2-5 Acceleration

Let's turn now to motion that is not at constant velocity. An example is the motion of an object you release from rest from some distance above the floor.

EXPLORATION 2.5A – Exploring the motion diagram of a dropped object

Step 1 - *Sketch a motion diagram for a ball that you release from rest from some distance above the floor, showing its position at regular time intervals as it falls.* The motion diagram in Figure 2.14 shows images of the ball that are close together near the top, where the ball moves more slowly. As the ball speeds up, these images get farther apart because the ball covers progressively larger distances in the equal time intervals. How do we know how much space to include between each image? One thing we can do is to consult experimental evidence, such as strobe photos of dropped objects. These photos show that the displacement from one time interval to the next increases linearly, as in Figure 2.14.

Step 2 - *At each point on the motion diagram, add an arrow representing the ball's velocity at that point. Neglect air resistance.* The arrows in Figure 2.14 represent the velocity of the ball at the various times indicated on the motion diagram. Because the displacement increases linearly from one time interval to the next, the velocity also increases linearly with time.

Figure 2.14: A motion diagram, and velocity vectors, for a ball released from rest at *t* = 0. The ball's position and velocity are shown at 0.1-second intervals.
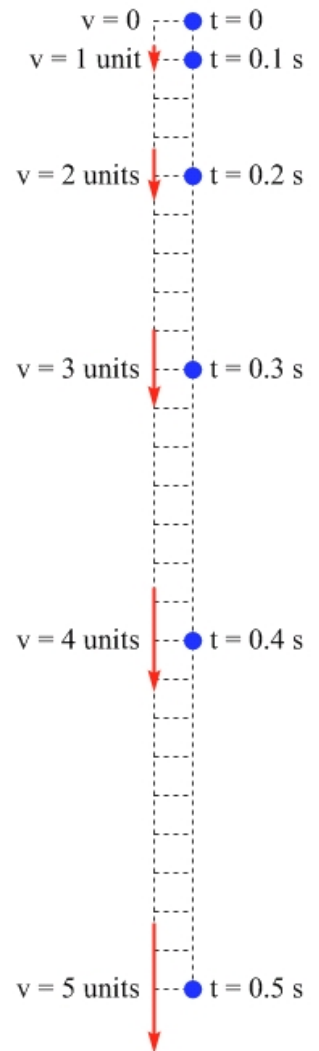
Key idea: For an object dropped from rest, the velocity changes linearly with time.
Related End-of-Chapter Exercises: 57 - 60

Another way to say that the ball's velocity increases linearly with time is to say that the rate of change of the ball's velocity, with respect to time, is constant:

$$\frac{\Delta \vec{v}}{\Delta t} = \text{constant} .$$

This quantity, the rate of change of velocity with respect to time, is referred to as the **acceleration**. Acceleration is related to velocity in the same way velocity is related to position.

Average acceleration: a vector representing the average rate of change of velocity with respect to time. The SI unit for acceleration is m/s^2.

$$\bar{\vec{a}} = \frac{\Delta \vec{v}}{\Delta t} = \frac{\text{change in velocity}}{\text{time interval}} .$$ (Equation 2.7: **Average acceleration**)

The direction of the average acceleration is the direction of the change in velocity.

> **Instantaneous acceleration:** a vector representing the rate of change of velocity with respect to time at a particular instant in time. The SI unit for acceleration is m/s^2.
>
> A practical definition of instantaneous acceleration at a particular instant is that it is the slope of the velocity-versus-time graph at that instant. Expressing this as an equation:
>
> $$\vec{a} = \frac{\Delta \vec{v}}{\Delta t},$$ (Equation 2.8: **Instantaneous acceleration**)
>
> where Δt is small enough that the acceleration can be considered to be constant over that interval.

Note that, on Earth, objects dropped from rest are observed to have accelerations of 9.8 m/s^2 straight down. This is known as $\vec{g}$, the acceleration due to gravity.

EXPLORATION 2.5B – Graphs in a constant-acceleration situation
        The graph in Figure 2.15 shows the velocity, as a function of time, of a bus moving in the positive direction along a straight road.

Figure 2.15: A graph of the velocity of a bus as a function of time.

Step 1 – *What is the acceleration of the bus? Sketch a graph of the acceleration as a function of time.* The graph in Figure 2.15 is a straight line with a constant slope. This tells us that the acceleration is constant because the acceleration is the slope of the velocity-versus-time graph. Using the entire 10-second interval, applying equation 2.8 gives:

$$\vec{a} = \frac{\Delta \vec{v}}{\Delta t} = \frac{+25 \text{ m/s} - (+5\text{ m}/\text{s})}{10 \text{ s}} = \frac{+20 \text{ m/s}}{10 \text{ s}} = +2.0 \text{ m/s}^2.$$

        The acceleration graph in Figure 2.16 is a horizontal line, because the acceleration is constant. Compare Figures 2.15 and 2.16 to the graphs for the red car in Figure 2.13. Note the similarity between the acceleration and velocity graphs in a constant-acceleration situation and the velocity and position graphs in a constant-velocity situation.

Figure 2.16: A graph of the acceleration of the bus as a function of time.

Step 2 - *What on the acceleration-versus-time graph is connected to the velocity?* The connection between velocity and acceleration is similar to that between position and velocity - the area under the curve of the acceleration graph is equal to the change in velocity, as shown in Figure 2.17. This follows from Equation 2.8, which in a constant acceleration situation can be written as $\Delta \vec{v} = \vec{a} \, \Delta t$.

Figure 2.17: In 10 seconds, the velocity changes by +20 m/s. This is the area under the acceleration-versus-time graph over that interval.
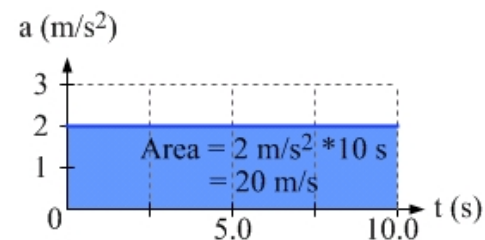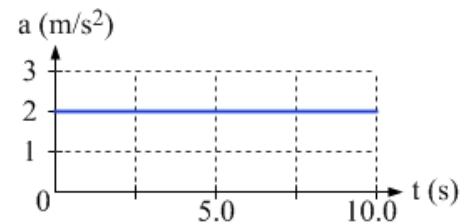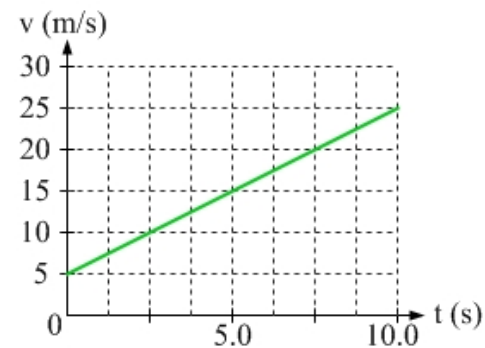
> **Key ideas**: The acceleration is the slope of the velocity-versus-time graph, while the area under the acceleration-versus-time graph for a particular time interval represents the change in velocity during that time interval.   **Related End-of-Chapter Exercises: 16 and 32.**

Essential Question 2.5: Consider the bus in Exploration 2.5B. Would the graph of the bus' position as a function of time be a straight line? Why or why not?

Answer to Essential Question 2.5: The position-versus-time graph is not a straight line, because the slope of such a graph is the velocity. The fact that the bus' velocity increases linearly with time means the slope of the position-versus-time graph also increases linearly with time. This actually describes a parabola, which we will investigate further in the next section.

2-6 Equations for Motion with Constant Acceleration

In many situations, we will analyze motion using a model in which we assume the acceleration to be constant. Let's derive some equations that we can apply in such situations. In general, at some initial time $t_i = 0$, the object has an initial position $\vec{x}_i$ and an initial velocity of $\vec{v}_i$, while at some (usually later) time t, the object's position is $\vec{x}$ and its velocity is $\vec{v}$.

Acceleration is related to velocity the same way velocity is related to position, so we can follow a procedure similar to that at the end of Section 2-3 to derive an equation for velocity.

Substitute $\vec{v}_f - \vec{v}_i$ for $\Delta\vec{v}$ in the rearrangement of Equation 2.8, $\Delta\vec{v} = \vec{a}\,\Delta t$.

This gives: $\qquad \vec{v}_f - \vec{v}_i = \vec{a}\,\Delta t = \vec{a}\left(t_f - t_i\right)$.

Generally, we define the initial time t_i to be zero: $\qquad \vec{v}_f - \vec{v}_i = \vec{a}\,t_f$.

Remove the "f" subscripts to make the equation as general as possible: $\quad \vec{v} - \vec{v}_i = \vec{a}\,t$. We also generally remove the vector symbols, although we must be careful to include signs.

$$v = v_i + at . \qquad \text{(Equation 2.9: \textbf{Velocity for constant-acceleration motion})}$$

A second equation comes from the definition of average velocity (Equation 2.2):

Average velocity $= \bar{v} = \dfrac{\Delta\vec{x}}{\Delta t} = \dfrac{\vec{x} - \vec{x}_i}{t - t_i} = \dfrac{\vec{x} - \vec{x}_i}{t}$.

If the acceleration is constant the average velocity is simply the average of the initial and final velocities. This gives, after again dropping the vector symbols:

$$\frac{v_i + v}{2} = \frac{x - x_i}{t} . \qquad \text{(Equation 2.10: \textbf{Connecting average velocity and displacement})}$$

Equation 2.10 is sometimes awkward to work with. If we substitute $v_i + at$ in for v (see Equation 2.9) in Equation 2.10, re-arranging produces an equation describing a parabola:

$$x = x_i + v_i t + \frac{1}{2}at^2 . \qquad \text{(Equation 2.11: \textbf{Position for constant-acceleration motion})}$$

We can derive another useful equation by combining equations 2.9 and 2.10 in a different way. Solving equation 2.9 for time, to get $t = \dfrac{v - v_i}{a}$, and substituting the right-hand side of that expression in for t in equation 2.10, gives, after some re-arrangement:

$$v^2 = v_i^2 + 2a\,\Delta x . \qquad \text{(Eq. 2.12: \textbf{Connecting velocity, acceleration, and displacement})}$$

<table>
<tr><td>

An important note about positive and negative signs.
When we make use of the equations on the previous page, we must be careful to include the appropriate positive or negative signs that are built into each of the variables. The first step is to choose a positive direction. If the initial velocity is in that direction, it goes into the equations with a positive sign. If the initial velocity is in the opposite direction (the negative direction), it goes into the equations with a negative sign. Apply a similar rule for the final velocity, the acceleration, the displacement, the initial position, and the final position. For all of those quantities, the sign is associated with the direction of the corresponding vector.

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Motion with constant acceleration is an important concept. Let's summarize a general, systematic approach we can apply to situations involving motion with constant acceleration.

A General Method for Solving a One-Dimensional Constant-Acceleration Problem
1. **Picture the scene.** Draw a diagram of the situation. Choose an origin to measure positions from, and a positive direction, and show these on the diagram.
2. **Organize what you know, and what you're looking for.** Making a table of data can be helpful. Record values of the variables used in the equations below.
3. **Solve the problem.** Think about which of the constant-acceleration equations to apply, and then set up and solve the problem. The three main equations are:

$v = v_i + at$.   (Equation 2.9: **Velocity for constant-acceleration motion**)

$x = x_i + v_i t + \dfrac{1}{2}at^2$.   (Equation 2.11: P**osition for constant-acceleration motion**)

$v^2 = v_i^2 + 2a\,\Delta x$.   (Equation 2.12: **Connecting velocity, acceleration, and displacement**)

4. **Think about the answer(s)**. Check your answers to see if they make sense.

EXAMPLE 2.6 – Working with variables
In physics, being able to work with variables as well as numbers is an important skill. This can also produce insights that working with numbers does not. Let's say an object is dropped from rest from the top of a building of height *H*, while another object is dropped from rest from the top of a building of height 4*H*. Assuming both objects fall under the influence of gravity alone (that is, they have the same acceleration), compare the times it takes them to reach the ground.

SOLUTION
$v_i = 0$, because the objects are dropped from rest. Take the initial position to be the top of the building in each case, so $x_i = 0$. This reduces Equation 2.11 to $x = at^2/2$. Because the acceleration is the same in each case, the equation tells us the position is proportional to the square of the time. To quadruple the final position, as we are doing, we need to increase the time by a factor of two. The fall from the building that is four times as high takes twice as long. This is illustrated by the motion diagram in Figure 2.18.

Figure 2.18: Falling from rest for double the time quadruples the distance traveled. The height of each of the four shaded regions is *H*, the height of the smaller building in Example 2.6.

Related End-of-Chapter Exercises: 52 and 55

Essential Question 2.6: Return to the situation described in Example 2.6. Compare the velocities of the objects just before they hit the ground.
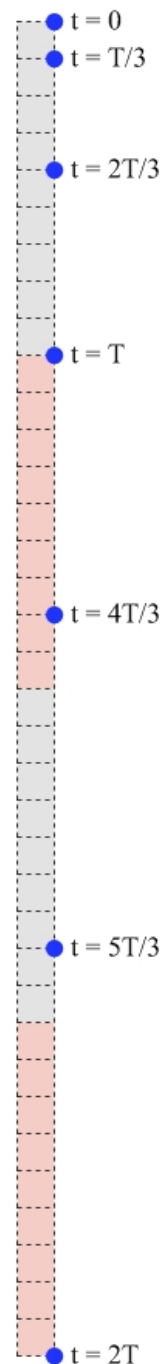
Answer to Essential Question 2.6: Using $v_i = 0$ reduces Equation 2.9 to $v = at$. Doubling the time doubles the final velocity, so the object dropped from the building that is four times higher has a final velocity twice as large as that of the other object. Equation 2.12 gives the same result.

2-7 Example Problem

Let's look at the various representations of motion with constant acceleration, considering the example of a ball tossed straight up in the air.

EXPLORATION 2.7 – A ball tossed straight up

You toss a ball straight up into the air. The ball takes 2.0 s to reach its maximum height, and an additional 2.0 s to return to your hand. You catch the ball at the same height from which you let it go, and the ball has a constant acceleration because it is acted on only by gravity. Consider the motion from the instant just after you release the ball until just before you catch it.

Step 1 – *Picture the scene – draw a diagram of the situation.* The diagram in Figure 2.19 shows the initial conditions, the origin, and the positive direction. We are free to choose either up or down as the positive direction, and to choose any reference point as the origin. In this case let's choose the origin to be the point from which the ball was released, and choose up to be positive.

Step 2 – *Organize the data.* Table 2.2 summarizes what we know, including values for the acceleration and the initial position. We need these values for the constant-acceleration equations. Because the ball moves under the influence of gravity alone, and we can assume the ball is on the Earth, the acceleration is the acceleration due to gravity, 9.8 m/s^2 directed down. Because down is the negative direction, we include a negative sign: $a = -9.8 \, \text{m/s}^2$.

Parameter	Value
Origin	Launch point
Positive direction	up
Initial position	$x_i = 0$
Initial velocity	$v_i = +\underline{} \, \text{m/s}$
Acceleration	$a = -9.8 \, \text{m/s}^2$
Position at $t = 4.0$ s	$x_{t=4\,\text{s}} = x_i = 0$
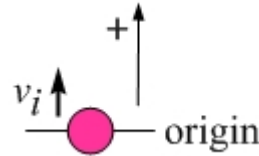
Figure 2.19: A diagram of the initial situation.

Table 2.2: Summarizing the information that was given about the ball.

Step 3 – *Solve the problem.* In this case, we want to draw graphs of the acceleration, velocity, and position of the ball, all as a function of time. To do this we should first write equations for the acceleration, velocity, and position. The acceleration is constant at $a = -9.8 \, \text{m/s}^2$. Knowing the acceleration allows us to solve for the initial velocity. One way to do this is to re-arrange Equation 2.9 to give $v_i = v - at$. Because $v = 0$ at $t = 2.0$ s (the ball is at rest for an instant when it reaches its maximum height) we get $v_i = v - at = 0 - (-9.8 \, \text{m/s}^2)(2.0 \, \text{s}) = +19.6 \, \text{m/s}$.

Knowing the initial velocity enables us to write equations for the ball's velocity (using Equation 2.9) and position (using Equation 2.11) as a function of time. The equations, and corresponding graphs, are part of the multiple representations of the motion shown in Figure 2.20. Note that the position versus time graph is parabolic, but the motion is confined to a line.
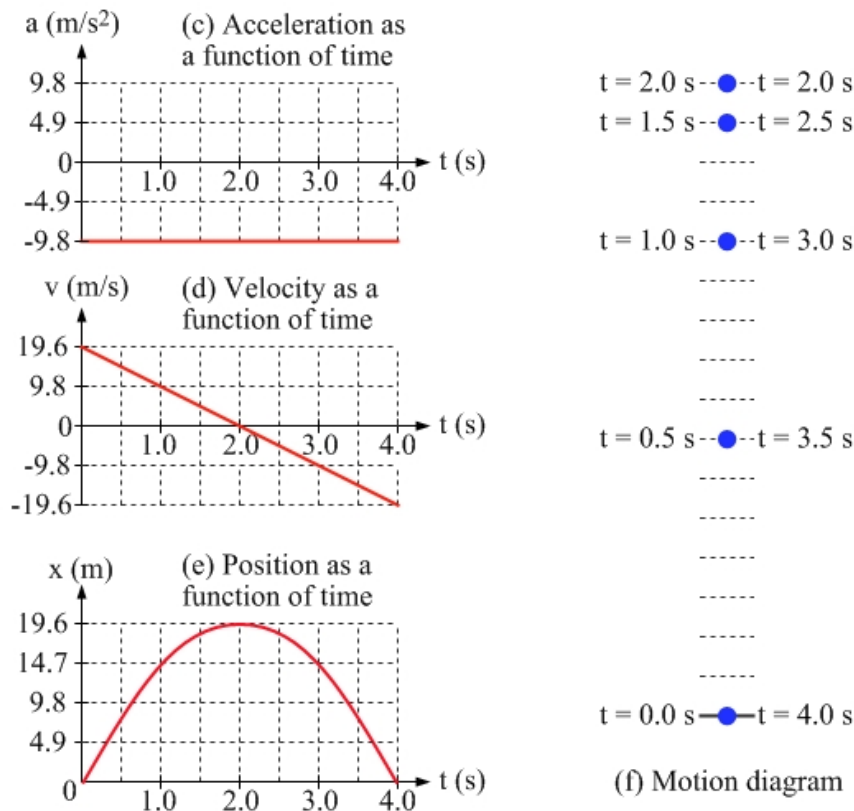
Step 4 - *Sketch a motion diagram for the ball.* The motion diagram is shown on the right in Figure 2.15. Note the symmetry of the up and down motions (this is also apparent from the graphs). The motion of the ball on the way down is a mirror image of its motion on the way up.

(a) Description of the motion: A ball you toss straight up into the air reaches its maximum height 2.0 s after being released, taking an addition 2.0 s to return to your hand. It experiences a constant acceleration from the moment you release it until just before you catch it.

(b) Equations (up is positive): Acceleration-versus-time: $a = -9.8 \text{ m/s}^2$

Velocity-versus-time: $v = +19.6 \text{ m/s} - (9.8 \text{ m/s}^2)t$

Position-versus-time: $x = +(19.6 \text{ m/s})t - (4.9 \text{ m/s}^2)t^2$

(c) Acceleration as a function of time

(d) Velocity as a function of time

(e) Position as a function of time

t = 2.0 s --●-- t = 2.0 s
t = 1.5 s --●-- t = 2.5 s

t = 1.0 s --●-- t = 3.0 s

t = 0.5 s --●-- t = 3.5 s

t = 0.0 s ─●─ t = 4.0 s

(f) Motion diagram

Figure 2.20: Multiple representations of a ball thrown straight up, including (a) a description in words; (b) equations for the ball's acceleration, velocity, and position; graphs giving the ball's (c) acceleration, (d) velocity, and (e) position as a function of time; and (f) a motion diagram. These different perspectives show how various aspects of the motion evolve with time.

Key ideas: At the Earth's surface the acceleration due to gravity has a constant value of $g = 9.8 \text{ m/s}^2$ directed down. We can thus apply constant-acceleration methods to situations involving objects dropped or thrown into the air. For an object that is thrown straight up the downward part of the trip is a mirror image of the upward part of the trip.
Related End-of-Chapter Exercises: 33, 61, and 62.

Essential Question 2.7: Consider again the ball in Exploration 2.7. The ball comes to rest for an instant at its maximum-height point. What is the ball's acceleration at that point?

Answer to Essential Question 2.7: A common misconception is that the ball's acceleration is zero at the maximum-height point. In fact, the acceleration is $\vec{g}$, 9.8 m/s² down, during the entire trip. This is what is shown on the acceleration graph, for one thing – the graph confirms that nothing special happens to the acceleration at $t = 2.0$ s, even though the ball is momentarily at rest. One reason for this is that the ball is under the influence of gravity the entire time.

2-8 Solving Constant-Acceleration Problems

Consider one more example of applying the general method for solving a constant-acceleration problem.

EXAMPLE 2.8 – Combining constant-acceleration motion and constant-velocity motion
A car and a bus are traveling along the same straight road in neighboring lanes. The car has a constant velocity of +25.0 m/s, and at $t = 0$ it is located 21 meters ahead of the bus. At time $t = 0$, the bus has a velocity of +5.0 m/s and an acceleration of +2.0 m/s².

When does the bus pass the car?

SOLUTION
1. **Picture the scene – draw a diagram.** The diagram in Figure 2.16 shows the initial situation, the positive direction, and the origin. Let's choose the positive direction to be the direction of travel, and the origin to be the initial position of the bus.

2. **Organize the data.** Data for the car and the bus is organized separately in Table 2.3, using subscripts C and B to represent the car and bus, respectively.

	Car	**Bus**
Initial position	$x_{iC} = +21\,\text{m}$	$x_{iB} = 0$
Initial velocity	$v_{iC} = +25\,\text{m/s}$	$v_{iB} = +5.0\,\text{m/s}$
Acceleration	$a_C = 0$	$a_B = +2.0\,\text{m/s}^2$

Table 2.3: Summarizing the information that was given about the car and the bus.
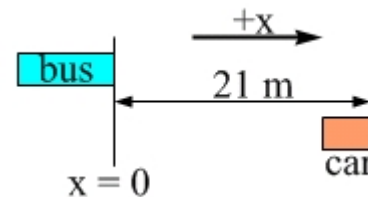
Figure 2.16: A diagram showing the initial positions of the car and the bus, the position of the origin, and the positive direction.

3. **Solve the problem.** Let's use Equation 2.8 to write expressions for the position of each vehicle as a function of time. Because we summarized all the data in Table 2.3 we can easily find the values of the variables in the equations.

For the car: $x_C = x_{iC} + v_{iC}\,t + \dfrac{1}{2}a_C\,t^2 = +21\,\text{m} + (25\,\text{m/s})t$.

For the bus: $x_B = x_{iB} + v_{iB}\,t + \dfrac{1}{2}a_B\,t^2 = 0 + (5.0\,\text{m/s})t + (1.0\,\text{m/s}^2)t^2$.

The bus passes the car when the vehicles have the same position. At what time does $x_C = x_B$? Set the two equations equal to one another and solve for this time (let's call it t_1).

$$+21\,\text{m} + (25\,\text{m/s})t_1 = (5.0\,\text{m/s})t_1 + (1.0\,\text{m/s}^2)t_1^2$$

Bringing everything to the left side gives: $(1.0\,\text{m/s}^2)t_1^2 - (20\,\text{m/s})t_1 - 21\,\text{m} = 0$.

We can solve this with the quadratic equation, where $a = 1.0 \text{ m/s}^2$, $b = -20 \text{ m/s}$, and $c = -21 \text{ m}$.

$$t_1 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{+(20 \text{ m/s}) \pm \sqrt{(400 \text{ m}^2/\text{s}^2) + (84 \text{ m}^2/\text{s}^2)}}{2.0 \text{ m/s}^2} = \frac{+(20 \text{ m/s}) \pm (22 \text{ m/s})}{2.0 \text{ m/s}^2} .$$

The equation gives us two solutions for t_1: Either $t_1 = +21$ s or $t_1 = -1.0$ s. Clearly the answer we want is +21 s. The other number does have a physical significance, however, so let's try to make some sense of it. The negative answer represents the time at which the car would have passed the bus if the motion conditions after $t = 0$ also applied to the period before $t = 0$.

4. **Think about the answer.** A nice way to check the answer is to plug $t_1 = +21$ s into the two position-versus-time equations from the previous step. If the time is correct the position of the car should equal the position of the bus. Both equations give positions of 546 m from the origin, giving us confidence that $t_1 = +21$ s is correct.

Note: This example is continued on the accompanying web site, solving for the time at which the car and the bus have the same velocity.

Related End-of-Chapter Exercises: 54 and 56.

Chapter Summary

Essential Idea: Describing motion in one dimension.
The motion of many objects (such as you, cars, and objects that are dropped) can be approximated very well using a constant-velocity or a constant-acceleration model.

Parameters used to Describe Motion
Displacement is a vector representing a change in position. If the initial position is $\vec{x}_i$ and the final position is $\vec{x}_f$ we can express the displacement as:

$$\Delta \vec{x} = \vec{x}_f - \vec{x}_i .$$ (Equation 2.1: **Displacement in one dimension**)

Average Velocity: a vector representing the average rate of change of position with respect to time. The MKS unit for velocity is m/s (meters per second).

$$\bar{\vec{v}} = \frac{\Delta \vec{x}}{\Delta t} = \frac{\text{net displacement}}{\text{time interval}} .$$ (Equation 2.2: **Average velocity**)

While velocity is a vector, and thus has a direction, speed is a scalar.

Average Speed $= \bar{v} = \dfrac{\text{total distance covered}}{\text{time interval}}$ (Equation 2.3: **Average speed**)

Instantaneous Velocity: a vector representing the rate of change of position with respect to time at a particular instant in time. The MKS unit for velocity is m/s (meters per second).

$$\vec{v} = \frac{\Delta \vec{x}}{\Delta t},$$ (Equation 2.4: **Instantaneous velocity**)

where Δt is small enough that the velocity can be considered to be constant over that interval.

Instantaneous Speed: the magnitude of the instantaneous velocity.

Average Acceleration: a vector representing the average rate of change of velocity with respect to time. The MKS unit for acceleration is m/s^2.

$$\bar{a} = \frac{\Delta \vec{v}}{\Delta t} = \frac{\text{change in velocity}}{\text{time interval}}.$$ (Equation 2.7: **Average acceleration**)

Instantaneous Acceleration: a vector representing the rate of change of velocity with respect to time at a particular instant in time.

$$\vec{a} = \frac{\Delta \vec{v}}{\Delta t},$$ (Equation 2.8: **Instantaneous acceleration**)

where Δt is small enough that the acceleration can be considered constant over that interval.

The **acceleration due to gravity,** $\vec{g}$, is 9.8 m/s^2, directed down, at the surface of the Earth.

A General Method for Solving a 1-Dimensional Constant-Acceleration Problem
1. **Picture the scene.** Draw a diagram of the situation. Choose an origin to measure positions from, and a positive direction, and show these on the diagram.
2. **Organize what you know, and what you're looking for.** Making a table of data can be helpful. Record values of the variables used in the equations below.
3. **Solve the problem.** Think about which of the constant-acceleration equations to apply, and then set up and solve the problem. The three main equations are:

$$v = v_i + at.$$ (Equation 2.9: **Velocity for constant-acceleration motion**)

$$x = x_i + v_i t + \frac{1}{2}at^2.$$ (Equation 2.11: **Position for constant-acceleration motion**)

$$v^2 = v_i^2 + 2a\,\Delta x.$$ (Equation 2.12: **Connecting velocity, acceleration, and displacement**)

4. **Think about the answer(s)**. Check your answers, and/or see if they make sense.

Graphs
The velocity is the slope of the position-versus-time graph; the displacement is the area under the velocity-versus-time graph. The acceleration is the slope of the velocity-versus-time graph; the change in velocity is the area under the acceleration-versus-time graph.

Constant Velocity and Constant Acceleration
The motion of an object at rest is a special case of constant-velocity motion. In constant-velocity motion the position-versus-time graph is a straight line with a slope equal to the velocity.

Constant-velocity motion is a special case of constant-acceleration motion. In one dimension, if the acceleration is constant and non-zero the position-versus-time graph is quadratic while the velocity-versus-time graph is a straight line with a slope equal to the acceleration.