A Guide to Differential Length, Area, and Volume

Cartesian	x	$\in(-\infty,\infty)$	dx
	y	$\in(-\infty,\infty)$	dy
	z	$\in(-\infty,\infty)$	dz
Cylindrical	r	$\in[0,\infty)$	dr
	θ	$\in[0,2\pi]$	$r\cdot d\theta$
	z	$\in(-\infty,\infty)$	dz
Spherical	r	$\in[0,\infty)$	dr
	θ	$\in[0,\pi]$	$r\cdot d\theta$
	ϕ	$\in[0,2\pi]$	$r\cdot\sin\theta\cdot d\phi$

If I want to form a differential area dA I just multiply the two differential lengths that from the area together. For example, if I wanted to from some differential area by sweeping out two angles θ and ϕ in spherical coordinates, my dA would be given by:

$$dA = r^2\sin\phi\cdot d\theta\cdot d\phi$$

Last let me consider the volume integral of some function f that is just a function of the radius (i.e. $f\equiv f(r)$).

$$\oiiint f(r)\cdot dV = \int_{r=0}^{R}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} f(r)\cdot r^2\sin\theta\cdot dr\cdot d\theta\cdot d\phi =$$

$$= \int_{\phi=0}^{2\pi} d\phi\int_{\theta=0}^{\pi}\sin\theta\cdot d\theta\int_{r=0}^{R} r^2 f(r)\cdot dr$$

$$= 4\pi\cdot\int_{r=0}^{R} r^2 f(r)\cdot dr$$

Hence, I have converted by volume integral into a regular old one-dimensional integral! The 4π that came from the $\int_{\phi=0}^{2\pi} d\phi\int_{\theta=0}^{\pi}\sin\theta\cdot d\theta$ is often referred to as the "solid angle". Since these two terms are cumbersome to write, that is, just too much to write for lazy physicists, the following short hand is often used:

$$\int_{\phi=0}^{2\pi} d\phi\int_{\theta=0}^{\pi}\sin\theta\cdot d\theta = \int d\Omega$$

where the limits of integration are understood. This $d\Omega$, or rather, the integral over it, is the mystifying solid angle, which, when explained, is (hopefully) not so mystifying!