Lecture: Exactly Solvable Models in Systems of Cold Bose Atoms

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2. Solution of quantum many body problem by Bethe Ansatz.

3. Physical properties derived from the Bethe Ansatz. Different limits.


5. Effective description of 1D systems. Bosonic Luttinger liquids.

The aim of these notes is to discuss the exact solvability of the 1D Bose gas with a delta-function interaction of arbitrary strength by Bethe Ansatz and to demonstrate its direct realisability in experimental systems of cold atoms. Also we discuss a connection of properties of 3D Bose gas with that of a gas confined in 1D and briefly describe an effective description on the basis of bosonic Luttinger liquids.
1. The 1D Bose gas is described by the canonical quantum Bose fields $\Psi(x, t)$ with canonical equal-time commutation relations:

\[
[\Psi(x, t), \Psi^\dagger(y, t)] = \delta(x - y) \quad (1)
\]
\[
[\Psi(x, t), \Psi(y, t)] = [\Psi^\dagger(x, t), \Psi^\dagger(y, t)] = 0. \quad (2)
\]

We consider equal-time quantities in the following and neglect the index $t$. The Hamiltonian of the model is

\[
H = \int dx \left[ \partial_x \Psi^\dagger(x) \partial_x \Psi(x) + c \Psi^\dagger(x) \Psi(x) \Psi(x) \Psi(x) \right] \quad (3)
\]

where $c$ is a coupling constant. The corresponding equation of motion

\[
i\partial_t \Psi = -\partial_x^2 \Psi + 2c \Psi^\dagger(x) \Psi(x) \Psi(x) \quad (4)
\]

is called the nonlinear Schrödinger equation. The Fock vacuum state $|0\rangle$ is defined by

\[
|0\rangle = \Psi(x) |0\rangle = 0.
\]

The state $|0\rangle$ is called pseudovacuum and it is to be distinguished from the physical vacuum, which is the ground state of the interacting system (Hamiltonian 3). Also

\[
\langle 0 | \Psi^\dagger(x) = 0, \quad \langle 0 | 0 \rangle = 1 \quad (6)
\]

The number of particles operator $N$ and momentum operator $P$ are

\[
N = \int \Psi^\dagger(x) \Psi(x) dx \quad (7)
\]
\[
P = -\frac{i}{2} \int (\Psi^\dagger(x) \partial_x \Psi(x) - [\partial_x \Psi^\dagger(x)] \Psi(x)) dx. \quad (8)
\]

These operator are integrals of motion (i.e. commute with the Hamiltonian (3))

\[
[H, N] = [H, P] = 0 \quad (9)
\]

Now, we can look for the common eigenfunctions $|\psi_N\rangle$ of operators $H, P, N$:

\[
\frac{1}{\sqrt{N!}} \int d^N x \chi_N(x_1, x_2, ..., x_N; \lambda_1, \lambda_2, ..., \lambda_N) \Psi^\dagger(x_1) \Psi^\dagger(x_2) ... \Psi^\dagger(x_N) |0\rangle = \langle \psi_N(\lambda_1, \lambda_2, ..., \lambda_N) \rangle = 0. \quad (10)
\]

These eigenfunctions satisfy

\[
\langle \psi_N(\lambda_1, \lambda_2, ..., \lambda_N) | \psi_M(\lambda_1', \lambda_2', ..., \lambda_M') \rangle = \delta_{NM} \delta(\lambda_1 - \lambda_1', ...) \quad (11)
\]
Here $\chi_N$ is a symmetric function of all $x_j$. The eigenvalue equation,

$$H|\psi_N\rangle = E_N|\psi_N\rangle, \quad P|\psi_N\rangle = p_N|\psi_N\rangle, \quad \hat{N}|\psi_N\rangle = N|\psi_N\rangle \quad (12)$$

result in the fact that $\chi_N$ is an eigenfunction of both quantum-mechanical Hamiltonian $\mathcal{H}_N$ and momentum $\mathcal{P}_N$

$$\mathcal{H}_N = \sum_{i=1}^{N} -\frac{\partial^2}{\partial x_i^2} + 2c \sum_{1 \leq k < j \leq N} \delta(x_k - x_j) \quad (13)$$

$$\mathcal{P}_N = \sum_{i=1}^{N} (-i \frac{\partial}{\partial x_i}) \quad (14)$$

$$\mathcal{H}_N \chi_N = E_N \chi_N \quad (15)$$

To demonstrate the formulas above lets take as an example the expression for $P$, Eq.(8) and integrate it by parts to get

$$P = i \int (\partial_x \Psi^\dagger(x)) \Psi(x) dx \quad (16)$$

Acting with this operator on the eigenfunction (10) gives

$$P|\psi_N(\lambda_1, \lambda_2, \ldots \lambda_N)\rangle = \frac{i}{\sqrt{N!}} \int d^N x \chi_N(x_1, x_2, \ldots, x_N; \lambda_1, \lambda_2, \ldots \lambda_N)[\partial_x \Psi^\dagger(x)] \times \sum_{k=1}^{N} \Psi^\dagger(x_1)\ldots[\Psi(x), \Psi^\dagger(x_k)]\ldots\Psi^\dagger(x_N)|0\rangle \quad (17)$$

where Eq.(5) was used. Now, Eq.(1) gives a $\delta$-function for the commutator which can be then integrated out to give

$$P|\psi_N(\lambda_1, \lambda_2, \ldots \lambda_N)\rangle = \frac{i}{\sqrt{N!}} \int d^N x \chi_N(x_1, x_2, \ldots, x_N; \lambda_1, \lambda_2, \ldots \lambda_N) \sum_{k=1}^{N} \Psi^\dagger(x_1) \frac{\partial}{\partial x_k} \Psi^\dagger(x_k \ldots \Psi^\dagger(x_N)|0\rangle \quad (18)$$

Now, we integrate by parts with respect to $x_k$ to get

$$P|\psi_N(\lambda_1, \lambda_2, \ldots \lambda_N)\rangle = \frac{1}{\sqrt{N!}} \int d^N x (-i \sum_{k=1}^{N} \frac{\partial}{\partial x_k} \chi_N(x_1, x_2, \ldots, x_N; \lambda_1, \lambda_2, \ldots \lambda_N)) \Psi^\dagger(x_1)\ldots\Psi^\dagger(x_N)|0\rangle \quad (19)$$

$$\text{Now, we integrate by parts with respect to } x_k \text{ to get}$$

$$P|\psi_N(\lambda_1, \lambda_2, \ldots \lambda_N)\rangle = \frac{1}{\sqrt{N!}} \int d^N x (-i \sum_{k=1}^{N} \frac{\partial}{\partial x_k} \chi_N(x_1, x_2, \ldots, x_N; \lambda_1, \lambda_2, \ldots \lambda_N)) \Psi^\dagger(x_1)\ldots\Psi^\dagger(x_N)|0\rangle \quad (20)$$
Thus the action of operator (8) on the state (10) is equivalent to the action of $P_N$ on $\chi_N$. The construction of quantum-mechanical Hamiltonian (13) is similar.

2. So the quantum field theory problem is reduced to a quantum-mechanical problem. The Hamiltonian $\mathcal{H}_N$ describing $N$ interacting Bose particles is repulsive for $c > 0$. Due to the symmetry of $\chi_N$ in all the $x_k$ variables it is sufficient to consider the following domain $R$ in the coordinate space:

$$R : 0 \leq x_1 \leq x_2 \leq \ldots \leq x_N \leq L$$  \hspace{1cm} (22)$$

which we call the fundamental region. Here $L$ is the size of the system. Inside this domain the function $\chi_N$ is an eigenfunction of the free Hamiltonian

$$\mathcal{H}_N^0 = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2}, \quad \mathcal{H}_N^0 \chi_N = E_N \chi_N.$$  \hspace{1cm} (23)$$

To get the condition on the wave function at the internal boundary inside the fundamental region we integrate the equation (13) over the variable $x_{j+1} - x_j$ (which is the center of mass coordinate for two adjacent variables) in the small vicinity $|x_{j+1} - x_j| < \epsilon \rightarrow 0$, considering all other $x_k \ (k \neq j, j+1)$ to be fixed in $R$. The result is

$$\left( \frac{\partial}{\partial x_{j+1}} - \frac{\partial}{\partial x_j} - c \right) \chi_N = 0, \quad x_{j+1} = x_j + 0$$  \hspace{1cm} (24)$$

Therefore the original Schrödinger equation (13) is now replaced by

$$\mathcal{H}_N^0 \chi_N = E_N \chi_N, \quad \mathcal{H}_N^0 = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2}$$  \hspace{1cm} \text{inside} \ R$$  \hspace{1cm} (25)$$

$$\left( \frac{\partial}{\partial x_{j+1}} - \frac{\partial}{\partial x_j} - c \right) \chi_N = 0, \quad x_{j+1} = x_j + 0$$  \hspace{1cm} \text{on the internal boundaries in} \ R \hspace{1cm} (26)$$

The periodic boundary condition reads in $R$:

$$\chi_N(0, x_2, \ldots, x_N) = \chi_N(x_2, \ldots, x_N, L)$$  \hspace{1cm} (27)$$

$$\frac{\partial}{\partial x} \chi_N(x, x_2, \ldots, x_N)\bigg|_{x=0} = \frac{\partial}{\partial x} \chi_N(x_2, \ldots, x_N, x)\bigg|_{x=L}$$  \hspace{1cm} (28)$$

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(To derive this, note that the original boundary condition is $\chi_N(0, x_2, ..., x_N) = \chi_N(L, x_2, ..., x_N)$, with a similar condition for derivative. Now although the argument of $\chi_N$ is not in $\mathbb{R}$, but by definition of Bose symmetry, $\chi_N(L, x_2, ..., x_N, L) = \chi_N(x_2, x_3, ..., x_N, L)$ which justify Eq.(83).

We now make the following Ansatz (substitution, guess), which is originally due to H. Bethe

$$\chi_N(x_1, ..., x_N) = \sum_P a(P) \exp(i \sum_{j=1}^N k_P x_j)$$

(31)

where the summation extends over all the permutations $P$ of $\{k\}$ and $a(P)$ are certain coefficients depending on $P$. This function is defined in $\mathbb{R}$. The extension outside this region follows from the requirement of total symmetry under all particle permutations.

Can we choose $a(P)$ so that $\chi_N$ satisfies the Schrödinger equation? Obviously, Eq.(25) can be satisfied, and

$$E_N = \sum_{i=1}^N k_i^2.$$  (32)

So, let us consider the Eq.(27) at $x_{j+1} = x_j$.

Let’s start with a case of two particles:

$$\left[-\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} + 2c\delta(x_1 - x_2)\right]\chi_2 = E_2\chi_2$$(33)

$$\left(\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_1}\right)\psi|_{x_2=x_1} = c\psi|_{x_2=x_1}$$ (34)

$$\chi_2(x_1, x_2) = a(12) \exp[i(k_1 x_1 + k_2 x_2)] + a(21) \exp[i(k_2 x_1 + k_1 x_2)]$$ (35)

Substituting (35) into (34) gives

$$[i(k_2 - k_1) - c]a(12) \exp(i(k_1 + k_2)x)$$

$$+ [i(k_1 - k_2) - c]a(21) \exp(i(k_1 + k_2)x) = 0$$ (36)

where $x_1 = x_2 = x$. Therefore

$$a(21) = -a(12) \frac{c - i(k_2 - k_1)}{c + i(k_2 - k_1)} = -a(12) \exp(i\theta_{21})$$ (38)
where

$$\theta_{21} = \theta(k_2 - k_1) = -2 \arctan\left(\frac{k_2 - k_1}{c}\right), \quad (39)$$

and therefore

$$\chi_2 = a(12)(\exp[i(k_1 x_1 + k_2 x_2)] + \exp[i(k_2 x_1 + k_1 x_2 + \theta_{21})] \quad (40)$$

In the case of three particles we have

$$[-\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} + 2c(\delta(x_1 - x_2) + \delta(x_2 - x_3) + \delta(x_1 - x_3))]\chi_3(41)$$

$$= E_3 \chi_3(42)$$

$$\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_1} \chi|_{x_2 = x_1} = c \chi|_{x_2 = x_1}(43)$$

$$\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_2} \chi|_{x_3 = x_2} = c \chi|_{x_3 = x_2}(44)$$

$$\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_1} \chi|_{x_3 = x_1} = c \chi|_{x_3 = x_1}(45)$$

$$\chi_3 = a(123) \exp[i(k_1 x_1 + k_2 x_2 + k_3 x_3)] + a(213) \exp[i(k_2 x_1 + k_1 x_2 + k_3 x_3)](46)$$

$$+ a(132) \exp[i(k_1 x_1 + k_3 x_2 + k_2 x_3)] + a(231) \exp[i(k_2 x_1 + k_3 x_2 + k_1 x_3)](47)$$

$$+ a(312) \exp[i(k_3 x_1 + k_1 x_2 + k_2 x_3)] + a(321) \exp[i(k_3 x_1 + k_2 x_2 + k_1 x_3)](48)$$

Substitute now this function into the equation for the (internal) boundary condition at, e.g. $x_1 = x_2$, and we obtain

$$i(k_2 - k_1)(a(123) - a(213))e^{i[(k_1 + k_2) x_2 + k_3 x_3]} \quad (49)$$

$$+ i(k_3 - k_2)(a(231) - a(321))e^{i[(k_2 + k_3) x_2 + k_1 x_3]} \quad (50)$$

$$+ i(k_3 - k_1)(a(132) - a(312))e^{i[(k_1 + k_3) x_2 + k_2 x_3]} \quad (51)$$

$$= c[a(123) + a(213)]e^{i[(k_1 + k_2) x_2 + k_3 x_3]} \quad (52)$$

$$+ (a(231) + a(321))e^{i[(k_3 + k_2) x_2 + k_1 x_3]} \quad (53)$$

$$+ (a(132) + a(312))e^{i[(k_3 + k_1) x_2 + k_2 x_3]} \quad (54)$$

therefore we get the following relations between the amplitudes

$$a(213) = -a(123)\frac{c - i(k_2 - k_1)}{c + i(k_2 - k_1)} = -a(123) \exp(i\theta_{21}) \quad (55)$$

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therefore we get the following relations between the amplitudes

\[
a(312) = -a(132) \frac{c - i(k_3 - k_1)}{c + i(k_3 - k_1)} = -a(132) \exp(i\theta_{31}) \quad (56)
\]

\[
a(321) = -a(231) \frac{c - i(k_3 - k_2)}{c + i(k_3 - k_2)} = -a(231) \exp(i\theta_{32}) \quad (57)
\]

Applying the condition at point \( x_2 = x_3 \) gives

\[
i(k_2 - k_1)(a(123) - a(132))e^{i[(k_3+k_2)x_2+k_1x_1]} + i(k_3 - k_1)(a(213) - a(231))e^{i[(k_1+k_2)x_2+k_2x_1]} + i(k_3 - k_1)(a(312) - a(321))e^{i[(k_1+k_2)x_2+k_3x_1]} = c[(a(123) + a(132))e^{i[(k_3+k_2)x_2+k_1x_1]} + (a(213) + a(231))e^{i[(k_1+k_2)x_2+k_1x_1]} + (a(312) + a(321))e^{i[(k_2+k_1)x_2+k_2x_1]}]
\]

therefore we get the following relations between the amplitudes

\[
a(132) = -a(123) \frac{c - i(k_3 - k_2)}{c + i(k_3 - k_2)} = -a(123) \exp(i\theta_{32}) \quad (64)
\]

\[
a(231) = -a(213) \frac{c - i(k_3 - k_1)}{c + i(k_3 - k_1)} = -a(213) \exp(i\theta_{31}) \quad (65)
\]

\[
a(321) = -a(312) \frac{c - i(k_2 - k_1)}{c + i(k_2 - k_1)} = -a(321) \exp(i\theta_{21}) \quad (66)
\]

Applying the condition at point \( x_1 = x_3 \) gives

\[
i(k_3 - k_1)(a(123) - a(321))e^{i[(k_1+k_3)x_1+k_2x_2]} + i(k_2 - k_1)(a(132) - a(231))e^{i[(k_1+k_2)x_1+k_3x_2]} + i(k_3 - k_2)(a(213) - a(312))e^{i[(k_1+k_2)x_1+k_1x_2]} = c[(a(123) + a(321))e^{i[(k_3+k_1)x_1+k_2x_2]} + (a(132) + a(231))e^{i[(k_1+k_1)x_1+k_1x_2]} + (a(213) + a(312))e^{i[(k_2+k_1)x_1+k_2x_2]}]
\]

therefore we get the following relations between the amplitudes

\[
a(312) = -a(213) \frac{c - i(k_3 - k_2)}{c + i(k_3 - k_2)} = -a(123) \exp(i\theta_{32}) \quad (73)
\]
\[ a(321) = -a(123) \frac{c - i(k_3 - k_1)}{c + i(k_3 - k_1)} = -a(213) \exp(i\theta_{31}) \quad (74) \]

\[ a(231) = -a(132) \frac{c - i(k_2 - k_1)}{c + i(k_2 - k_1)} = -a(231) \exp(i\theta_{21}) \quad (75) \]

Therefore one can observe that

\[ a(132) = (-1)a(123)e^{i\theta_{32}} \quad (76) \]
\[ a(213) = (-1)a(123)e^{i\theta_{21}} \quad (77) \]
\[ a(312) = (-1)^2a(123)e^{i\theta_{21}}e^{i\theta_{32}} \quad (78) \]
\[ a(231) = (-1)^2a(123)e^{i\theta_{21}}e^{i\theta_{31}} \quad (79) \]
\[ a(321) = (-1)^3a(123)e^{i\theta_{21}}e^{i\theta_{32}}e^{i\theta_{31}}. \quad (80) \]

This allow to express the wave function in terms of phase shifts \( \theta \)'s and the amplitude \( a(123) \) which can be absorbed into normalization factor of the wave function. More important observation which is valid for arbitrary number of particles is that if we have two amplitudes \( a(Q) \) and \( a(P) \) (where \( P, Q \) are collective indexes) such that \( P \) corresponds to transposition \( k_1, ..., k_p, k_q, k_3, ..., k_n \) of momenta and \( Q \) corresponds to transposition \( k_1, ..., k_q, k_p, k_3, ..., k_n \) then

\[ a(Q) = -a(P) \frac{c - i(k_q - k_p)}{c + i(k_q - k_p)} = -a(P) \exp(i\theta_{qp}) \quad (82) \]

The fact that the scattering amplitudes are related to each other by a multiplication on a two-particle phase shift leads to the conclusion that any \( N \)-particle scattering process can be factorized into combination of a two-particle scattering processes. This observation is crucial for the exact solvability of the problem. On a more formal level this results into the so-called Yang-Baxter equation between scattering matrices of different scattering events.

The Bethe Ansatz for the wave function (31) determines only the form of the wave function. The allowed values of momenta \( k \)'s will be determined from the periodic boundary condition which is equivalent to

\[ (-1)^{N-1}e^{-ik_jL} = \exp(i\sum_{s=1}^{N} \theta_{sj}) \quad (83) \]
which can be understood intuitively as follows: imagine we bring one particle (say, particle $j$) around the circle through all the other particles. Because the scattering can be factorized on a product of two-particle scattering we obtain a product of two-particle factors $\exp(i\theta_{sj})$. The requirement for the wave function to be single-valued gives Eq.(83).

Dividing two successive equations from Eq.(83) and equating exponents we have

$$ (k_{j+1} - k_j)L = \sum_{s=1}^{N} (\theta_{sj} - \theta_{s(j+1)}) + 2\pi n_j $$  \hspace{1cm} (84)

where $j = 1, 2, ..., N - 1$ and $n$-integers depending on $j$.

3. To make some progress we consider now the system of large number of particles $N$ and large length $L$ while keeping the density of the gas to be a constant, $\rho = N/L$. In this limit $k_{j+1} - k_j << 2\pi/L$. Then we can use a Taylor expansion to get

$$ \theta(k_s - k_j) - \theta(k_s - k_{j+1}) = -\frac{2c(k_{j+1} - k_j)}{c^2 + (k_s - k_j)^2} + O(1/L^2). $$  \hspace{1cm} (85)

Therefore

$$ (k_{j+1} - k_j) = -2c(k_{j+1} - k_j) \frac{1}{L} \sum_{s=1}^{N} \frac{1}{c^2 + (k_s - k_j)^2} + \frac{2\pi}{L} + O(1/L^2). $$  \hspace{1cm} (86)

Define now the function $f(k)$ by the following equation:

$$ k_{j+1} - k_j = \frac{1}{L} f(k_j). $$  \hspace{1cm} (87)

The sum may be approximated by the integral and we get from (84)

$$ 2\pi f(k) = 1 + 2c \int_{-B}^{B} \frac{f(p)}{c^2 + (p - k)^2} dp $$  \hspace{1cm} (88)

where $B$ is some cut-off. The meaning of the function $f(k)$ is that for a large system $L f(k)dk$ is equal to the number of $k$’s in the interval $(k, k + k + dk)$. The consistency condition we have therefore is

$$ \int_{-B}^{B} f(k)dk = \rho = \frac{N}{L} $$  \hspace{1cm} (89)
which is fixed in our **thermodynamic limit** \( N \to \infty, N \to \infty \). The equation above defines the cut-off \( B \) self-consistently.

The energy now is

\[
E = \sum_{j=1}^{N} k_j^2 = \frac{N}{\rho} \sum_{-B}^{B} f(k)k^2 dk
\]  

(90)

The change of variables

\[
k \equiv Bx, \quad c \equiv B\lambda, \quad f(Bx) \equiv g(x)
\]  

(91)

brings the Bethe-Ansatz equations into the following form

\[
2\pi g(y) = 1 + 2\lambda \int_{-1}^{1} \frac{g(x)dx}{x^2 + (x - y)^2}
\]  

(92)

\[
e(\gamma) = \frac{\gamma^3}{\lambda^3} \int_{-1}^{1} g(x)x^2 dx
\]  

(93)

\[
\lambda = \gamma \int_{-1}^{1} g(x)dx.
\]  

(94)

Here several comments are in order. **First**, we introduced the dimensionless coupling constant

\[
\gamma = \frac{c}{\rho}
\]  

(95)

which is given by the ratio of the interaction energy and the kinetic energy. In dimensional units

\[
\gamma = \frac{mg_{1D}}{\hbar^2 \rho}, \quad g_{1D} = \frac{\hbar^2 c}{m}, \quad c = \frac{2}{a_{1D}}
\]  

(96)

where \( a_{1D} \) is a one-dimensional scattering length. The dimensionless coupling \( \gamma \) characterizes 1D Bose physics entirely. **Second**, the energy \( E \) is defined as

\[
E = N\rho^2 e(\gamma)
\]  

(97)

\( (E = \hbar^2 \rho^2 e(\gamma)/2m \) in dimensional units. **Third**, the solution of the Eq. (92)-(94) goes as follows: (a) solve (92) for fixed \( \lambda \); (b) use (94) to determine
\( \lambda \) as a function of \( \gamma \); (c) Eq. (93) gives the energy \( e(\gamma) \); (d) find the cut-off \( B = B(\gamma) = \rho \gamma \lambda^{-1} \).

**Different limits.**

\( \gamma \) large. In this case the denominator of the kernel in (92) can be replaced by \( \lambda^2 \). Therefore we get

\[
2\pi g(y) = 1 + \frac{2}{\lambda} \int_{-1}^{1} g(x) dx
\]

which has the following solution (check by a direct substitution):

\[
g(x) = \frac{\lambda}{2\pi \lambda - 4}
\]

Then we obtain that \( \lambda = (\gamma + 2)/\pi \) and

\[
e(\gamma) = \frac{\pi^2}{3} \left( \frac{\gamma}{\gamma + 2} \right)^2, \quad B = \frac{\pi \rho \gamma}{\gamma + 2}
\]

The chemical potential can be computed as well

\[
\mu = \frac{\partial E}{\partial N} = \rho^2 (3e(\gamma) - \gamma \frac{de(\gamma)}{d\gamma}) = \frac{3\gamma + 2}{\gamma + 2} \rho^2 e(\gamma)
\]

The limit \( \gamma = \infty \) is called Tonks-Girardeau limit. In this case the behavior of the system of (strongly interacting) bosons is equivalent to the behavior of the system of spinless fermions which respect the Fermi-Dirac distribution (there are no doubly occupied sites). It can also be considered as a gas of hard spheres. In this limit the wave function take a particularly simple form:

\[
\chi_N(x_1...x_N) = |\text{det}(\phi_i(x_j))|
\]

where \( i, j = 1, ..., N \) and \( \phi_i(x_j) \) are single-particle fermionic orbitals. The \text{det} is nothing but the Slater determinant and the modulus sign gives the proper symmetrization needed to have a right symmetry under permutations. This connection demonstrate a general property of 1D systems: the statistics is not so strict as in higher dimensions and one freely switch between fermionic and bosonic descriptions of interacting 1D systems.

\( \gamma \) small. In the weak-coupling limit one can assume that the mean-field theory can be justified. This is indeed true for the ground state properties
Consider first the result from the Bogoliubov theory, and then compare it with the result from exact solution. In the Bogoliubov theory the spectrum of the elementary excitations in our units is given by

$$\epsilon(p) = \rho^2 \frac{p^2}{\rho} (p^2 + 4\gamma)^{1/2}.$$ \hspace{1cm} (103)

The ground state energy is

$$E = \frac{1}{2} N^2 \left( \frac{2c}{L} \right) + \frac{1}{2} \sum_p [\epsilon(p) - p^2 - N\left( \frac{2c}{L} \right)] = N\rho^2 \sqrt{\gamma} \left( 1 - \frac{4}{3\pi} \sqrt{\gamma} \right).$$ \hspace{1cm} (104)

On the other hand the kernel of Eq.(92) goes to $2\pi \delta(x - y)$ as $\lambda \to 0$. Therefore the equation becomes $2\pi g(x) = 2\pi g(x) + 1$ which has no solution. The careful analysis of corrections to the kernel leads to the following result

$$g(x, \lambda) \sim \frac{1}{2\pi \lambda} (1 - x^2)^{1/2}$$ \hspace{1cm} (105)

which then gives $\lambda = \sqrt{\gamma}/2$ and

$$e(\gamma) = \gamma, \quad B = 2\rho \sqrt{\gamma}$$ \hspace{1cm} (106)

and for the chemical potential $\mu = 2\rho^2 \gamma$.

The case of arbitrary $\gamma$ can be treated by direct solution of the integral equation (92).

4. The realization of 1D Bose system in the experiments of D. Weiss (see Ref. [2]) with trapped cold bosonic atoms demonstrates unambiguously that the system is indeed described by the Lieb-Liniger model and it shows a perfect fit to the energy obtained from Eq. (93). One should make few comments here: first the experimental results are expressed as a one-dimensional temperature $T_{1D} = 2E/(Nk_B)$ as a function of one-dimensional effective parameter $\gamma$. The 1D parameter $\gamma$ is related to the properties of three-dimensional gas as follows [3]

$$\gamma = \frac{2a_{3D}}{a_{1D}^2 \rho_{1D}} \left( 1 - C \frac{a_{3D}}{a_{\perp}} \right)^{-1}$$ \hspace{1cm} (107)

where $a_{3D}$ is the s-wave scattering length of a 3D gas, $a_{\perp} = (\hbar/(m\omega_{\perp}))^{1/2}$ is the confinement length scale in the transversal direction of the trap with
frequency $\omega_\perp$. Finally, $C = 1.4603$. Therefore experimentally one can change the coupling $\gamma$ by changing the density $\rho$ or by tuning the transversal confinement $\omega_\perp$.

Using Bethe Ansatz one can compute the correlation functions of different quantities. For example, coming back to the Eq. (3) and fields $\Psi(x)$, one can show that at long distances

$$<\Psi^\dagger(x)\Psi(0)> = Ax^{-1/\theta}$$

$$<\Psi^\dagger(x)\Psi(x)\Psi^\dagger(0)\Psi(0)> = \rho^2 + \frac{A}{x^2} + \frac{C\cos(2\pi\rho x)}{x^\theta}$$

(109)

(110)

where $\theta = 4\pi\rho/v_F$ and $A, C$ are known constants. Here, as usual $v_F = (\partial e(\lambda)/\partial k(\lambda))|_{\lambda = B}$. All the properties of a system described above were limited by a zero temperature only. One can do computations at non-zero temperatures as well.

5. The effective description of a system, known as a bosonic Luttinger liquid, goes as follows. Assuming small deviations of a density operator $\rho(x) = \Psi^\dagger(x)\Psi(x)$ around a mean value $\rho_0 = N/L$ one can write $\rho(x) = \rho_0 + \Pi(x)$. The bosonic field operator $\Psi(x)$ can be decomposed in a usual way, $\Psi^\dagger(x) = \sqrt{xe^{-i\phi(x)}}$. From the canonical commutation relations (1) one can derive

$$[\Pi(x), \phi(x')] = i\delta(x-x')$$

(111)

Using these canonically conjugated fields one can rewrite Eq. (3) in the following form

$$H_{eff} = \frac{v_s}{2} \int dx \left[ \frac{\pi}{K} \Pi^2(x) + \frac{K}{\pi} (\partial_x \phi)^2 \right]$$

(112)

where we defined the Luttinger parameters $K = \sqrt{v_J/v_N}$ and the sound velocity $v_s = \sqrt{v_N v_J}$ in terms of the density stiffness $v_N$ and the phase stiffness $v_J$. The Luttinger parameters $K, v_s$ can be computed from the Bethe Ansatz using the fact that

$$v_N = \frac{L}{\pi\hbar} \frac{\partial^2 E}{\partial N^2}, \quad v_s = \sqrt{\frac{L}{m\rho} \frac{\partial^2 E}{\partial L^2}}$$

(113)
The effective theory gives right behavior of the asymptotic correlation functions. In particular, a momentum distribution function, measurable experimentally, gives

\[ n(p) = \frac{1}{L} \int_0^L dx \int_0^L dx' < \Psi^\dagger(x)\Psi(x') > e^{ip(x-x')} \sim p^{-\beta} \quad (114) \]

where \( \beta = 1 - 1/(2K) \). This result is applicable in the limit \( p << \rho_0 \).

References