Challenge Problem Set 1 Solutions
PY 211

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• 1.1)
A string of total mass, M, and length, ℓ, is situated on a table such that, initially, a small amount, δ, hangs over the edge as shown in Figure 1. At what time, t, does the string slide all the way off of the table? Assume that the table is high enough off the ground such that the string does not come into contact with the ground before sliding off the table. Further assume that the mass of the string is uniformly distributed (i.e. we may write $\lambda = \frac{M}{\ell}$). (Hint: Use Newton's Second Law and note that the mass of the left-hand side of the equation does *not* equal the mass on the right-hand side of the equation.)

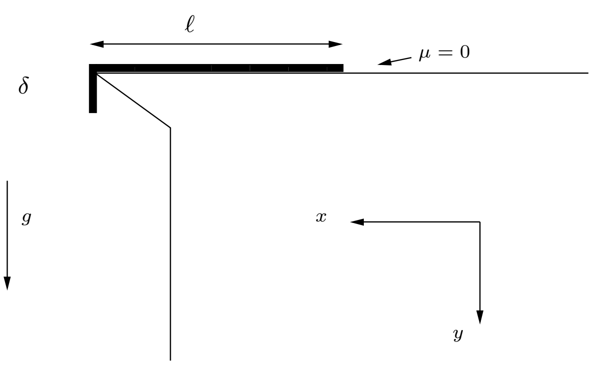

Figure 1:

We begin with Newton's Second Law.

$$\begin{aligned} F &= Ma && (1) \\ F(x) &= M\left(\frac{y}{\ell}\right)g && (2) \\ a &= \frac{d^2y}{dt^2} && (3) \\ \frac{d^2y}{dt^2} &= y\frac{g}{\ell} && (4) \end{aligned}$$

This equation tells us that the second derivative of some function is proportional to itself. We know two functions like that: e^x and e^{-x}. So, let's try the following:

$$y(t) = Ae^{\sqrt{\frac{g}{\ell}}t} + Be^{-\sqrt{\frac{g}{\ell}}t} \quad (5)$$

A and B are unknown constansts that we can determine by demanding that $y(t=0) = \delta$ and $v(t=0) = 0$.

$$\begin{aligned} y(t=0) &= \delta && (6) \\ \delta &= A + B && (7) \\ v(t=0) &= 0 && (8) \\ 0 &= A - B && (9) \\ A &= B && (10) \\ 2A &= \delta && (11) \\ & && (12) \end{aligned}$$

Thus, we know:

$$A = B = \frac{delta}{2} \quad (13)$$

and therefor:

$$\begin{aligned} y(t) &= \frac{\delta}{2}\left(e^{\sqrt{\frac{g}{\ell}}t} + e^{-\sqrt{\frac{g}{\ell}}t}\right) && (14) \\ cosh(x) &= \frac{e^x + e^{-x}}{2} && (15) \\ y(t) &= \delta cosh\left(\sqrt{\frac{g}{\ell}}t\right) && (16) \end{aligned}$$

Finally, if we want to determine how long it takes for the string to fall off the table, we set $y = \ell$ and solve for t.

$$\ell = \delta cosh\left(\sqrt{\frac{g}{\ell}}t\right) \tag{17}$$

$$\frac{\ell}{\delta} = cosh\left(\sqrt{\frac{g}{\ell}}t\right) \tag{18}$$

$$cosh^{-1}\left(\frac{\ell}{\delta}\right) = \sqrt{\frac{g}{\ell}}t \tag{19}$$

$$t = \sqrt{\frac{g}{\ell}}cosh^{-1}\left(\frac{\ell}{\delta}\right) \tag{20}$$

The arc-hyperbolic cosine (Figure 2) is a trnasendental function (like arc-cosine) and so we cannot simplify our result any further. Suppose, however, that $\delta = 1cm$ and $\ell = 1m$ then $t = 16.6s$.

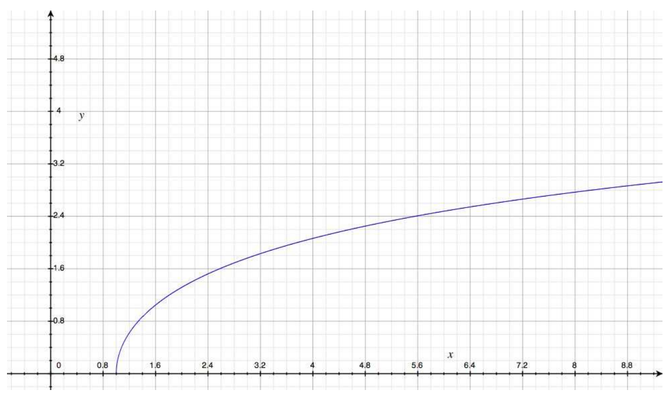

Figure 2: $y(x) = cosh^{-1}(x)$

- 1.2)

A particle slides along a one dimensional surface given by the equation $y(x) = x^2$. Assume the surface is frictionless.

(i) Show that the force, F, on a particle of mass m can be expressed as $F = mv\frac{dv}{dx}$. (Hint: This is a one-liner)

(ii) Find the force on the particle as a function of x (Hint: All forces in this problem are conservative).

(iii) If $x(t = 0) = -d$, determine the time, t, at which the particle arrives at the origin ($x = 0$).

(iv) If $y(x) \to x^{2n}$ where n is an integer greater than one, qualitatively, how do you expect t to change? Why?

Please note that the equation $y(x) = x^2$ does not make sense dimensionally. That's my error, designing a poorly stated problem. So, just imagine there is a constant in front of x (i.e. $y(x) = ax^2$) where a has units of inverse length.
(i)

$$
\begin{aligned}
F &= mv\frac{dv}{dx} && (21)\\
&= m\left(\frac{dx}{dt}\right)\left(\frac{dv}{dx}\right) && (22)\\
&= m\frac{dv}{dt} && (23)\\
&= ma && (24)
\end{aligned}
$$

Admittedly, this requires some "sloppy" treatment of differentials that would drive a mathematician *mad*; however, we are not mathematicians, and so, we can get away with it! A more careful analysis would reveal what we did is OK, provided both $\frac{dv}{dt}$ and $\frac{dv}{dx}$ are well enough behaved, where we use "well enough behaved" in the mathematical sense. (OK, so it was a four-liner!)
(ii)

First, let's use conservation of energy.

$$\frac{1}{2}mv^2 = mg\left(y_i - y_f\right) \tag{25}$$

$$\frac{1}{2}mv^2 = mg(x_i)^2 - mgx^2 \tag{26}$$

$$v = \sqrt{2g(x_i^2 - x^2)} \tag{27}$$

Now, let's find $\frac{dv}{dx}$.

$$\frac{dv}{dx} = \frac{d}{dx}\left(\sqrt{2g(x_i^2 - x^2)}\right) \tag{28}$$

$$\frac{dv}{dx} = \frac{1}{2}\left(2g(x_i^2 - x^2)\right)^{-\frac{1}{2}}(-4gx) \tag{29}$$

$$\frac{dv}{dx} = \frac{-2gx}{\sqrt{2g(x_i^2 - x^2)}} \tag{30}$$

Therefor:

$$F(x) = mv\frac{dv}{dx} \tag{31}$$

$$F(x) = m\left(\sqrt{2g(x_i^2 - x^2)}\right)\left(\frac{-2gx}{\sqrt{2g(x_i^2 - x^2)}}\right) \tag{32}$$

$$F(x) = -2mgx \tag{33}$$

(iii)

Using Newton's Second Law, let's find the time, t.

$$F = m\frac{d^2x}{dt^2} \tag{34}$$

$$= -2mgx \tag{35}$$

$$m\frac{d^2x}{dt^2} = -2mgx \tag{36}$$

We see that the second derivative of some function is proportional to minus itself. We know two functions like that, *cosine* and *sine*! So, let's try the following.

$$x(t) = Asin\left(\sqrt{2g}t\right) + Bcos\left(\sqrt{2g}t\right) \tag{37}$$

A and B are unknown constants that we can determine by demanding that at $t = 0$, $x = -d$ and $v = 0$.

$$x(t=0) = -d \tag{38}$$
$$-d = B \tag{39}$$
$$v(t=0) = \frac{dv(t=0)}{dt} \tag{40}$$
$$v(t=0) = 0 \tag{41}$$
$$0 = A \tag{42}$$

So, we now know $x(t)$.

$$x(t) = -dcos\left(\sqrt{2g}t\right) \tag{43}$$

So, now let's set $x = 0$ and solve for the time, t!

$$0 = -dcos\left(\sqrt{2g}t\right) \tag{44}$$
$$\frac{\pi}{2} = \sqrt{2g}t \tag{45}$$
$$t = \frac{\pi}{2\sqrt{2g}} \tag{46}$$
$$t = \frac{\pi}{\sqrt{8g}} \tag{47}$$

It is iteresting to note that the solution is *independent* of d.

(iv)

If the mass were traveling along a curve $y = x$, I know it would take a time, $t = \sqrt{\frac{2d}{gsin\theta}}$ to traverse the distance. If I take $sin\theta = 1$ (which, physically, is nonsense!) then my time is $t = \sqrt{\frac{2d}{g}}$ which is actually faster than the time given in Equation 47. However, if $sin\theta$ is smaller, than the time given in Equation 47 can be made the lesser of the two times. Thus, while my first instinct may be to think larger n's correspond to shorter times, I think I more clever analysis is needed to say anything for sure. The shape that minimizes the time is the solution to the brachistochrone problem, a very famous problem. If you are unfamiliar with it, I would recommend googleing it, as it is a very interesting and rich problem.

• 1.3)
A box of mass m is situated on top of a sphere of radius R as shown in Figure 3. At what angle, θ, does the box lose contact with the sphere? Assume the box is given a small nudge to get it moving, but that this nudge provides negligible kinetic energy. Further assume that there is not friction between the box and the sphere and that the sphere is held in place during the whole process. Qualitatively, how would the angle change if there were friction in the system?

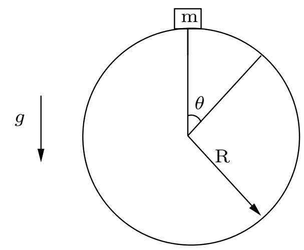

Figure 3:

Let's take the positive y-axis to point from the initial location of the mass through the center of the sphere. Then:

$$\frac{1}{2}mv^2 = mgy \tag{48}$$

We know that the mass will lose contact with the sphere when:

$$\frac{v^2}{R} > F_N \tag{49}$$

Thus:

$$\begin{aligned} \frac{v^2}{R} &= F_n & (50) \\ F_n = mgcos\theta v^2 &= 2gy & (51) \\ \frac{2gy}{R} &= mgcos\theta & (52) \end{aligned}$$

From trigonometry, we know:

$$\begin{aligned} cos\theta &= \frac{R-y}{R} && (53) \\ y &= R(1-cos\theta) && (54) \end{aligned}$$

Using this new expression for y:

$$\begin{aligned} 2g(1-cos\theta) &= mgcos\theta && (55) \\ 1-cos\theta &= \frac{1}{2}cos\theta && (56) \\ \frac{3}{2}cos\theta &= 1 && (57) \\ cos\theta &= \frac{2}{3} && (58) \\ \theta &\approx 48^\circ && (59) \end{aligned}$$

The amazing thing about this result is that it is independent of the mass of the object, the gravitational constant (g), and the radius of the sphere! Therefore, if I conduct the experiment on Earth with a ball of radius one meter and I repeat the experiment on the moon with a ball of radius 1 kilometer, I will measure the same angle!
If there is friction in the system, it will take a longer time to gain the necessary velocity to fall of the ball, and therefore, a larger angle is expected!