical distance of *h*, and that the initial velocity of ball B is directed horizontally with a velocity of v_i. Table 4.5 shows how we organize the data. Note how the data for the *x*-direction (horizontal) motion for ball *B* are kept separate from the data for the *y*-direction (vertical) motion.

Component	**Ball *B*, *x* direction**	**Ball *B*, *y* direction**	**Ball *A*, *y* direction**
Initial position	$x_{iB} = 0$	$y_{iB} = 0$	$y_{iA} = 0$
Final position	$x_B = ?$	$y_B = +h$	$y_A = +h$
Initial velocity	$v_{ixB} = +v_i$	$v_{iyB} = 0$	$v_{iyA} = 0$
Final velocity	$v_{xB} = +v_i$	$v_{yB} = ?$	$v_{yA} = ?$
Acceleration	$a_{xB} = 0$	$a_{yB} = +g$	$a_{yA} = +g$

Table 4.5: Organizing the data for ball *A* (dropped from rest) and ball *B* (with an initial velocity that is horizontal). Note that everything is the same for the two balls in the *y*-direction, which is vertical.

One of the most common errors in analyzing a projectile-motion situation is to mix up information from the *x* and *y* directions, such as by using the acceleration due to gravity as the acceleration in the horizontal direction. Organizing the data into a table like the one above makes such errors far less likely. Including the appropriate sign on all vectors is another way to reduce errors, because it reminds us to think about which sign is correct and whether we really want a + or a –. A statement like $y_B = +h$ tells us that the final vertical position of ball *B* is a distance *h* from the origin in the positive *y*-direction.

Can we use the data from Table 4.5 to justify the conclusion from Exploration 4.4 that the two balls reach the ground at the same time? Absolutely. The appropriate motion diagrams are shown in Figure 4.12. Analyzing the *y*-direction subproblem for ball *B*, we can use equation 4.2y to find an expression for the time to reach the ground.

$$\vec{y} = \vec{y}_i + \vec{v}_{iy} t + \frac{1}{2} \vec{a}_y t^2 ,$$

$$+h = 0 + 0 + \frac{1}{2} g t^2 .$$

This gives $t^2 = +\dfrac{2h}{g}$.

Therefore, the time for ball B to reach the ground is $t = \sqrt{\dfrac{2h}{g}}$.

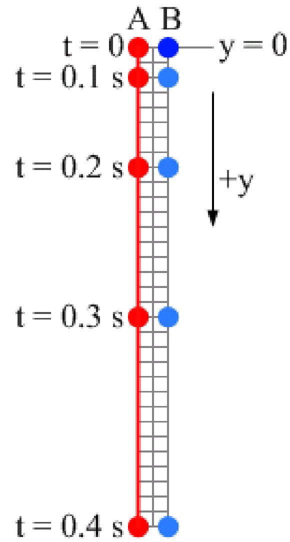


Figure 4.12: A motion diagram for the vertical components of the motion for the balls.

Does the answer make any sense? First, it does have the right units. Second, it says that if we increase *h* the ball takes longer to reach the ground, which makes sense. Third, it says that the larger the acceleration due to gravity the smaller the time the ball takes to reach the ground, which also sounds right. Note that if we solve for the time ball *A* takes to reach the ground we get exactly the same result, because *A* has the same initial position, final position, initial vertical velocity, and vertical acceleration as *B*.

Step 3 – ***Find an expression for the horizontal distance traveled by ball B before it reaches the ground.*** Even though we're dealing with the *x* subproblem, we can use the time from the *y* subproblem – that is often key to solving projectile motion problems. The motion diagram for the *x*-direction motion is shown in Figure 4.13. One way to find the horizontal distance that ball *B* travels is to use Equation 4.3x (see Table 4.4 in Section 4.4 for the equations).

$$x = x_i + v_{ix} t + \frac{1}{2} a_x t^2 ,$$

$$x = 0 + v_i t + 0 = +v_i \sqrt{\frac{2h}{g}} .$$

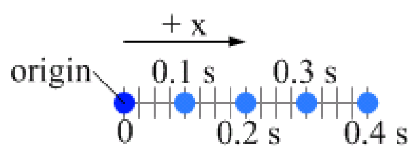


Figure 4.13: A motion diagram for the horizontal component of ball *B*'s motion.

Again, be careful not to mix the *x* information with the *y* information. Here, for instance, we can use Table 4.5 to remind us that the acceleration in the *x* direction is zero. The motion diagram in Figure 4.13, showing constant-velocity motion, confirms that the acceleration in the *x* direction is zero.

Key idea for projectile motion: One way to solve a projectile-motion problem is to break the two-dimensional problem into two independent one-dimensional problems, ***linked by the time***, and apply the one-dimensional constant-acceleration methods from Chapter 2.
Related End-of-Chapter Exercises: 44, 45.

Essential Question 4.5 When a sailboat is at rest, a beanbag you release from the top of the mast lands in a bucket that is on the deck at the base of the mast. Will the beanbag still land in the bucket if you release the beanbag from rest when the sailboat is moving with a constant velocity?