Quantum geometry, non-adiabatic response and emergent macroscopic dynamics. Lecture notes. Karpacz Winter School, 2014

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I. GAUGE POTENTIALS IN CLASSICAL AND QUANTUM HAMILTONIAN SYSTEMS.

A. Classical Hamiltonian systems.

Hamiltonian systems, are generally defined by specifying a set of canonical variables $p_j, q_j$ satisfying canonical relations

$$\{p_i, q_j\} = \delta_{ij},$$  \hspace{1cm} (1)

where $\{ \ldots \}$ denotes the Poisson bracket.

$$\{A(\vec{p}, \vec{q}), B(\vec{p}, \vec{q})\} = \sum_j \frac{\partial A}{\partial p_j} \frac{\partial B}{\partial q_j} - \frac{\partial B}{\partial p_j} \frac{\partial A}{\partial q_j}.$$  \hspace{1cm} (2)
It is easy to check that any orthogonal transformation

\[ Q = R(\lambda)q, \quad P = R(\lambda)p \quad (3) \]

preserves both the Poisson brackets and the symplectic operator. A general class of transformations which preserve the Poisson brackets are known as canonical transformations and can be expressed through the generating functions (Landau and Lifshitz, 1982). It is easy to check that infinitesimal canonical transformations can be generated by gauge potentials

\[ \begin{align*}
q_j(\lambda + \delta \lambda) &= q_j(\lambda) - \frac{\partial A_\lambda(\lambda, \vec{p}, \vec{q})}{\partial p_j} \delta \lambda, \\
p_j(\lambda + \delta \lambda) &= p_j(\lambda) + \frac{\partial A_\lambda(\lambda, \vec{p}, \vec{q})}{\partial q_j} \delta \lambda, 
\end{align*} \quad (4) \]

where \( \lambda \) parametrizes the canonical transformation and the gauge potential \( A_\lambda \) is some function of canonical variables and the parameters. Then up to the terms of the order of \( \delta \lambda^2 \) the transformation above preserves the Poisson brackets

\[ \{p_i(\lambda + \delta \lambda), q_j(\lambda + \delta \lambda)\} = \delta_{ij} + \delta \lambda \left( \frac{\partial^2 A_\lambda}{\partial p_j \partial q_i} - \frac{\partial^2 A_\lambda}{\partial p_j \partial q_i} \right) + O(\delta \lambda^2) = \delta_{ij} + O(\delta \lambda^2). \quad (6) \]

**Exercises.**

(i) Show that the generator of translations \( \vec{q}(X) = \vec{q}_0 - \vec{X} \) is the momentum operator: \( \vec{A}_\vec{X}(\vec{q}, \vec{p}) = \vec{p} \). You need to treat \( \vec{X} \) as a three component parameter \( \vec{\lambda} \). Note that the number of particles (and thus phase space dimension) can be much higher than three.

(ii) Show that the generator of the rotations around z-axis:

\[ \begin{align*}
q_x(\theta) &= \cos(\theta)q_{x0} - \sin(\theta)q_{y0}, \quad q_y(\theta) = \cos(\theta)q_{y0} + \sin(\theta)q_{x0}, \\
p_x(\theta) &= \cos(\theta)p_{x0} - \sin(\theta)p_{y0}, \quad p_y(\theta) = \cos(\theta)p_{y0} + \sin(\theta)p_{x0},
\end{align*} \]

is the angular momentum operator: \( A_\theta = p_x q_y - p_y q_x \).

(iii) Find the gauge potential \( A_\lambda \) corresponding to the orthogonal transformation (3).

Hamiltonian dynamics is a particular canonical transformation parametrized by time

\[ \begin{align*}
\frac{dq_j}{dt} &= \{H, q_j\} = \frac{\partial H}{\partial p_j}, \quad \frac{dp_j}{dt} = \{H, p_j\} = -\frac{\partial H}{\partial q_j},
\end{align*} \quad (7) \]

Clearly these Hamiltonian equations are equivalent to Eqs. (5) with the convention \( A_t = -H \). This observation shows that the gauge potentials \( A_\lambda \) are generators of motion in the parameter space.
One can extend canonical transformations to the complex variables. Instead of doing this in all
generality we will focus on particular phase space variables which are complex wave amplitudes.
E.g. for Harmonic oscillators for each normal mode with the Hamiltonian
\[ H_k = \frac{p_k^2}{2m} + \frac{m\omega_k^2}{2} q_k^2 \]  
we can define new linear combinations
\[ p_k = i \sqrt{\frac{m\omega_k}{2}} (a_k^* - a_k), \quad q_k = \sqrt{\frac{1}{2m\omega_k}} (a_k + a_k^*) \]  
or equivalently
\[ a_k^* = \frac{1}{\sqrt{2}} \left( q_k \sqrt{m\omega_k} - i \sqrt{\frac{m\omega_k}{2}} p_k \right), \quad a_k = \frac{1}{\sqrt{2}} \left( q_k \sqrt{m\omega_k} + i \sqrt{\frac{m\omega_k}{2}} p_k \right). \]  
Let us now compute the Poisson brackets of the complex wave amplitudes
\[ \{a_k, a_k\} = \{a_k^*, a_k^*\} = 0, \quad \{a_k, a_k^*\} = i. \]  
To avoid dealing with the imaginary Poisson brackets it is convenient to introduce new coherent
state Poisson brackets
\[ \{A, B\}_c = \sum_k \frac{\partial A}{\partial a_k} \frac{\partial B}{\partial a_k^*} - \frac{\partial B}{\partial a_k} \frac{\partial A}{\partial a_k^*}. \]  
From this definition it is immediately clear that
\[ \{a_k, a_q^*\}_c = \delta_{kq}. \]  
Comparing this relation with Eq. (11) we see that standard and coherent Poisson brackets differ
by the factor of \(i\):
\[ \{\ldots\} = i\{\ldots\}_c. \]  
Infinitesimal canonical transformations preserving the coherent state Poisson brackets can be also
defined using the guage potentials:
\[ i \frac{\partial a_k}{\partial \lambda} = - \frac{\partial A_\lambda}{\partial a_k^*}, \quad i \frac{\partial a_k^*}{\partial \lambda} = \frac{\partial A_\lambda}{\partial a_k}. \]

**Exercise.** Check that any unitary transformation \(\tilde{a}_k = U_{k,k'} a_{k'}\), where \(U\) is a unitary matrix,
preserves the coherent state Poisson bracket, i.e. \(\{\tilde{a}_k, \tilde{a}_q^*\}_c = \delta_{k,q}\). Verify that the Bogoliubov
transformation
\[ \gamma_k = \cosh(\theta_k) a_k + \sinh(\theta_k) a_k^*, \quad \gamma_k^* = \cosh(\theta_k) a_k^* + \sinh(\theta_k) a_k, \]
with $\theta_k = \theta_{-k}$ also preserves the coherent state Poisson bracket, i.e.

$$\{\gamma_k, \gamma_{-k}\}_c = \{\gamma_{-k}, \gamma_k^*\}_c = 0,$$

$$\{\gamma_k, \gamma_k^*\}_c = 1.$$  \hfill (17)

Assume that $\theta_k$ are known functions of some parameter $\lambda$, e.g. the interaction strength. Find the gauge potential $A_\lambda = \sum_k A_k$, which generates such transformations.

Let us write the Hamiltonian equations of motion for the new coherent variables. Using that

$$\frac{da}{dt} = \partial a / \partial t - \{a, H\} = \partial a / \partial t - i\{a, H\}_c$$

and using that our variables do not explicitly depend on time (such dependence would amount to going to a moving frame, which we will not consider here) we find

$$i\frac{da_k}{dt} = \{a_k, H\}_c = \frac{\partial H}{\partial a_k^*}, \quad i\frac{da_k^*}{dt} = \{a_k^*, H\}_c = -\frac{\partial H}{\partial a_k}$$  \hfill (19)

These equations are also known as Gross-Pitaevskii equations. Note that these equations are arbitrary for arbitrary Hamiltonians and not restricted to Harmonic systems.

And finally let us write down the Liouville equations of motion for the probability distribution $\rho(q, p, t)$ or $\rho(a, a^*, t)$. The latter just express incompressibility of the probability flow, which directly follows conservation of the phase space volume $d\Gamma = dqdp$ or $d\Gamma = da da^*$ for arbitrary canonical transformations including time evolution and from the conservation of the total probability $\rho d\Gamma$:

$$0 = \frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} - \{\rho, H\} = \frac{\partial \rho}{\partial t} - i\{\rho, H\}_c,$$  \hfill (20)

or equivalently

$$\frac{\partial \rho}{\partial t} = \{\rho, H\}, \quad i\frac{\partial \rho}{\partial t} = -\{\rho, H\}_c$$  \hfill (21)

**B. Unitary transformations in quantum systems.**

Analogue of canonical transformations in classical mechanics are unitary transformations in quantum mechanics. In classical systems these transformations reflect the freedom of choosing canonical variables while in quantum systems they reflect the freedom of choosing basis states.

The wave function representing some state can be always expanded in some basis:

$$|\psi\rangle = \sum_n \psi_n |n\rangle_0,$$  \hfill (22)

where $|n\rangle_0$ is some fixed, parameter independent basis. One can always make a unitary transformation to some other basis $|n(\lambda)\rangle = U_{nm}(\lambda) |n_0\rangle$ or equivalently $|n_0\rangle = U_{nm}^* |m(\lambda)\rangle$.  \hfill (22)
same state \(|\psi\rangle\) can be written as
\[
|\psi\rangle = \sum_n \psi_n U^*_m |m(\lambda)\rangle = \sum_m \tilde{\psi}_m(\lambda) |m(\lambda)\rangle,
\] (23)
where \(\tilde{\psi}_m(\lambda) = U^*_m \psi_n = U^\dagger \psi\). We can introduce the gauge potential by analogy with the classical systems as the generator of infinitesimal unitary transformations. Namely
\[
i \partial_\lambda \tilde{\psi}(\lambda) = -iU^\dagger \partial_\lambda U U^\dagger \psi = -iU^\dagger \partial_\lambda \tilde{U} \psi = -A_\lambda \tilde{\psi},
\] (24)
where we introduced the gauge potential
\[
A_\lambda = iU^\dagger \partial_\lambda U.
\] (25)

Note that up to the sign the gauge potential plays the role similar to the Hamiltonian, i.e. it generates the motion in the parameter space. It is easy to check that the gauge potential is a Hermitian operator
\[
A_\lambda = iU^\dagger \partial_\lambda U = A_\lambda,
\] (26)
where we used that
\[
\partial_\lambda U^\dagger = -U^\dagger \partial_\lambda U U^\dagger.
\] (27)

The gauge potential can be also represented through the matrix elements:
\[
\langle n_0 | A_\lambda | m_0 \rangle = i \langle n_0 | U^\dagger \partial_\lambda U | m_0 \rangle = i \langle n(\lambda) \partial_\lambda | m(\lambda) \rangle.
\] (28)

Note that the diagonal elements of the gauge potentials in the basis of some Hamiltonian, i.e. the basis where \(|n(\lambda)\rangle\) represent eigenstates of some Hamiltonian \(H(\lambda)\), represent the Berry connections, which we will discuss in detail later. So the Berry connections are direct analogues of energies representing the expectation values of the gauge potentials (Hamiltonians) within some states. With this definition one can extend the notion of the Berry connections to arbitrary states, not necessarily the eigen states of some Hamiltonians.

**Exercises**

i) Verify that the gauge potential corresponding to the translations: \(\tilde{p}(x) = \psi(\lambda + x)\) is the momentum operator. Similar verify that the gauge potential for rotations is the angular momentum operator.

ii) Consider quantum generalization of the Bogoliubov transformations discussed in the previous section. Show that the quantum and classical gauge potentials coincide if we identify complex amplitudes \(a_k\) and \(a_k^*\) with the annihilation and creation operators respectively.
C. Hamiltonian dynamics in the moving frame. Galilean transformation.

Gauge potentials are closely integrated into the Hamiltonian dynamics. In particular, they naturally appear in gauge theories like electromagnetism to enforce the gauge invariance. We will come to this issue later. For now we simply note that the equations of motion should be invariant under the gauge transformations. Indeed we can describe the same system using an arbitrary set of the canonical variables in the classical language or an arbitrary basis in the quantum language.

Let us consider now the classical equations of motion of some system described by some Hamiltonian, possibly time and parameter dependent (the system can be either single- or many-particle)

\[
\frac{dq}{dt} = \{H, q\}, \quad \frac{dp}{dt} = \{H, p\}
\]  

(29)

Now let us go to the moving frame, i.e. let us find the analogous e. o. m. in terms of canonical variables \(q(\lambda, t), p(\lambda, t)\). Then obviously

\[
\frac{dq}{dt} = \frac{\partial q}{\partial t} + \dot{\lambda} \frac{\partial q}{\partial \lambda}.
\]  

(30)

Note that \(dq/dt\) is the full derivative in the moving frame, \(\partial_t q\) is the derivative in the lab frame taken at constant \(\lambda\) and thus described by the Hamiltonian equations of motion. Using the definition of the gauge potential we find

\[
\frac{dq}{dt} = \{H, q\} - \dot{\lambda} \{A_\lambda, q\}, \quad \frac{dp}{dt} = \{H, p\} - \dot{\lambda} \{A_\lambda, p\}.
\]  

(31)

We thus see that the equations of motion in the moving frame preserve their Hamiltonian nature. The new moving Hamiltonian is given by the generalized Galilean transformation

\[
\tilde{H} = H - \dot{\lambda} A_\lambda.
\]  

(32)

If \(\lambda\) stands for a say \(x\)-coordinate of the reference frame then as we discussed \(A_\lambda A_\lambda = P_x\) and the expression above reduces to the standard Galilean transformation. If \(\lambda\) stands for an angle of the reference frame that \(A_\lambda\) is the angular momentum and the transformation of the Hamiltonian reduces to the Hamiltonian in the rotating frame. Similarly one can check that standard gauge transformations in electromagnetism can be also understood through going to a moving frame with respect to momentum.

Exercise. **Find the gauge potential corresponding to the translations of the momentum** \(p = p + \lambda(t)\). **Find the gauge potential describing this transformation. Show that the moving frame Hamiltonian with the Galilean term accounts to the standard gauge transformation in the electromagnetism where** \(A_x \rightarrow A_x + \partial_x f, \ \phi \rightarrow \phi - \partial_t f\), **where** \(A_x\) and \(\phi\) are the \(x\)-component of the vector...
potential and the scalar potential. You may either work in the units $e = c = 1$ or insert them explicitly.

It is interesting that the Galilean transformation can be understood from the extended variational principle, where equations of motion can be obtained by extremizing the action in the extended parameter space-time

$$S = \int [p\,dq - H\,dt + A_\lambda d\lambda] \quad (33)$$

with respect to all possible trajectories $p(\lambda, t), q(\lambda, t)$ satisfying the initial conditions. Extremizing the action at constant time $t$ clearly gives back the canonical transformations (5). Extremizing this action with respect to time reproduces the Hamiltonian equations of motion. If we extremize the action along some space time trajectory $\lambda(t)$ such that $d\lambda = \dot{\lambda}dt$ we will clearly reproduce the Hamiltonian equations of motion with the Galilean term (32).

Very similar analysis goes through for the quantum systems. Thus the Schroedinger equation

$$i \partial_t \psi = H \psi \quad (34)$$

after transformation to the moving frame: $\psi = U(\lambda)\tilde{\psi}$ reads

$$i\dot{\lambda}(\partial_\lambda U)\tilde{\psi} + iU\partial_t \tilde{\psi} = HU \tilde{\psi} \quad (35)$$

Multiplying both sides of this equation by $U^\dagger$ and moving the first term in the L.H.S. of this equation to the right we find

$$i \partial_t \tilde{\psi} = [U^\dagger HU - \dot{\lambda}A_\lambda] \tilde{\psi}. \quad (36)$$

Here $U^\dagger HU$ is the original Hamiltonian written in the rotating basis while $-\dot{\lambda}A_\lambda$ is the original Galilean term.

To be more specific let us consider a simple example of a spin one half particle in an external field. The Hamiltonian in the fixed (lab) frame is:

$$H(\lambda) = -\Delta \left( \cos(\lambda) \sigma_z + \sin(\lambda) \sigma_x \right)$$

with eigenstates

$$|gs_\lambda\rangle = \begin{pmatrix} \cos \left( \frac{\lambda}{2} \right) \\ \sin \left( \frac{\lambda}{2} \right) \end{pmatrix}, \quad |ex_\lambda\rangle = \begin{pmatrix} -\sin \left( \frac{\lambda}{2} \right) \\ \cos \left( \frac{\lambda}{2} \right) \end{pmatrix}$$
We will consider a very specific unitary transformation, which will be very important throughout this course, namely the transformation, which diagonalizes the instantaneous Hamiltonian, $H(\lambda)$. Clearly this transformation is simply the rotation around the $y$-axis $\lambda$:

$$U(\lambda) = \exp \left[ -i \frac{\sigma_y \lambda}{2} \right] = \begin{pmatrix} \cos(\lambda/2) & -\sin(\lambda/2) \\ \sin(\lambda/2) & \cos(\lambda/2) \end{pmatrix}$$

Note that $U(\lambda_0 = 0) = I$ is the identity matrix. Then the Gauge potential is:

$$\mathcal{A} = i U(\lambda) (\partial_\lambda U(\lambda)) = \frac{\sigma_y}{2},$$

which in this example is $\lambda$-independent. Note that the Pauli matrix $\sigma_y$ acts in the $\lambda$-independent lab basis, which coincides with the basis of the instantaneous Hamiltonian

$$\tilde{H} \equiv U^\dagger(\lambda) H(\lambda) U(\lambda) = -\Delta \sigma_z.$$ 

Clearly this Hamiltonian has eigenstates:

$$|\tilde{gs}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\tilde{es}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

which are identical to the eigenstates at $\lambda_0 = 0$:

$$|\tilde{gs}\rangle = |gs_{\lambda_0}\rangle, \quad |\tilde{es}\rangle = |ex_{\lambda_0}\rangle$$

By direct inspection one can verify that

$$\frac{1}{2} \langle \tilde{n} | \sigma_y | \tilde{m} \rangle = i \langle n | \partial_\lambda | m \rangle, \quad (37)$$

where $n, m = \{ gs, es \}$.

II. GEOMETRY OF THE GROUND STATE MANIFOLD. FUBINI-STUDY METRIC AND THE BERRY CURVATURE.

In the previous section we treated quantum and classical systems on equal footing. In this section we will focus on the geometric properties of the ground state manifold thus will focus on exclusively quantum systems. In the next section, where we will study relation between geometry and dynamics we will again consider both classical and quantum systems.

The first notion of the quantum geometric tensor appeared in 1980 in Ref. (Provost and Vallee, 1980). Formally the geometric tensor is defined on any manifold of states characterized by some
parameter $\vec{\lambda}$. \footnote{Starting from this section we will assume that the parameters can be multi-component.} For concreteness in this section we will consider the family of ground states of some Hamiltonian $H(\vec{\lambda})$. We will assume that the ground state is either non-degenerate or in the case of degeneracy it is not connected to another ground state by local perturbations. The geometric tensor naturally appears when one considers the distance between nearby states $|\psi_0(\vec{\lambda})\rangle$ and $|\psi_0(\vec{\lambda} + \delta\vec{\lambda})\rangle$ and expands it to the lowest order in $\delta\vec{\lambda}$:

$$d \equiv 1 - f^2 = 1 - |\langle \psi_0(\vec{\lambda})|\psi_0(\vec{\lambda} + \delta\vec{\lambda})\rangle|^2,$$

where $f = |\langle \psi_0(\vec{\lambda})|\psi_0(\vec{\lambda} + \delta\vec{\lambda})\rangle|$ is the so called fidelity of the ground state. Note that $1 - f^2$ is always positive therefore at small $\delta\vec{\lambda}$ the Taylor expansion of this quantity should not contain any first order terms in $\delta\vec{\lambda}$ and rather start with the quadratic term:

$$1 - f^2 \approx \delta\lambda_\alpha \chi_{\alpha\beta} \delta\lambda_\beta,$$

where $\chi_{\alpha\beta}$ is the symmetric positive definite geometric tensor. To find this tensor explicitly let us note that $1 - F^2$ is nothing but the probability that the system is excited if we do a quantum quench where the parameter suddenly changes from $\vec{\lambda}$ to $\vec{\lambda} + \delta\vec{\lambda}$. Indeed $F^2$ is simply the probability to remain in the ground state after this quench. The amplitude of going to the excited state is given by

$$a_n = \langle \psi_n(\vec{\lambda} + \delta\vec{\lambda})|\psi_0(\vec{\lambda})\rangle \approx \delta\lambda_\alpha \langle n|\tilde{\partial}_\alpha|0\rangle,$$

where to shorten the notations we introduced $\partial_\alpha \equiv \partial_{\lambda_\alpha}$ and the arrow over the derivative illustrates that it acts on the left. Also to simplify notations we use $|n\rangle$ instead of $|\psi_n(\vec{\lambda})\rangle$. Differentiating the orthonormality condition $\langle n|m \rangle$ with respect to $\lambda_\alpha$ we find

$$\langle n|\tilde{\partial}_\alpha|m \rangle + \langle n|\partial_\alpha|m \rangle = 0.$$
Note that we can skip the tilde in the basis because in the leading order in $\delta \vec{\lambda}$ the basis does not change. Thus we see that

$$1 - F^2 = \sum_{n \neq 0} |a_n|^2 = \sum_{n \neq 0} \delta \lambda_\alpha \delta \lambda_\beta \langle 0 | A_\alpha n \langle n | A_\beta | 0 \rangle = \delta \lambda_\alpha \delta \lambda_\beta \langle 0 | A_\alpha A_\beta | 0 \rangle_c,$$

where the subindex $c$ implies that we are taking the connected correlation function (or equivalently the covariance):

$$\langle 0 | A_\alpha A_\beta | 0 \rangle_c = \langle 0 | A_\alpha A_\beta | 0 \rangle - \langle 0 | A_\alpha | 0 \rangle \langle 0 | A_\beta | 0 \rangle.$$

This covariance precisely determines the geometric tensor introduced by Provost and Vallee

$$\chi_{\alpha \beta} = \langle 0 | A_\alpha A_\beta | 0 \rangle_c.$$

In terms of many-body wave functions the geometric tensor can be expressed through the overlap of derivatives:

$$\chi_{\alpha \beta} = \langle 0 | \partial_\alpha \partial_\beta | 0 \rangle_c = \langle \partial_\alpha \psi_0 | \partial_\beta \psi_0 \rangle_c = \langle \partial_\alpha \psi_0 | \partial_\beta \psi_0 \rangle - \langle \partial_\alpha \psi_0 \rangle \langle \psi_0 | \partial_\beta \psi_0 \rangle.$$

The last term in this expression is necessary to enforce invariance of the distance under arbitrary global phase transformations of the wave function $\psi_0(\vec{\lambda}) \rightarrow \exp[i\phi(\vec{\lambda})] \psi_0(\vec{\lambda})$, which should not affect the notion of the distance between different ground states.

Note that in general the geometric tensor is not symmetric. Indeed because the operators $A_\alpha$ are Hermitian one can show that

$$\chi_{\alpha \beta} = \chi_{\beta \alpha}^*.$$  

Only the symmetric part of $\chi_{\alpha \beta}$ determines the distance between the states. Indeed in the quadratic form

$$d = \delta \lambda_\alpha \chi_{\alpha \beta} \delta \lambda_\beta$$

one can always symmetrize the indexes $\alpha$ and $\beta$ so that the antisymmetric part drops out. Nevertheless, as we will see later, both the symmetric and the anti-symmetric parts of the geometric tensor are very important so we will introduce both

$$g_{\alpha \beta} = \frac{\chi_{\alpha \beta} + \chi_{\beta \alpha}}{2} = \frac{1}{2} \langle 0 | A_\alpha A_\beta + A_\beta A_\alpha | 0 \rangle = \Re \langle 0 | A_\alpha A_\beta | 0 \rangle.$$
is called the Fubini-Study metric tensor. The imaginary part of the geometric tensor is known as the Berry curvature (Berry, 1984). It plays a crucial role in all quantum topological phenomena:

\[ F_{\alpha\beta} = i(\chi_{\alpha\beta} - \chi_{\beta\alpha}) = -2\Im \chi_{\alpha\beta} = i\langle 0| [A_\alpha, A_\beta]|0 \rangle. \]  

(51)

Let us note that the Berry curvature can be expressed through the derivatives of the Berry connections:

\[ F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha, \]  

(52)

where

\[ A_\alpha = \langle 0| A_\alpha |0 \rangle = i\langle 0| \partial_\alpha |0 \rangle. \]  

(53)

Indeed direct differentiation gives

\[ \partial_\alpha A_\beta - \partial_\beta A_\alpha = i\langle 0| \overrightarrow{\partial_\alpha} \partial_\beta |0 \rangle - i\langle 0| \overrightarrow{\partial_\beta} \partial_\alpha |0 \rangle + i\langle 0| \partial_\alpha^2 |0 \rangle - i\langle 0| \partial_\beta^2 |0 \rangle = i(\chi_{\alpha\beta} - \chi_{\beta\alpha}). \]  

(54)

The Berry connection and the Berry curvature are key geometric characteristics of quantum states with the broken time reversal symmetry. Indeed representing the ground state wave function as

\[ \psi_0 = |\psi_0\rangle \exp[i\phi] \]  

(55)

we find that

\[ A_\alpha = -\partial_\alpha \phi \]  

(56)

Therefore the integral of \( A_\alpha \) over a closed path represents the total phase (Berry phase) accumulated by the wave function during the adiabatic evolution

\[ \gamma = \oint_{\vec{l}} \partial_\alpha \phi d\lambda_\alpha = -\oint_{\vec{l}} A_\alpha d\lambda_\alpha \]  

(57)

By the Stokes theorem the same phase can be represented as the integral of the Berry curvature over the surface enclosed by the contour \( \vec{l} \):

\[ \gamma = \int_S F_{\alpha\beta} d\lambda_\alpha \wedge d\lambda_\beta, \]  

(58)

where the wedge product implies that the integral is directed.

---

\footnote{Sometimes in literature by the Fubini-Study metric one understands the complete metric in the projective Hilbert space where the number of parameters \( \lambda_\alpha \) coincides with the dimension of the Hilbert space.}
To get an intuition about the Berry curvature and then the metric tensor let us consider two simple examples. First, following the original paper by Berry, let us consider the Aharonov-Bohm geometry, namely a particle confined in a deep potential in a presence of a solenoid. The Hamiltonian for this system is

$$
\mathcal{H} = \frac{\left(\vec{p} - \frac{e}{c} \vec{\Lambda}(\vec{r})\right)^2}{2m} + V(\vec{r} - \vec{R}),
$$

where $\vec{\Lambda}$ is the vector potential (we use $\vec{\Lambda}$ to avoid confusion with the Berry connection) and $V(\vec{r} - \vec{R})$ is a confining potential near some point $\vec{R}$ outside the solenoid. Outside the solenoid there is no magnetic field so the vector potential can be written as a gradient of the magnetic potential

$$
\vec{\Lambda} = \vec{\nabla} \Phi \Rightarrow \Phi = \int \vec{\Lambda} d\vec{l}
$$

Note that this expression valid even if the magnetic field is non-zero but then the potential $\Phi$ is path dependent. It is well known that the vector potential can be eliminated by a gauge transformation:

$$
\psi = \tilde{\psi} \exp \left[ i \frac{e}{c \hbar} \Phi \right].
$$

Then the Hamiltonian for $\tilde{\psi}$ becomes independent of the vector potential and thus the wave function $\tilde{\psi}$ is real. In this case the Berry connection with respect to $\vec{R}$, the position of the trap, is

$$
-\partial_{R} \phi = -\frac{e}{c \hbar} \partial_{R} \Phi = \frac{e}{c \hbar} \partial_{r} \Phi = \frac{e}{c \hbar} \vec{\Lambda}(\vec{R})
$$

In general one needs to average the vector potential over the wave function $\tilde{\psi}(\vec{r})$ but assuming that it is localized near $\vec{R}$ it is not important. So we see that up to the fundamental constants the Berry connection plays the role of the vector potential, hence the Berry phase place the role of the flux and the Berry curvature (curl of the Berry connection) plays the role of the magnetic field. This analogy is very useful when we think about general parameter space and as we will see later this analogy is not coincidental. Like the magnetic field the Berry curvature is the source of the Lorentz force.

Now let us consider the next illustrative example of a spin one half in an external magnetic field. We will analyze geometry shown in Fig. 1. The Hamiltonian is given by

$$
\mathcal{H} = -\hbar \vec{\sigma}
$$

Because the ground state does not depend on the magnitude of the magnetic field we can choose parameters to be the angles $\theta$ and $\phi$. We already discussed the ground and excited states for this
model at $\phi = 0$. For general $\phi$ we have

$$|\psi_0\rangle = \begin{pmatrix} \cos(\theta/2)e^{i\phi/2} \\ \sin(\theta/2)e^{-i\phi/2} \end{pmatrix}, \quad |\psi_1\rangle = \begin{pmatrix} -\sin(\theta/2)e^{i\phi/2} \\ \cos(\theta/2)e^{-i\phi/2} \end{pmatrix}. $$

Direct evaluation of the geometric tensor for the ground state gives

$$\chi_{\theta\theta} = \frac{1}{4}, \quad \chi_{\phi\phi} = \frac{1}{4} \sin^2(\theta), \quad \chi_{\theta\phi} = -\frac{i}{4} \sin(\theta)$$

This expressions can be either computed by direct differentiation of the ground state wave function with respect to $\theta$ and $\phi$ or from calculating the covariance matrix of the gauge potentials

$$A_\theta = i\partial_\theta = \frac{1}{2}\tau_y, \quad A_\phi = i\partial_\phi = \frac{1}{2}(\cos(\theta)\tau_z + \sin(\theta)\tau_x)$$

Recall that the Pauli matrices here act in the basis of instantaneous eigenstates where the Hamiltonian is $\mathcal{H} = -h\tau_z$. The equations above generalize to particles with arbitrary spin where instead of spin one half operators like $1/2\tau_y$ one uses the angular momentum operators like $S_y$.

From the expression for the geometric tensor we see that the nonzero metric tensor components are

$$g_{\theta\theta} = \frac{1}{4}, \quad g_{\phi\phi} = \frac{1}{4} \sin^2 \theta,$$
and the Berry curvature is
\[ F_{\theta \phi} = \frac{1}{2} \sin(\theta). \] (66)

It is interesting to point that for the excited state the metric tensor is the same while the Berry curvature has an opposite sign.

Originally Provost and Vallee thought that the metric tensor is a nice mathematical object, which is however unmeasurable. Contrary it was immediately understood that the Berry curvature is responsible for many different physical phenomena like Aharonov-Bohm effect or the quantum Hall effect. Later in the course we will discuss in a greater detail the relation between the geometric tensor and the dynamical response. For now let us show that the geometric tensor is measurable and related to the standard dynamical response coefficients. Indeed let us differentiate the relation
\[ \mathcal{H}|n\rangle = E_n|n\rangle \] (67)
with respect to \( \lambda \) and multiply both sides on \( \langle m | \), where \( m \neq n \). For simplicity we will assume there are no degeneracies. Then we find
\[ \langle m | \partial_\lambda \mathcal{H}|n\rangle + E_m \langle m | \partial_\lambda |n\rangle = E_n \langle m | \partial_\lambda |n\rangle. \]

Thus we find
\[ \langle m | \partial_\lambda |n\rangle = \frac{\langle m | \partial_\lambda \mathcal{H}|n\rangle}{E_n - E_m}. \] (68)

Using this result we can rewrite the components of the geometric tensor as
\[ \chi_{\alpha \beta} = \sum_{n \neq 0} \langle 0 | \partial_\alpha |n\rangle \langle n | \partial_\beta |0\rangle = -\sum_{n \neq 0} \langle 0 | \partial_\alpha |n\rangle \langle n | \partial_\beta |0\rangle = \sum_{n \neq 0} \frac{\langle 0 | \partial_\alpha \mathcal{H}|n\rangle \langle n | \partial_\beta \mathcal{H}|0\rangle}{(E_n - E_0)^2} \] (69)

Now let us use the following trick standard in connecting the Lehmann’s representation of some observable to the non-equal time correlation functions:
\[ \frac{1}{(E_n - E_0)^2} = \int_0^\infty d\omega \frac{1}{\omega^2} \delta(E_n - E_0 - \omega) = \int_0^\infty d\omega \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{-i(E_n - E_0 - \omega)t}. \] (70)

In this integral we can always add \( \exp[-\epsilon|t|] \) for the convergence. Next we note that
\[ \langle 0 | e^{iE_0 t} \partial_\alpha \mathcal{H} e^{-iE_n t} |n\rangle = \langle 0 | (\partial_\alpha \mathcal{H})(t) |n\rangle \] (71)
is the matrix element of the operator \( \partial_\alpha \mathcal{H} \) in the Heisenberg representation. Plugging this into Eq. (69) we find
\[ \chi_{\alpha \beta} = \int_0^\infty \frac{d\omega}{\omega^2} S_{\alpha \beta}(\omega), \] (72)
where

\[ S_{\alpha\beta}(\omega) = \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{i\omega t} \langle 0 | (\partial_{\alpha} \mathcal{H})(t) \partial_{\beta} \mathcal{H}(0) | 0 \rangle_c. \]  

(73)

This object \( S_{\alpha\beta}(\omega) \) is a standard response coefficient. Its symmetric part, determining the metric tensor, appears in the noise, full counting statistics and the absorption spectrum (through the fluctuation-dissipation relations). Its antisymmetric part determining the Berry curvature appears in standard Kubo linear response. This implies that both the metric tensor and the Berry curvature are measurable quantities in standard experimental setups.

### III. GEOMETRIC INVARIANTS: GEOMETRY OF THE XY CHAIN.

Both the Berry curvature and the Fubini-Study metric tensor define geometric (topological) invariants. There are many of those. In these lecture notes we will concentrate on two most important ones: (first) Chern number and the Euler characteristic. Because the Chern number has been extensively discussed in literature in many different contexts we will mention it only briefly and will concentrate on the Euler characteristics, which has been discussed much less with respect to the physical systems. Also in this section we will focus exclusively on two-dimensional manifolds. Geometry and topology of higher-dimensional manifolds is much more complex and is often understood through various two-dimensional cuts in any case.

#### A. Basic definitions of the Euler characteristic and the first Chern number.

The Euler characteristic of the metric manifold \( \mathcal{M} \) is an integer equal to the integrated Gauss curvature over the manifold with an additional boundary term:

\[ \xi(\mathcal{M}) = \frac{1}{2\pi} \left[ \int_{\mathcal{M}} K dS + \oint_{\partial\mathcal{M}} k_g dl \right], \]  

(74)

A standard notation for the Euler characteristic is \( \chi \) but because we reserved this symbol for the geometric tensor we will use \( \xi \) instead. The two terms on the left side of Eq. (74) are the bulk and boundary contributions to the Euler characteristic of the manifold. We refer to the first term,

\[ \xi_{\text{bulk}}(\mathcal{M}) = \frac{1}{2\pi} \int_{\mathcal{M}} K dS, \]  

(75)

and the second term,

\[ \xi_{\text{boundary}}(\mathcal{M}) = \frac{1}{2\pi} \oint_{\partial\mathcal{M}} k_g dl, \]  

(76)
as the bulk and boundary Euler integrals, respectively. These terms, along with their constituents – the Gaussian curvature \( K \), the geodesic curvature \( k_g \), the area element \( dS \), and the line element \( dl \) – are geometric invariants, meaning that they remain unmodified under any change of variables. More explicitly, if the metric is written in first fundamental form as

\[
ds^2 = E d\lambda_1^2 + 2 F d\lambda_1 d\lambda_2 + G d\lambda_2^2 ,
\]

then these invariants are given by (Kreyszig, 1959)

\[
K = \frac{1}{\sqrt{g}} \left[ \frac{\partial}{\partial \lambda_2} \left( \sqrt{g} \Gamma^2_{11} \right) - \frac{\partial}{\partial \lambda_1} \left( \sqrt{g} \Gamma^2_{12} \right) \right],
\]

\[
k_g = \sqrt{g} G^{3/2} \Gamma^1_{22},
\]

\[
dS = \sqrt{g} d\lambda_1 d\lambda_2,
\]

\[
dl = \sqrt{G} d\lambda_2 ,
\]

where \( k_g \) and \( dl \) are given for a curve of constant \( \lambda_1 \). The metric determinant \( g \) and Christoffel symbols \( \Gamma^k_{ij} \) are

\[
g = EG - F^2
\]

\[
\Gamma^k_{ij} = \frac{1}{2} g^{km} \left( \partial_j g_{im} + \partial_i g_{jm} - \partial_m g_{ij} \right) ,
\]

where \( g^{ij} \) is the inverse of the metric tensor \( g_{ij} \). As we see the explicit expressions for the Euler characteristic are quite cumbersome but they are known and unique functions of the metric tensor. A simple intuitive understanding of the Gauss curvature in two dimensions comes from

\[
K = \frac{1}{R_1 R_2} ,
\]

where \( R_1 \) and \( R_2 \) are the principal radii of curvature, which are the minimal and the maximal radii of the circles touching the surface. The geodesic curvature is a curvature of the boundary projected to the tangent plane. Thus e.g. for a sphere the geodesic curvature of a great circle is zero. In simple words the geodesic curvature measures the distance of the boundary from the geodesic. For the manifolds without boundaries like a torus or a sphere the Euler characteristic simply counts the number of holes in the manifold. Thus for a sphere the Euler characteristic is \( \xi = 2 \) for a torus \( \chi = 0 \) and each additional hole gives an extra contribution of \( -2 \).

The other important topological characteristic is the (first) Chern number, which is defined through the Berry curvature. To understand where it comes from let us consider a closed manifold like shown in Fig. 1 and choose an arbitrary closed contour on that sphere like a dashed line. Let
us compute the Berry phase (flux) along this contour by two ways:

$$\gamma_{\text{top}} = \int_{S_{\text{top}}} F_{\alpha\beta} d\lambda_\alpha d\lambda_\beta, \quad \gamma_{\text{bottom}} = -\int_{S_{\text{bottom}}} F_{\alpha\beta} d\lambda_\alpha d\lambda_\beta, \quad (81)$$

where the minus sign in the second term is because the top and bottom surfaces of the sphere bounded by the curve have opposite orientations with respect to this curve, which can be checked by e.g. thumb rule. Recall that $\gamma$ represents the physical phase acquired by the wave function during the (adiabatic) motion in the parameter space. Since the wave function is unique the two phases should be identical up to an overall constant $2\pi n$. Thus we find that

$$2\pi n = \gamma_{\text{top}} - \gamma_{\text{bottom}} = \oint_S F_{\alpha\beta} d\lambda_\alpha \wedge d\lambda_\beta \quad (82)$$

The integer $n$ is precisely the Chern number $\text{ch}_1$ so we get

$$\text{ch}_1 = \frac{1}{2\pi} \oint_S F_{\alpha\beta} d\lambda_\alpha \wedge d\lambda_\beta \quad (83)$$

As we found earlier for a spin one half particle in a magnetic field $F_{\theta\phi} = 1/2 \sin(\theta)$ so in this case

$$\text{ch}_1 = \frac{1}{2\pi} \int_0^{2\pi} d\phi \int_0^\pi d\theta \frac{1}{2} \sin(\theta) = 1 \quad (84)$$

One can easily check that for a particle with a spin $S$ this Chern number is equal to $2S$.

Exercise. Prove the statement above.

B. Geometric structure of the XY chain.

Let us now analyze geometric invariants for a particular XY chain model. This model is sufficiently simple so that all calculations can be done explicitly analytically but yet it has a very rich phase diagram with two different phase transitions and multicritical points. The content of this chapter is adopted from Ref. (Kolodrubetz et al., 2013).

Quantum XY chain described by the Hamiltonian

$$\mathcal{H} = -\sum_j \left[ J_x \sigma_x^j \sigma_x^{j+1} + J_y \sigma_y^j \sigma_y^{j+1} + h \sigma_z^j \right], \quad (85)$$

where $J_{x,y}$ are exchange couplings, $h$ is a transverse field, and the spins are represented as Pauli matrices $\sigma^{x,y,z}$. It is convenient to re-parameterize the model in terms of new couplings $J$ and $\gamma$ as

$$J_x = J \left( \frac{1+\gamma}{2} \right), \quad J_y = J \left( \frac{1-\gamma}{2} \right), \quad (86)$$

FIG. 2 Ground state phase diagram of the XY Hamiltonian (Eq. (87)) for $\phi = 0$. The rotation parameter $\phi$ modifies the Ising ferromagnetic directions, otherwise maintaining all features of the phase diagram. As a function of transverse field $h$ and anisotropy $\gamma$, the ground state undergoes continuous Ising-like phase transitions between paramagnet and ferromagnet at $h = \pm 1$ and anisotropic transitions between ferromagnets aligned along X and Y directions (X/Y-FM) at $\gamma = 0$. These two types of phase transition meet at multi-critical points.

where $J$ is the energy scale of the exchange interaction and $\gamma$ is its anisotropy. We add an additional tuning parameter $\phi$, corresponding to simultaneous rotation of all the spins about the z-axis by angle $\phi/2$. While rotating the angle $\phi$ has no effect on the spectrum of $\mathcal{H}$, it does modify the ground state wave function. To fix the overall energy scale, we set $J = 1$.

The Hamiltonian described above can be written as

$$\mathcal{H}(h, \gamma, \phi) = -\sum_j \left[ \sigma_j^+ \sigma_{j+1}^- + \text{h.c.} \right] - \gamma \sum_j \left[ e^{i\phi} \sigma_j^+ \sigma_{j+1}^- + \text{h.c.} \right] - h \sum_j \sigma_j^z. \quad (87)$$

Since the Hamiltonian is invariant under the mapping $\gamma \rightarrow -\gamma$, $\phi \rightarrow \phi + \pi$, we generally restrict ourselves to $\gamma \geq 0$, although we occasionally plot the superfluous $\gamma < 0$ region when convenient. This model has a rich phase diagram (Damle and Sachdev, 1996; Mukherjee et al., 2011), as shown in Fig. 2. There is a phase transition between paramagnet and Ising ferromagnet at $|h| = 1$ and $\gamma \neq 0$. There is an additional critical line at the isotropic point ($\gamma = 0$) for $|h| < 1$. The two transitions meet at multi-critical points when $\gamma = 0$ and $|h| = 1$. Another notable line is $\gamma = 1$, which corresponds to the transverse-field Ising (TFI) chain. Finally let us note that there are two other special lines $\gamma = 0$ and $|h| > 1$ where the ground state is fully polarized along the magnetic field and thus $h$-independent. Thus this line is characterized by vanishing susceptibilities.
including vanishing metric along the \( h \)-direction. One can show that such state is fully protected by the rotational symmetry of the model and can be terminated only at the critical (gapless) point (Kolodrubetz et al., 2013). The phase diagram is invariant under changes of the rotation angle \( \phi \).

Rewriting the spin Hamiltonian in terms of free fermions via a Jordan-Wigner transformation, \( \mathcal{H} \) can be mapped to an effective non-interacting spin one-half model (Sachdev, 1999) with

\[
\mathcal{H} = \sum_k \mathcal{H}_k; \quad (88)
\]

\[
\mathcal{H}_k = -\begin{pmatrix}
  h - \cos(k) & \gamma \sin(k) e^{i\phi} \\
  \gamma \sin(k) e^{-i\phi} & -[h - \cos(k)]
\end{pmatrix}
\]

The ground state of \( \mathcal{H}_k \) is a Bloch vector with azimuthal angle \( \phi \) and polar angle \( \theta_k \) with

\[
\theta_k = \tan^{-1} \left[ \frac{\gamma \sin(k)}{h - \cos(k)} \right]. \quad (89)
\]

Because the Hamiltonian effectively describes an independent set of two-level systems we can immediately extend the results from the previous section to find the expression for different metric tensor components.

In particular differentiating the ground state in each momentum sector with respect to \( h \) we find

\[
\partial_h |g_{s_k}\rangle = \frac{\partial_h \theta_k}{2} \begin{pmatrix}
  -\sin \left( \theta_k \frac{\gamma}{2} e^{i\phi/2} \\
  \cos \left( \theta_k \frac{\gamma}{2} e^{-i\phi/2} \right)
\end{pmatrix} = -\frac{\partial_h \theta_k}{2} |e_{s_k}\rangle. \quad (90)
\]

The same derivation applies to the anisotropy \( \gamma \), since changing either \( \gamma \) or \( h \) only modifies \( \theta_k \) and not \( \phi \). Thus we find

\[
A_\lambda = \frac{1}{2} \sum_k (\partial_\lambda \theta_k) \tau_k^\lambda, \quad (91)
\]

where \( \lambda = \{h, \gamma\} \) and \( \tau_k^{x,y,z} \) are Pauli matrices that act in the instantaneous ground/excited state basis. Similarly, for the parameter \( \phi \), we find that

\[
A_\phi = -\frac{1}{2} \sum_k [\cos(\theta_k) \tau_k^z + \sin(\theta_k) \tau_k^x]. \quad (92)
\]

Using that

\[
g_{\mu\nu} = \frac{1}{2} \langle g_{s} | A_\mu A_\nu + A_\nu A_\mu | g_{s}\rangle \quad (93)
\]
we find
\[ gh_h = \frac{1}{4} \sum_k \left( \frac{\partial \theta_k}{\partial h} \right)^2, \quad g_{\gamma \gamma} = \frac{1}{4} \sum_k \left( \frac{\partial \theta_k}{\partial \gamma} \right)^2, \quad gh_\gamma = \frac{1}{4} \sum_k \frac{\partial \theta_k}{\partial h} \frac{\partial \theta_k}{\partial \gamma}, \quad g_{\phi \phi} = \frac{1}{4} \sum_k \sin^2(\theta_k), \quad (94) \]

The remaining two components of the metric tensor \( g_{h \phi} = g_{\gamma \phi} \) are equal to zero.

The expressions for the metric tensor can be evaluated in the thermodynamic limit, where the summation becomes integration over momentum space. It is convenient to divide all components of the metric tensor by the system size and deal with intensive quantities \( g_{\mu \nu} \rightarrow g_{\mu \nu}/L \). Then one calculates these integrals to find that

\[
\begin{align*}
g_{\phi \phi} &= \frac{1}{8} \left\{ \begin{array}{ll}
\frac{|\gamma|}{|\gamma|+1}, & |h| < 1 \\
\frac{\gamma^2}{1-\gamma^2} \left( \frac{|h|}{\sqrt{h^2-1+\gamma^2}} - 1 \right), & |h| > 1
\end{array} \right. \\
g_{hh} &= \frac{1}{16} \left\{ \begin{array}{ll}
\frac{|\gamma|}{|\gamma|+1}, & |h| < 1 \\
\frac{1}{|h|\gamma^2} \left( \frac{1}{(h^2-1)(h^2-1+\gamma^2)^{3/2}} \right), & |h| > 1
\end{array} \right. \\
g_{\gamma \gamma} &= \frac{1}{16} \left\{ \begin{array}{ll}
\frac{2}{(1-\gamma^2)^2} \left[ \frac{|h|}{\sqrt{h^2-1+\gamma^2}} - 1 \right], & |h| < 1 \\
\frac{1}{|h|\gamma^2} \left( \frac{1}{(1-\gamma^2)(h^2-1+\gamma^2)^{3/2}} \right), & |h| > 1
\end{array} \right. \\
g_{h \gamma} &= \frac{1}{16} \left\{ \begin{array}{ll}
0, & |h| < 1 \\
\frac{-|h|\gamma}{h(h^2-1+\gamma^2)^{3/2}}, & |h| > 1
\end{array} \right. \\
\end{align*}
\quad (95) \]

Using the metric tensor we can visualize the ground state manifold by building an equivalent (i.e., isometric) surface and plotting its shape. It is convenient to focus on a two-dimensional manifold by fixing one of the parameters. We then represent the two-dimensional manifold as an equivalent three-dimensional surface. To start, let’s fix the anisotropy parameter \( \gamma \) and consider the \( h-\phi \) manifold. Since the metric tensor has cylindrical symmetry, so does the equivalent surface. Parameterizing our shape in cylindrical coordinates and requiring that

\[ dz^2 + dr^2 + r^2 d\phi^2 = gh_h dh^2 + g_{\phi \phi} d\phi^2, \quad (96) \]

we see that

\[ r(h) = \sqrt{g_{\phi \phi}}, \quad z(h) = \int_0^h dh_1 \sqrt{gh_h(h_1) - \left( \frac{dr(h_1)}{dh_1} \right)^2}. \quad (97) \]

Using Eq. (95), we explicitly find the shape representing the XY chain. In the Ising limit \( (\gamma = 1) \),
FIG. 3 Equivalent graphical representation of the phase diagram of the transverse field Ising model \((\gamma = 1)\) in the \(h - \phi\) plane. The ordered ferromagnetic phase maps to a cylinder of constant radius. The disordered paramagnetic phases \(h > 1\) and \(h < -1\) map to the two hemispherical caps. The inset shows how the cylindrical coordinates \(z\) and \(r\) depend on the transverse field \(h\).

we get

\[
\begin{align*}
  r(h) &= \frac{1}{4} |h| < 1, \\
  z(h) &= \frac{\arcsin(h)}{4} \\
  r(h) &= \frac{1}{4|h|} |h| > 1, \\
  z(h) &= \frac{\pi |h|}{8} + \frac{\sqrt{h^2 - 1}}{4|h|}.
\end{align*}
\]

The phase diagram is thus represented by a cylinder of radius 1/4 corresponding to the ferromagnetic phase capped by the two hemispheres representing the paramagnetic phase, as shown in Fig. 3. It is easy to check that the shape of each phase does not depend on the anisotropy parameter \(\gamma\), which simply changes the aspect ratio and radius of the cylinder. Because of the relation \(r(h) = \sqrt{g_{\phi\phi}}\) this radius vanishes as the anisotropy parameter \(\gamma\) goes to zero. By an elementary integration of the Gaussian curvature, the phases have bulk Euler integral 0 for the ferromagnetic cylinder and 1 for each paramagnetic hemisphere. These numbers add up to 2 as required, since the full phase diagram is homeomorphic to a sphere. From Fig. 3, it is also clear that the phase boundaries at \(h = \pm 1\) are geodesics, meaning that the geodesic curvature (and thus the boundary contribution \(\xi_{\text{boundary}}\)) is zero for a contour along the phase boundary. This boundary integral protects the value of the bulk integral and vice versa.

In the Ising limit \((\gamma = 1)\), the shape shown in Fig. 3, can also be easily seen from computing the
curvature $K$ using Eq. (78). Within the ferromagnetic phase, the curvature is zero – no surprise, given that the metric is flat by inspection. The only shape with zero curvature and cylindrical symmetry is a cylinder. Similarly, within the paramagnet, the curvature is a constant $K = 16$, like that of a sphere. Therefore, to get cylindrical symmetry, the phase diagram is clearly seen to be a cylinder capped by two hemispheres.

We can also reconstruct an equivalent shape in the $\gamma - \phi$ plane. In this case we expect to see a qualitative difference for $|h| > 1$ and $|h| < 1$ because in the latter case there is an anisotropic phase transition at the isotropic point $\gamma = 0$, while in the former case there is none. These two shapes are shown in Fig. 4. The anisotropic phase transition is manifest in the conical singularity developing at $\gamma = 0$.\footnote{We note a potential point of confusion, namely that a naive application of Eq. (78) would seem to indicate that the curvature is a constant $K = 4$ in the ferromagnetic phase for $\gamma > 0$, in which case the singularity at $\gamma = 0$ is not apparent. However, a more careful derivation shows that the curvature is indeed singular at $\gamma = 0$: $K = 4 - 8(1 - \gamma) \frac{\partial^2}{\partial \gamma^2} |\gamma| = 4 - 16\delta(\gamma)$, where $\delta(\gamma)$ is the Dirac delta function.}

The singularity at $\gamma = 0$ yields a non-trivial bulk Euler integral for the anisotropic phase transition. To see this, consider the bulk integral

$$
\xi_{\text{bulk}}(\epsilon) = \lim_{L \to \infty} \int_0^{2\pi} d\phi \int_0^\infty d\gamma \sqrt{g(\gamma, \phi)} K(\gamma, \phi). \quad (99)
$$

In the limit $\epsilon \to 0^+$, this integral has a discontinuity as a function of $h$ at the phase transition, as seen in Fig. 4. Thus, $\xi_{\text{bulk}} \equiv \xi_{\text{bulk}}(\epsilon = 0^+)$ can be used as a geometric characteristic of the anisotropic phase transition. Direct calculation shows that $\xi_{\text{bulk}} = 1/\sqrt{2}$ in the ferromagnetic phase and $\xi_{\text{bulk}} = 1$ in the paramagnetic phase. This non-integer geometric invariant is due to the existence of a conical singularity.

A careful analysis shows that in both cases the bulk Euler characteristics are protected by the universality of the transition. I.e. if one adds extra terms to the Hamiltonian, which do not qualitatively affect the phase diagram the bulk Euler characteristic does not change. The details of the proof are available in Ref. (Kolodrubetz et al., 2013). But the basic idea is very simple. The sum of the bulk and the boundary Euler characteristics is protected by the geometry of the parameter manifold. As long as the boundary of the manifold coincides with the phase boundary all components of the metric tensor become universal (Campos Venuti and Zanardi, 2007). Therefore it is not surprising that the geodesic curvature also becomes universal and thus the boundary Euler characteristic is protected. As a result the bulk Euler characteristic is protected too. It is interesting that unlike critical exponents the bulk Euler characteristic truly characterizes the phase transition and does not depend on the parametrization. One can also analyze the Euler
FIG. 4 (insets) Equivalent graphical representation of the phase diagram of the XY model in the $\gamma - \phi$ plane, where $\gamma \in [0, \infty)$ and $\phi \in [0, 2\pi]$. The right inset shows the paramagnetic disordered phase and the left inset represents the ferromagnetic phase. It is clear that in the latter case there is a conical singularity developing at $\gamma = 0$ which represents the anisotropic phase transition. The plots show bulk Euler integral $\xi_{\text{bulk}}(\epsilon)$, demonstrating the jump in $\xi_{\text{bulk}}$ at the phase transition between the paramagnet and ferromagnet in the limit $\epsilon \to 0^+$. 

characteristic and the Gauss curvature of the in the $h - \gamma$ plane. One finds additional nonintegrable curvature singularities near the anisotropic phase transition and near the multi-critical point.

IV. GAUGE POTENTIALS AND NON-ADIABATIC RESPONSE.

A. Dynamical Quantum Hall effect.

We already noted in the first section that the gauge potentials appear in the Galilean term in the moving Hamiltonian:

$$H_{\text{eff}} = U^\dagger H U - \dot{\lambda}_\alpha A_\alpha = \tilde{H} - \dot{\lambda}_\alpha A_\alpha.$$  \hspace{1cm} (100)

The Hamiltonian $\tilde{H}$ is diagonal and thus only produces shifts in the energies but does not couple them (i.e. does not have any off-diagonal matrix elements) so it is not responsible for the transitions between levels. Conversely the Galilean term is in general off-diagonal and thus causes the transitions between levels. Near the adiabatic limit the Galilean term is small and thus can be treated as a perturbation.
Let us first consider the setup where the system is initially prepared in equilibrium (for concreteness in the ground state) at some initial value of the coupling $\vec{\lambda}_0 \equiv \vec{\lambda}(t = 0)$. Then the coupling starts changing in time. To avoid need of worrying about initial transients, which can be done but makes derivations more involved, we will assume that the rate of change of the coupling is a smooth function of time. Under this smooth transformation in the leading order in $\dot{\lambda}$ the system follows the ground state of the moving Hamiltonian $H_{\text{eff}}$. One can worry whether the adiabatic theorem applies to this Hamiltonian, which is still time-dependent. Later we will give a rigorous derivation of the result (alternatively see Ref. (Gritsev and Polkovnikov, 2012) for more details). Loosely speaking the Galilean term removes the leading non-adiabatic contribution from the Hamiltonian. The higher order corrections will be of the higher order in $|\dot{\lambda}|$. When we do the measurement on the system, on the other hand, we do it in the lab frame. One can thus view the instantaneous measurement process as sudden quench where the rate $\dot{\lambda}$ is quenched to zero. Of course there is no real quench but all measurements are done in the fixed basis where the Galilean term is absent.

Within the first order of perturbation theory the transition amplitudes to the excited states due to such quench are given by

$$a_n = -\dot{\lambda}_\alpha \frac{\langle n|\mathcal{A}_\alpha|0\rangle}{E_n - E_0}$$

Next we compute the leading order correction to the observables due to this quench. It is convenient to represent observables as generalized forces conjugate to some other coupling $\lambda_\beta$:

$$M_\beta = -\partial_\beta H.$$  \hspace{1cm} (102)

Matrix elements of these objects already appeared in the definition of the geometric tensor so it is convenient to continue dealing with them. Note that any observable can be represented as some generalized force. E.g. the magnetization is conjugate to the magnetic field, current is conjugate to a vector potential etc. Then we find that

$$M_\beta = \langle \psi|\mathcal{M}_\beta|\psi\rangle \approx M_\beta^{(0)} - \dot{\lambda}_\alpha \sum_{n \neq 0} (a_n^* \langle n|\partial_\beta H|0\rangle + a_n \langle 0|\partial_\beta H|n\rangle) = M_\beta^{(0)} +$$

$$i \dot{\lambda}_\alpha \sum_{n \neq 0} \frac{\langle 0|\partial_\beta H|n\rangle \langle n|\partial_\beta H|0\rangle - \langle 0|\partial_\beta H|n\rangle \langle n|\partial_\alpha |0\rangle}{E_n - E_0} = M_\beta^{(0)} + F_{\beta\alpha} \dot{\lambda}_\alpha,$$  \hspace{1cm} (103)

where $M_\beta^{(0)}$ is the adiabatic generalized force evaluated in the instantaneous ground state. The last equality immediately follows from Eq. (68). The equation above constitutes the dynamical Quantum Hall effect. It shows that the leading non-adiabatic (Kubo) correction to the generalized force comes from the product of the Berry curvature and the rate of change of the parameter $\vec{\lambda}$. 


Let us illustrate that this relation reduces to the standard integer quantum Hall effect. To show this we will make two assumptions: (i) the ground state of the system is not-degenerate (degeneracies can lead to the fractional QHE) and (ii) the Hamiltonian of the system can be represented in the form

\[ \mathcal{H} = \sum_{j=1}^{N} \left( \frac{\vec{p}_j - \frac{e}{c} \vec{A}_j}{2m_j} \right)^2 + V(\vec{r}_1, \vec{r}_2, \ldots \vec{r}_N), \]  

(104)

where \( V \) is arbitrary momentum independent potential which can include both interactions between particles and the external potential. We remind that we use the \( \vec{A} \) notation for the vector potential to avoid the confusion with the gauge potential. Let us assume that the vector potential consists of some static part representing a static magnetic field and extra dynamic part representing electric field in the system (recall that in the Coulomb gauge \( \vec{E} = \frac{1}{c} \partial_t \vec{A} \)). We will choose the components of the time-dependent vector potential (divided by speed of light) as our parameters, i.e.

\[ \lambda_x = \frac{1}{c} A_x, \quad \lambda_y = \frac{1}{c} A_y. \]  

(105)

Note that the generalized force with respect to \( \lambda_y \) is

\[ \mathcal{M}_y = -\partial_{\lambda_y} \mathcal{H} = \sum_j \frac{e}{m_j} \left( p_j^{(y)} - \frac{e}{c} A_j^{(y)} \right) = \mathcal{J}_y, \]  

(106)

which is the current operator along the \( y \)-direction. Clearly in the absence of the electric field there is no current so the dynamical Hall relation reads

\[ J_y = h F_{\lambda_x, \lambda_y} E_x, \]  

(107)

where we explicitly included the factor of \( h \) omitted in Eq. (103). This factor must be there to fix the dimensions, or alternatively can be obtained by noting that the gauge potentials are actually \( A_\alpha = i\hbar \partial_\alpha \). To find the Hall conductivity we note that the total current is related to the two-dimensional current density via

\[ J_y = L_x L_y j_y, \]  

(108)

where \( L_x \) and \( L_y \) are the dimensions of the sample. Therefore the Hall conductivity \( \sigma_{yx} = j_y / E_x \) is related to the Berry curvature via

\[ \sigma_{xy} = \frac{h F_{\lambda_x, \lambda_y}}{L_x L_y}. \]  

(109)

Now let us use the fact that the Chern number related to the integral of the Berry curvature over the closed manifold is an integer. Because of the assumed form of the Hamiltonian the uniform
vector potential can be gauged away by absorbing it into the phase of the wave function like in the Aharonov-Bohm geometry. This is possible because the spatially uniform vector potential corresponds to the zero magnetic field. Thus the acquired phase by the wave-function is

\[ \phi = \frac{e}{\hbar} \int (\Lambda_x dx + \Lambda_y dy) = e(\lambda_x L_x + \lambda_y L_y) \]  

(110)

Clearly this phase acquires winding of 2\(\pi\) if \(\lambda_x\) changes between 0 and \(2\pi \hbar/(e L_x)\) and similarly for \(\lambda_y\). So the parameter manifold is effectively a torus of the area \(4\pi^2 \hbar^2/(e^2 L_x L_y)\). Because the Berry curvature is independent of the vector potential (since it only affects the overall phase of the wave function) the Chern number is just the product of the Berry curvature and the area divided by \(2\pi\)

\[ \text{ch}_1 = \frac{1}{2\pi} F_{\lambda_x \lambda_y} \frac{4\pi^2 \hbar^2}{e^2 L_x L_y} \]  

(111)

or equivalently

\[ F_{\lambda_x \lambda_y} = \frac{e^2 L_x L_y}{2\pi \hbar^2} \text{ch}_1. \]  

(112)

Combining this with the expression for the Hall conductivity we find

\[ \sigma_{xy} = \frac{\hbar}{L_x L_y} \frac{e^2 L_x L_y}{2\pi \hbar^2} \text{ch}_1 = \frac{e^2}{2\pi \hbar} \text{ch}_1, \]  

(113)

which precisely constitutes the integer quantum Hall effect. This simple derivation highlights that the quantum Hall effect can be thought of as the linear non-adiabatic response in the system. Note that this derivation does not require assuming any particular boundary conditions. We relied on the periodicity of the wave function in the vector-potential space not real space.

The second example we discuss is again a spin in a time-dependent magnetic field. Namely we will assume that the parameters \(\theta\) and \(\phi\) characterizing the angles of the external magnetic field change in time. Suppose that the spin is prepared in the ground state along the magnetic field and then the latter starts changing in time e.g. along the \(\theta\)-direction. The generalized force along the orthogonal \(\phi\)-direction is just the magnetization. In the adiabatic limit it is clearly zero since in this case the magnetization simply follows the magnetic field. The leading non-adiabatic correction is then given by the Berry curvature:

\[ m_\phi \approx F_{\phi \theta} \dot{\theta}. \]  

(114)

Likewise

\[ m_\theta \approx F_{\theta \phi} \dot{\phi}. \]  

(115)
in experiments it is easier to measure the magnetization along \( y \)-direction. Then if the magnetic field has a time dependent \( x \)-component and a time independent \( z \)-component we have

\[
m_y = F_{yx} \dot{h}_x, \tag{116}
\]

where

\[
F_{yx} = \frac{1}{2\hbar^2} \cos(\theta). \tag{117}
\]

From this it is easy to recover \( F_{\theta\phi} \) by a standard transformation to the spherical coordinates:

\[
F_{\theta\phi} = \hbar^2 F_{yx} \tan(\theta) = \frac{1}{2} \sin(\theta). \tag{118}
\]

In Fig. 5 we show numerically computed dependence of the transverse \( y \)-magnetization on the rate of change of the magnetic field \( v \) for a particular protocol

\[
\mathcal{H} = -\sigma_z - h_x(t)\sigma_x, \tag{119}
\]

where \( h_x(t) = 0.5 + vt \). The transverse magnetization is computed at time \( t = 0 \) and the initial condition corresponds to a ground state at large negative time \( t = -100/v \). As it is evident from

![Graph](image)

**FIG. 5** Dependence of the transverse magnetization on the rate of change of the magnetic field along the \( x \)-direction (see text for details)

the figure at slow rates dependence of the transverse magnetization on the rate is linear and the slope is exactly given by the Berry curvature. Integrating the Berry curvature over the angles of the field one can measure the Chern number.
It is interesting that even within such a simple system one can already observe the topological phase transition where the Chern number changes from 0 to 1. For this we can consider a slight modification into the Hamiltonian by adding a constant static magnetic field along the $z$ direction.

$$\mathcal{H} = \frac{1}{2} [h_0 \sigma_z + h_1 \cos(\theta) \sigma^x + h_1 \sin(\theta) \cos(\phi) \sigma^y + h_1 \sin(\theta) \sin(\phi) \sigma^y].$$

Then as one changes the magnetic field $h_1$ along the sphere of constant radius at fixed $h_0$ we can have two different scenarios. First when $h_0 < h_1$ corresponds to the total magnetic field encircling the origin $h = 0$ and thus producing the Chern number equal to one. The second scenario is realized when $h_0 > h_1$. Then the total magnetic field does not enclose the origin and the Chern number is zero. The easiest way to see this is to take the limit $h_1 \to 0$ and recall that the Chern number can not change unless the surface crosses a gapless crossing point. This phase transition was recently observed in experiments on superconducting qubits (M. Schroer et. al., unpublished).

Recall that the Chern number is the total flux of the Berry curvature (magnetic field) and it only depends on whether the gapless crossing point $h = 0$ is enclosed by the integration surface or not. This situation is very similar to electrostatics, where the flux of electric field through a closed surface only depends on the total charge inside this surface. Thus this crossing point plays the role of the magnetic monopole. Monopoles appear in more complicated systems and can not simply disappear under the small perturbations (because the Chern number is protected in any gapped phase). Thus contrary to the common wisdom the level crossing points are protected and if one introduces some generic perturbation they just move around in the parameter space.

This simple transition is equivalent to the transition from the normal to the topological insulator realized e.g. in the Haldane model. We will not go into the details of this model, we only point that the model represents a single particle band with two sublattices. As a result at each Bloch momentum $\vec{k}$ the Hamiltonian is effectively described by the $2 \times 2$ matrix, which is equivalent to spin one half system in a magnetic field, which depends on $\vec{k}$. One defines the Chern number of the band as

$$\frac{1}{2\pi} \int \int dk_x dk_y F_{k_x,k_y},$$

where

$$F_{k_x,k_y} = i(\langle \partial_{k_x} \psi | \partial_{k_y} \psi \rangle - \langle \partial_{k_y} \psi | \partial_{k_x} \psi \rangle).$$

Because the Brillouin zone is a closed manifold (equivalent to torus) this band Chern number is either zero or one depending on the parameters of the Hamiltonian. The transition between the
FIG. 6 Schematic representation of the mapping of the effective magnetic field in the Haldane model to the spin one half system. In the topologically nontrivial phase the magnetic field wraps around the sphere

bands with different Chern numbers is called topological phase transition. In Fig. 6 we pictorially show how the mapping of the Haldane model to the spin one half particle in external field works. In the topologically non-trivial phase the effective magnetic field wraps around the sphere, which in our language corresponds to $h_1 > h_0$ and vice versa. It is interesting that we were able to define the topological phase transition for just one spin. This works because we consider not one ground state but rather the whole manifold of ground states, which maps to a completely filled band of an insulator.

We used a simple single-particle problem as an illustration. The situation becomes much more interesting if we consider interacting systems. In particular, following Ref. (Gritsev and Polkovnikov, 2012) we quote the numerical results for the Chern number computed through the non-adiabatic response for a disordered spin chain:

$$\mathcal{H} = -\hbar \sum_{j=1}^{N} \zeta_j \vec{\sigma}_j - J \sum_{j=1}^{N-1} \eta_j \vec{\sigma}_j \vec{\sigma}_{j+1},$$

(122)

where $\zeta_j$ and $\eta_j$ are drawn from a uniform distribution in the interval $[0.75, 1.25]$. We fix the $|\vec{h}| = 1$ and look into the Berry curvature associated with angles of the magnetic field $\theta$ and $\phi$ as a function of $J$ (see Fig. 7). Because of the $SU(2)$ invariance of the system like for a single spin the Chern number and the Berry curvature are simply different by a factor of 2. At large negative $J$
the system minimizes the total spin to $S = 1/2$ the Berry curvature is also $1/2$ corresponding to the Chern number equal to one. At small $J$ the system becomes polarized and the Chern number is 9. In between the Berry curvature and thus the Chern number changes in steps. If one breaks the $SU(2)$ invariance considering e.g. anisotropic interactions the quantization of the Berry curvature disappears while the Chern number remains quantized. The minimal model for observing this is a two-spin system, which was recently realized experimentally also using superconducting qubits (P. Roushan et. al., unpublished).

**B. General approach to non-adiabatic response. Emergent Newtonian dynamics.**

In this section we extend the analysis of a previous section to a more general class of systems, which are not necessarily in a ground state and which might have gapless excitations. This chapter closely follows Ref. (D’Alessio and Polkovnikov, 2014). Our starting point will be von Neumann’s equation for the density matrix in the moving frame

$$i\frac{d\rho}{dt} = \left[H - \lambda_\nu A_\nu, \rho\right],$$
where we remind that $\mathcal{H}$ is the diagonal Hamiltonian in the instantaneous basis, similarly all other observables appearing here are also rotated into the instantaneous frame. We will then use the standard time-dependent perturbation theory (Kubo formalism), where the Galilean term plays a role of the perturbation. Namely we will go to the interaction picture (the Heisenberg representation with respect to $\tilde{\mathcal{H}}$), where the von Neumann’s equation becomes

$$i \frac{d\rho_H}{dt} = -\dot{\lambda}_\nu [A_{H,\nu}(t), \rho_H], \quad (123)$$

which is equivalent to the integral equation

$$\rho_H(t) = \rho_H(0) + i \int_0^t dt' \dot{\lambda}_\nu(t') [A_{H,\nu}(t'), \rho_H(t')] \quad (124)$$

Note that if the moving Hamiltonian $H$ is time dependent if the spectrum explicitly depends on $\vec{\lambda}$. However, this dependence is trivial because it amounts to phase factors $\phi_n = \int_0^t E_n(t') dt'$ instead of $\phi_n = E_n t$. For example

$$\langle \tilde{n} | O_H(\lambda(t)) | \tilde{m} \rangle = \exp \left[ i \int_0^t dt' (E_n(t') - E_m(t')) \right] \langle \tilde{n} | O(\lambda(t)) | \tilde{m} \rangle, \quad (125)$$

where $|\tilde{n}\rangle$ denotes the co-moving basis states. We emphasize that this expression is not the same as the Heisenberg representation with respect to the original Hamiltonian $H(\lambda(t))$. The representation we use is perhaps more correctly termed as the adiabatic Heisenberg representation since it uses adiabatic energy levels $\epsilon_n(t)$, while all the transitions (off-diagonal terms) are treated perturbatively. Clearly if $\epsilon_n$ is time independent the expression (125) reduces to the conventional Heisenberg representation of the operator $O(\lambda(t))$. In the leading order in perturbation theory we can substitute the stationary density matrix into the R.H.S. of the integral equation (124):

$$\rho_H(t) = \rho_H(0) + i \int_0^t dt' \dot{\lambda}_\nu(t') [A_{H,\nu}(t'), \rho_0], \quad (126)$$

where we used $\rho_H^0 = \rho^0$. From this we can find the linear response correction to the generalized forces:

$$\langle M_\nu(t) \rangle = M_\nu(t) + i \int_0^t dt' \dot{\lambda}_\mu(t') \langle [M_{H,\nu}(t), A_{H,\mu}(t')] \rangle_0.$$
where we recall that by definitions \( \mathcal{M}_\nu(t) \equiv \langle \mathcal{M}_\nu(t) \rangle_0 \). Evaluating this expression in the co-moving basis and using Eqs. (42) and (68) we arrive at

\[
\langle \mathcal{M}_\nu(t) \rangle = M_\nu(t) - \int_0^t dt' \lambda_\mu(t') \sum_{n \neq m} \frac{\rho_n^0 - \rho_m^0}{E_m - E_n} e^{i(E_m - E_n)(t - t')} \langle \tilde{m}_i | \mathcal{M}_\nu(\lambda(t)) | \tilde{n} \rangle \langle \tilde{n} | \mathcal{M}_\mu(\lambda(t')) | \tilde{m} \rangle
\]

\[
= M_\nu(t) - \int_0^t dt' \lambda_\mu(t') \sum_{n \neq m} \frac{\rho_n^0 - \rho_m^0}{E_m - E_n} \langle \tilde{m}_i | \mathcal{M}_{H,\nu}(t) | \tilde{n} \rangle \langle \tilde{n} | \mathcal{M}_{H,\mu}(t') | \tilde{m} \rangle
\]

\[
= M_\nu(t) - \int_0^t dt' \int_0^\beta d\tau \lambda_\mu(t') \sum_{n \neq m} \rho_m^0 \langle \tilde{m}_i | \mathcal{M}_{H,\nu}(t) | \tilde{n} \rangle \langle \tilde{n} | \mathcal{M}_{H,\mu}(t' + i\tau) | \tilde{m} \rangle
\]

\[
= M_\nu(t) - \int_0^t dt' \int_0^\beta d\tau \lambda_\mu(t') \langle \mathcal{M}_{H,\nu}(t) \mathcal{M}_{\mu}(t' + i\tau) \rangle_0,
\]

which gives the microscopic force in the most general form. To go from the second to the third line in this equation we used that for the thermal distribution, \( \rho_m^0 = Z^{-1} \exp[-\beta E_m] \), we have

\[
\int_0^\beta d\tau \rho_m^0 e^{-\langle E_n - E_m \rangle \tau} = \frac{\rho_n^0 - \rho_m^0}{E_m - E_n}
\]

It is convenient to change the integration variable in Eq. (127) from \( t' \) to \( t - t' \) and use the fact that in equilibrium the non-equal time correlation functions are translationally invariant in time. Then

\[
\langle \mathcal{M}_\nu(t) \rangle \approx M_\nu(t) - \int_0^t dt' \int_0^\beta d\tau \lambda_\mu(t - t') \langle \mathcal{M}_{H,\nu}(t') \mathcal{M}_{\mu}(i\tau) \rangle_0 \approx
\]

\[
M_\nu - \int_0^t dt' \lambda_\mu(t - t') \sum_{n \neq m} \rho_n^0 - \rho_m^0 e^{i(E_n - E_m) t'} \langle m_\lambda | \mathcal{M}_\nu | n_\lambda \rangle \langle n_\lambda | \mathcal{M}_{\mu} | m_\lambda \rangle
\]

Now we will use the time scale separation. Namely we recall that by the assumption \( \tilde{\lambda} \) represents slow variables in the system. Mathematically this statement implies that the non-equal time correlation function of the generalized forces \( \langle \mathcal{M}_{H,\nu}(t') \mathcal{M}_{\mu}(i\tau) \rangle_0 \) decays much faster than the characteristic rate of change of \( \tilde{\lambda}(t) \). Because we are interested in long time dynamics of the system we can extend the integration limit over \( t' \) to \( \infty \). Effectively this corresponds to neglecting short time transients. Because of the time scale separation it is natural to expand \( \lambda_\mu(t - t') \approx \hat{\lambda}_\mu(t) - t' \hat{\lambda}_\mu(t) + \ldots \) As we will see shortly it is important to keep the first two terms in this expansion and all other terms, in most cases, describe unessential subleading corrections. Then we find

\[
\langle \mathcal{M}_\nu(t) \rangle = M_\nu + F_{\nu\mu} \hat{\lambda}_\mu - \eta_{\nu\mu} \hat{\dot{\lambda}}_\mu - \kappa_{\nu\mu} \hat{\ddot{\lambda}}_\mu - F'_{\nu\mu} \hat{\dddot{\lambda}}_\mu,
\]

\[\text{An important exception is a motion of a charged object in a vacuum, where the friction force is proportional to the third derivative of the coordinate. In this case one has to add the next term to this expansion.}\]
where we split the coefficients in front of $\dot{\lambda}_{\mu}$ and $\ddot{\lambda}_{\mu}$ into symmetric $\eta_{\nu\mu}$, $\kappa_{\nu\mu}$ and anti-symmetric $F_{\nu\mu}$ and $F'_{\nu\mu}$ components.

To find the explicit expressions for these coefficients we will use the identities

$$\int_0^\infty \exp[i(E_m - E_n)t']dt' = -iP \left( \frac{1}{E_m - E_n} \right) + \pi \delta(E_n - E_m),$$

(131)

where $P$ stands for the principal value. Note that the first Principal value term is antisymmetric under the permutation of indexes $n$ and $m$, while the second is symmetric. Because, as it is evident from Eq. (129), the permutation of $n$ and $m$ is equivalent to the permutation of $\nu$ and $\mu$ we see that the principal value determines the antisymmetric coefficient $F_{\mu\nu}$ and the second, symmetric term determines $\eta_{\nu\mu}$. Therefore

$$F_{\mu\nu} = -i \sum_{n \neq m} \frac{\rho_0^m - \rho_0^n}{(E_m - E_n)^2} \langle m_{\lambda} | M_{\nu} | n_{\lambda} \rangle \langle n_{\lambda} | M_{\mu} | m_{\lambda} \rangle$$

$$= i \sum_{n \neq m} \rho_0^n \langle m_{\lambda} | M_{\nu} | n_{\lambda} \rangle \langle n_{\lambda} | M_{\mu} | m_{\lambda} \rangle - \langle m_{\lambda} | M_{\mu} | n_{\lambda} \rangle \langle n_{\lambda} | M_{\nu} | m_{\lambda} \rangle \frac{1}{(E_n - E_m)^2}. \tag{132}$$

If we compare this expression with Eq. (69) we will recognize that $F_{\mu\nu}$ is just the thermal average of the Berry curvature over the energy eigenstates. This justifies that we use the same notation $F_{\mu\nu}$ here. At zero temperature this expression obviously reduces to the ground state Berry curvature. Similarly

$$\eta_{\nu\mu} = \pi \beta \sum_{n \neq m} \rho_0^n \langle m_{\lambda} | M_{\nu} | n_{\lambda} \rangle \langle n_{\lambda} | M_{\mu} | m_{\lambda} \rangle \delta(E_n - E_m),$$

where we used that

$$\frac{\rho_0^m - \rho_0^n}{E_m - E_n} \to \rho_0^n \beta$$

(133)

when $E_m \to E_n$. As we will see shortly $\eta_{\nu\mu}$ represents the friction force on the system. It is non-zero only if the system has gapless excitation. In particular, at zero temperature, unless the system is quantum critical, the friction coefficient is always zero because one can not satisfy the $\delta$ function constraint.

In the similar spirit one can derive the other two coefficients. Now we will use that

$$- \int_0^\infty t' \exp[i(E_m - E_n)t']dt' = \frac{1}{(E_n - E_m)^2} - \pi \delta'(E_n - E_m), \tag{134}$$

which can be checked by adding small imaginary part into the integral and sending it to zero. Now the off-shell term is symmetric, while the on-shell term is antisymmetric. The first off-shell term
defines the coefficient $\kappa_{\nu\mu}$, which as we will see shortly determines the mass renormalization

$$\kappa_{\nu\mu} = \sum_{n \neq m} \frac{\rho_n^0 - \rho_m^0}{(\epsilon_m - \epsilon_n)^3} \langle m_\lambda | M_\nu | n_\lambda \rangle \langle n_\lambda | M_\mu | m_\lambda \rangle = \sum_{n \neq m} \frac{\rho_n^0 - \rho_m^0}{(\epsilon_m - \epsilon_n)^3} \langle m_\lambda | \overleftarrow{\partial_{\lambda_\nu}} | n_\lambda \rangle \langle n_\lambda | \overrightarrow{\partial_{\lambda_\mu}} | m_\lambda \rangle \tag{135}$$

At low temperatures $\beta \to \infty$ this expression reduces to

$$\kappa_{\nu\mu} \approx \sum_{m \neq 0} \frac{(\langle 0_\lambda | M_\nu | m_\lambda \rangle \langle m_\lambda | M_\mu | 0_\lambda \rangle + \nu \leftrightarrow \mu)}{(E_m - E_0)^3}. \tag{136}$$

while at high temperatures (or near the classical limit) we find

$$\kappa_{\nu\mu} \approx \frac{\beta}{2} \sum_{m \neq 0} \rho_m \left( \langle m_\lambda | \overleftarrow{\partial_{\lambda_\nu}} \overrightarrow{\partial_{\lambda_\mu}} | m_\lambda \rangle_c + \nu \leftrightarrow \mu \right) = \beta g_{\nu\mu} \tag{137}$$

where $g_{\nu\mu}$ is the Fubini-Study metric tensor of energy eigenstates weighted with the thermal distribution. This is a natural finite temperature generalization of the zero-temperature metric tensor.

Let us mention that in traditional units the expressions for the mass Eqs. (135) - (137) should be multiplied by $\hbar^2$, which follows from the correct definition of the Gauge potentials $i \partial_{\lambda_\mu} \rightarrow i \hbar \partial_{\lambda_\mu}$.

We also point that the mass tensor $\kappa$ can be written through the integrated connected imaginary time correlation function of the gauge potentials $A_\nu$ and $A_\mu$:

$$\kappa_{\nu\mu} = \frac{1}{2} \int_0^\beta d\tau \langle A_{H,\nu}(\tau) A_{H,\mu}(0) + \nu \leftrightarrow \mu \rangle. \tag{138}$$

At high temperatures the imaginary time integral reduces to a factor $\beta$ and this result clearly reduces to Eq. (137). And finally let us quote the expression for the antisymmetric tensor $F'$:

$$F'_{\nu\mu} = i\pi \sum_{n \neq m} \frac{\rho_n^0 - \rho_m^0}{E_n - E_m} \langle m_\lambda | M_\nu | n_\lambda \rangle \langle n_\lambda | M_\mu | m_\lambda \rangle \delta'(E_n - E_m) \tag{139}$$

As $\eta$ this tensor is responsible for dissipation, but usually it is subleading and we will not discuss it further. We only point that as $F$ this tensor is always zero if the instantaneous Hamiltonian respects time-reversal symmetry.

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