Quantum Mechanics I, Sheet 1, Spring 2015

February 18, 2015 (EP, Auditoire Stuckelberg)

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Tutorial: O. Peil (oleg.peil@unige.ch)

I. PROBABILITY DISTRIBUTION

Let us consider a random variable $x \in \mathbb{R}$ and a probability distribution defined by its probability density function $P(x) : \mathbb{R} \to [0, 1]$, with

$$\int_{-\infty}^{\infty} P(x) \, \mathrm{d}\, x = 1.$$

The expectation value of a function $O(x)$ in a variable x is given by

$$\langle O(x) \rangle = \int_{-\infty}^{\infty} P(x) O(x) \, \mathrm{d}\, x.$$

In particular, one can introduce the expectation value of the random variable itself (or the mean value of the distribution), $\langle x \rangle$, as well as its variance (the variance of the distribution)

$$(\delta x)^2 := \left\langle (x - \langle x \rangle)^2 \right\rangle = \left\langle x^2 \right\rangle - \langle x \rangle^2.$$

- Evaluate the mean value and the variance of the Gaussian distribution

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2} \right).$$

II. FOURIER TRANSFORM

Consider the Fourier transform of a function $f(x)$,

$$\mathcal{F}[f](k) \equiv \hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) \, \mathrm{d}\, x,$$

with the inverse transform

$$\mathcal{F}^{-1}[\hat{f}](x) \equiv f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \hat{f}(k) \, \mathrm{d}\, k.$$

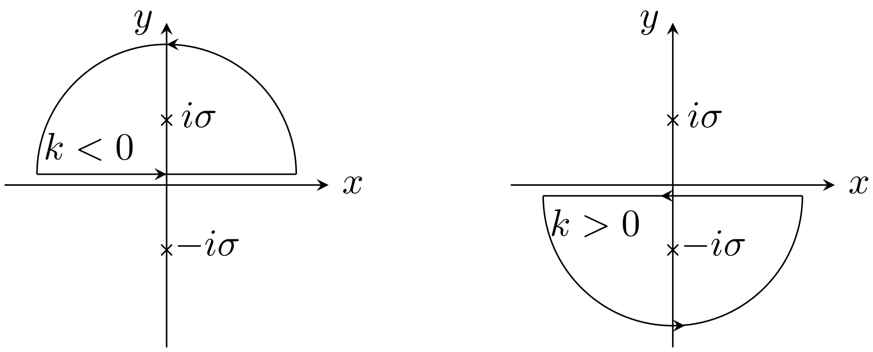
1. For a well-behaved function, $f(x) \to 0$ as $x \to \pm\infty$, with a derivative $f'(x) \equiv \mathrm{d}\, f / \mathrm{d}\, x$ prove

$$\mathcal{F}[f'](k) = ik\hat{f}(k).$$

2. Evaluate the Fourier transform of the Lorentz distribution,

$$f(x) = \frac{1}{\pi} \frac{\sigma}{x^2 + \sigma^2}.$$

[*Hint:* Use contour integration with a semi-circular part of the contour encompassing the lower complex semi-plane for $k > 0$ and the upper semi-plane for $k < 0$ (see the figure below).]



III. DIRAC COMB

Consider an infinite train of δ-pulses:

$$f(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT),$$

where T is the periodicity.

1. Since $f(t)$ is a periodic function, it can be represented as a Fourier-series

$$f(t) = \sum_{m=-\infty}^{m=\infty} c_m e^{i2\pi m \frac{t}{T}}.$$

Find the coefficients c_m.

2. Using the above expression evaluate the Fourier transform of $f(t)$ (in the time domain), $\hat{f}(\omega) = \mathcal{F}[f](\omega)$. Sketch the obtained function of frequency.

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Sheet 2

March 4, 2015 (EP, Auditoire Stuckelberg)

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I. THE PHOTOELECTRIC EFFECT

A graduate student measures the maximum energy of photoelectrons from an aluminim plate. For a radiation wavelength of 200 nm, the maximum energy is 2.3 eV. When the wavelength is changed to 258 nm, the measured energy of the electrons is 0.90 eV. From those data, calculate Plank's constant and the work function of aluminum.

II. THE COMPTON EFFECT

A 100-MeV photon collides with a proton at rest. What is the maximum possible energy loss for the photon?

III. THE BOHR ATOM AND CORRESPONDENCE PRINCIPLE

The energy for a harmonic oscillator is given by $p^2/2m + m\omega^2 r^2/2$. Use Bohr quantization rules with the angular momentum and quantum number n to calculate the energy levels for this system. Restrict your analysis to circular orbits. Is the correspondence principle satisfied for all values of n?

IV. WAVE PACKETS AND PROBABILITY INTERPRETATION

Consider the wave function

$$\psi(x) = \begin{cases} N & |x| \leq x_0 \\ 0 & |x| > x_0 \end{cases}$$

a) Plot $\psi(x)$ and find N such that the function is normalised, i.e., $\int_{-\infty}^{\infty} dx\, |\psi(x)|^2 = 1$.

b) The wave function $\psi(x)$ can be considered as a wave packet with amplitudes $\phi(p)$ defined implicitly by a relation $\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dp\ \phi(p) e^{ipx/\hbar}$. Find $\phi(p)$ and show that it is normalised. (Hint: use $\int_{-\infty}^{\infty} dx \frac{sin^2 x}{x^2} = \pi$ for the normalisation.)

c) Compute the average value $\langle x \rangle$, $\langle x^2 \rangle$ and the variance $(\Delta x)^2 \equiv \langle x^2 \rangle - \langle x \rangle^2$, where $\langle x^n \rangle = \int_{-\infty}^{\infty} dx\ x^n\, |\psi(x)|^2$.

d) Compute $\langle p \rangle$, $\langle p^2 \rangle$ and the variance of the momentum $(\Delta p)^2 \equiv \langle p^2 \rangle - \langle p \rangle^2$, where $\langle p^n \rangle = \int_{-\infty}^{\infty} dp\ p^n\, |\phi(p)|^2$.

e) Check whether your results found in **c)** and **d)** are consistent with the Heisenberg uncertainty relation?

Quantum Mechanics I, Sheet 3, Spring 2015

March 10, 2015 (EP, Auditoire Stuckelberg)

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I. MOMENTUM REPRESENTATION

Consider a wave function $\psi(x) = Ae^{-\mu|x|}$.

1. Find the normalization coefficient A.
2. Calculate the wave function in momentum space $\phi(p)$.
3. Evaluate the norm of this function,

$$\int_{-\infty}^{\infty} |\phi(p)|^2 \, \mathrm{d}\, p$$

II. PARTICLE IN A BOX

Consider a particle in a potential of the form

$$V(x) = \begin{cases} \infty, & |x| > a \\ 0, & |x| < a. \end{cases}$$

1. Draw a potential and write out the time-independent Schrödinger equation and boundary conditions for the wave function of the particle.
2. The even (+) and odd (−) eigenfunctions of the Schrödinger equation for this potential can be written as follows:

$$\begin{aligned} \psi_+(x) &= \frac{1}{\sqrt{a}} \cos\frac{\pi x}{2a}(2n+1), && n = 0, 1, 2, \ldots \\ \psi_-(x) &= \frac{1}{\sqrt{a}} \sin\frac{\pi x}{2a}(2n), && n = 1, 2, 3, \ldots \end{aligned}$$

   Using the Schrödinger equation find eigenvalues corresponding to the above eigenfunctions.

3. What is energy of the ground state, E_0? Write the wave function of the ground state.
4. Assume that the particle is initially in the ground state. Then the boundaries of the box are suddenly moved to $x = \pm b$ ($b > a$).

a. Find the probability that a particle will be found in the ground state for the new potential.

b. Find the probability that a particle will be found in the first excited state. Explain the answer.

III. PROBABILITY CURRENT

Consider a normalized wave function $\psi(x)$. Assume that the system is in the state described by the wave function

$$\Phi(x) = C_1\psi(x) + C_2\psi^*(x),$$

where C_1 and C_2 are two known complex numbers.
A complex function $\psi(x)$ can be generally expressed in terms of two real functions $f(x)$, $\theta(x)$ as follows

$$\psi(x) = f(x)e^{i\theta(x)}.$$

1. Obtain an expression for the probability current density $j(x)$ for the state $\Phi(x)$ in terms of functions $f(x)$ and $\theta(x)$.

2. Calculate the expectation value $\langle p\rangle$ of the momentum in the state $\Phi(x)$ and show that

$$\langle p\rangle = m\int_{-\infty}^{+\infty} j(x)\mathrm{d}x.$$

   To obtain this result, one has to assume that the function $f(x)$ vanishes at infinity. Show that both the probability current and the momentum vanish if $|C_1| = |C_2|$.

Quantum Mechanics I, Spring 2015

Sheet 4

March 18, 2015 (EP, Auditoire Stuckelberg)

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I.

An operator O is said to be *linear* if, for any functions $f(x)$ and $g(x)$ and for any constants a and b,

$$O\left(af(x) + bg(x)\right) = aOf(x) + bOg(x).$$

(a) Which of the following operators are linear?

$$\begin{array}{ll}
I = \text{identity} & If(x) = f(x) \\
S = \text{squares } f(x) & Sf(x) = f^2(x) \\
D = \partial_x = \partial/\partial x & Df(x) = \partial f(x)/\partial x \\
I_x = \int_0^x dx' & I_x f(x) = \int_0^x dx' f(x') \\
A = \text{adds } 3 & Af(x) = f(x) + 3 \\
P = \text{maps to } g(x) & Pf(x) = g(x) \\
T = \text{translates by } L & Tf(x) = f(x - L)
\end{array}$$

(b) Let A and B be linear operators, and let C denote their commutator, i.e.

$$C \equiv [A, B] = AB - BA.$$

Show that C is also a linear operator.

II.

In this problem, we will study in more details the Translation operator and commutation relations. For clarity, we will denote operators with a "hat". Remember that the position operator, $\hat{x}$, acts on functions of position, $f(x)$, as

$$\hat{x}f(x) = xf(x).$$

On the other hand, the "derivative" operator, $\hat{D}$, acts on functions of position, $f(x)$, as

$$\hat{D}f(x) = \partial_x f(x),$$

while the “translate-by-L” operator, $\hat{T}_L$, acts on functions of position, $f(x)$, as

$$\hat{T}_L f(x) = f(x - L).$$

(a) Show that

$$[\hat{T}_L, \hat{x}] = -L\hat{T}_L.$$

Note: Two operators ($\hat{A}$, $\hat{B}$) are equal if $\hat{A}f(x) = \hat{B}f(x)$ for any function $f(x)$.

(b) Show that $\hat{T}_L$ **commutes with** the derivative operator, i.e. that

$$[\hat{T}_L, \hat{D}] = 0.$$

(c) Show that

$$\hat{T}_L = e^{-L\hat{D}}.$$

Hint: use the Taylor expansion of $e^x = 1 + x + x^2/2 + \ldots$

(d) Use (a) and (c) to show that

$$[\hat{D}, \hat{x}] = \hat{I}$$

where $\hat{I}$ is the identity operator defined in the first problem.

(e) If $\hat{T}_L f(x) = f(x - L)$, how does $\hat{T}_L$ act of $\tilde{f}(k)$, the fourier transform of $f(x)$? In other words, what modification of $\tilde{f}(k)$ corresponds to translating $f(x)$ by L?

(f) Use parts (c) and (e) to determine how $\hat{D}$ acts on $\tilde{f}(k)$.

(g) Use the definition of the position operator and the definition of the fourier transform to determine how $\hat{x}$ acts on the fourier transform, $\tilde{f}(k)$, of $f(x)$.

(h) Verify that the commutation relation $[\hat{D}, \hat{x}] = \hat{I}$ holds whether acting on a function $f(x)$ or its fourier transform, $\tilde{f}(k)$. Comment.

III.

A particle of mass m moves in one dimension under the influence of a potential $V(x)$. Suppose it is in an energy eigenstate $\psi(x) = (\gamma^2/\pi)^{1/4} \exp(-\gamma^2 x^2/2)$ with energy $E = \hbar^2\gamma^2/2m$.

(a) Find the mean position of the particle.

(b) Find the mean momentum of the particle.

(c) Find $V(x)$.

IV.

Suppose a particle is in an eigenstate of the infinite well of width L. (i) Show that we know its energy exactly. (ii) I now argue that since the energy in the box is purely kinetic, then we know the particle’s momentum exactly well. From this argument, I claim that there is a contradiction with the Heisenberg uncertainty relation because the uncertainty in the particle position is finite ($\Delta x < L$). Give a physicist’s proof that there is something wrong with my reasoning.

V.

A particle of mass m is confined in a one-dimensional region $0 \leq x \leq a$ of the infinite well (Particle in a box, Gasiorowicz eq. 3-13). The time-independent Schrodinger equation for $0 < x < a$ is

$$\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + E\psi = 0. \tag{1}$$

The normalized eigenfunctions of the Hamiltonian are

$$u_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right) \tag{2}$$

and the energy eigenvalues are

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}, \quad n = 1, 2, 3, \ldots. \tag{3}$$

At $t = 0$, the normalized wave function of the particle is

$$\psi(x, t=0) = \sqrt{8/5a}\left[1 + \cos\left(\frac{\pi x}{a}\right)\right] \ \sin(\pi x/a). \tag{4}$$

(a) What is the wave function at a later time $t = t_0$? Write down the expression for $\psi(x, t_0)$.
(b) What is the average energy ($\langle H \rangle$) of the system at $t = 0$ and at $t = t_0$?
(c) What is the probability that the particle is found in the left half of the box (i.e., in the region $0 \leq x \leq a/2$) at $t = t_0$?
Hint: According to the spectral theorem in quantum mechanics, predicting the measurement of energy involves expressing the wave function as a superposition of orthonormalized eigenfunctions of the Hamiltonian, and interpreting the coefficients of each eigenfunction as the probability amplitude to measure the associated eigenvalue. Specifically, use the fact that any wave function $\psi(x,t)$ can be expanded in $u_n(x)$:

$$\psi(x,t) = \sum A_n u_n(x) e^{-iE_n t/\hbar}, \tag{5}$$

where

$$A_n = \int_0^a dx u_n^*(x)\psi(x). \tag{6}$$

Quantum Mechanics I, Sheet 5, Spring 2015

March 26, 2015 (EP, Auditoire Stuckelberg)

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I. HALF HARMONIC OSCILLATOR

Consider a particle in a 1D potential:

$$V(x) = \begin{cases} \frac{1}{2}\omega^2 x^2, & x > 0 \\ \infty, & x < 0. \end{cases}$$

1. What are the boundary conditions for the wavefunction in this case?
2. What are the eigenenergies of the particle?
3. Write out the ground state wavefunction.

II. THREE-DIMENSIONAL POTENTIAL

Consider a Schrödinger equation in 3D:

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + V(x,y,z)u(x,y,z) = Eu(x,y,z),$$

with the potential

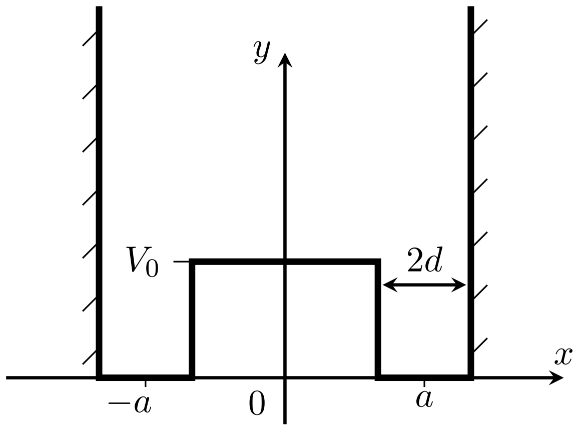
$$V(x,y,z) = \frac{m}{2}\omega^2(x^2+y^2) + V(z),$$

$$V(z) = \begin{cases} 0, & \text{for } |z| < a \\ \infty, & \text{otherwise.} \end{cases}$$

1. Write out the ground state wavefunction $u_0(x,y,z)$. $\left[\textit{Hint}\text{: Use the separation of variables.}\right]$
2. Find the eigenenergies.
3. Consider a case when $a \gg \sqrt{\hbar/m\omega}$, i.e. when the potential is strongly "squeezed" in x,y-directions. What is the structure of energy levels in this case? Sketch first several energy levels and indicate the degeneracies (i.e. the number of different eigenstates with the same energy).

III. POTENTIAL WITH A BARRIER

Consider a potential representing an infinite well with a barrier of height V_0 in the middle (see Figure). Let the energy of the particle be $0 < E < V_0$, in which case one can define the parameters



$$
\begin{aligned}
k^2 =& \frac{2mE}{\hbar^2}, \\
q^2 =& \frac{2m}{\hbar^2}\left(V_0 - E\right).
\end{aligned}
$$

1. Write out the boundary conditions for the wave-function.

2. Construct the matching conditions for the wave-function and find the equation for the bound state energies in terms of parameters k and q. $\Big[$ *Hint: Be careful with manipulating* sin *and* cos *functions; when you divide an expression by a value of a function mind that the value might be zero.* $\Big]$

3. Sketch the eigenfunctions for the two lowest states.

4. Carefully consider a limit $V_0 \to \infty$, which is equivalent to $q \to \infty$.

5. Rewrite the equation for the eigenenergies for the case $E > V_0$. Note that $\sinh ix = i \sin x$.

IV. * DELTA-POTENTIAL AS A LIMITING CASE OF A SQUARE WELL

Consider a square well potential defined as usual

$$
V(x) = \begin{cases} -V_0, & |x| < a \\ 0, & |x| > a. \end{cases}
$$

Delta-potential can be considered as a limiting case of a square-well potential when $V_0 \to \infty$, $a \to 0$ in such a way that $2aV_0 = \text{const}$, i.e. the volume of the well is kept constant.

Starting from the equation for the eigenenergies for a square well (see the book or your lecture notes) derive the energy of the bound state of the delta-potential by taking the limit.

* Problems marked with an asterisk $(*)$ are optional.

Quantum Mechanics I, Sheet 6, Spring 2015

March 31, 2015 (EP, Auditoire Stuckelberg)

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I. HERMITIAN OPERATORS

An operator A is called Hermitian if it is equal to its Hermitian conjugate, $A^\dagger = A$.
Consider operators of position, $\mathbf{r}$ (components x, y, z), and momentum p_x, p_y, p_z in 3D. They are Hermitian operators obeying the commutation relation

$$[r_\alpha, p_\beta] = i\hbar\delta_{\alpha,\beta},$$

where $\alpha, \beta = x, y, z$.
Check whether the following operators are Hermitian:

a. $p_y\, y$

b. zp_y

c. $p_x + ix$

II. COMMUTATION RELATIONS

1. For a set of arbitrary (non-commuting) operators A, B, C prove the following identities:

   a. $[A, BC] = [A, B]C + B[A, C]$

   b. $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$

2. The angular momentum operator is defined as

$$\mathbf{L} = \mathbf{r} \times \mathbf{p},$$

   with components

$$L_x = yp_z - zp_y, \quad L_y = zp_x - xp_z, \quad L_z = xp_y - yp_x.$$

   Evaluate the following commutators:

   a. $[L_x, L_y]$

   b. $[L_y, {L_z}^2]$

   c. $[L_y, \mathbf{L}^2]$

III. UNITARY OPERATORS AND TIME EVOLUTION

An operator U is called unitary if

$$UU^{\dagger} = 1.$$

a. Prove that all eigenvalues of a unitary operator U must have the form $e^{i\alpha}$, where $\alpha \in \mathbb{R}$.

b. Let H be a Hamiltonian with eigenvalues E_n. Prove that $e^{-iHt/\hbar}$ is unitary and find its eigenvalues.

c. Show that if H is time-independent the solution $\Psi(t)$ of the Schrödinger equation can be obtained in the form $\Psi(t) = e^{-iHt/\hbar}\Psi(t = 0)$.

d. Use the previous results to show that time evolution in quantum mechanics is reversible.

IV. ZERO-POINT ENERGY FROM HEISENBERG UNCERTAINTY

Consider the Hamiltonian for the quantum harmonic oscillator,

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2,$$

and find the minimum average energy $\langle H \rangle$ using the uncertainty relation for x and p.

V. EHRENFEST THEOREM

Given a Hamiltonian H and a wave function $\psi(x,t)$ satisfying the Schrödinger equation for H, prove the Ehrenfest theorem describing the evolution of an observable A:

$$\frac{\mathrm{d}\,\langle A \rangle}{\mathrm{d}\,t} = \left\langle \frac{\partial A}{\partial t} \right\rangle + \frac{1}{i\hbar}\langle [A, H] \rangle.$$

where the expectation values are taken with respect to $\psi(x,t)$.

Quantum Mechanics I, Spring 2015

Sheet 7

April 15, 2015 (EP, Auditoire Stuckelberg)

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Tutorial: J. Lavoie (jonathan.lavoie@unige.ch)

I.

In some physical contexts, the following operator may arise

$$\hat{Q} \equiv i\frac{d}{d\phi},$$

where ϕ is the usual polar coordinate in two dimensions. Is this operator hermitian? Find its eigenfunctions and eigenvalues.
(Hint: Work with functions $f(\phi)$ on the finite interval $0 \leq \phi \leq 2\pi$ and assume that $f(\phi + 2\pi) = f(\phi)$)

II.

Linear operators that represent observables have the special property that their expectation value for all of the states that form the linear space must be real. Such operators are called hermitian. The reality of the expectation values

$$\langle A \rangle^* = \langle A \rangle$$

translates into the statement that for any wave function $\psi(x)$,

$$\int_{-\infty}^{\infty} dx\psi^*(x)A\psi(x) = \left[\int_{-\infty}^{\infty} dx\psi^*(x)A\psi(x)\right]^* = \int_{-\infty}^{\infty} dx(A\psi(x))^*\psi(x). \tag{1}$$

Using equation (1), prove that for a hermitian operator A,

$$\int_{-\infty}^{\infty} dx\phi^*(x)A\psi(x) = \int_{-\infty}^{\infty} dx(A\phi(x))^*\psi(x) \tag{2}$$

for any admissible pair of wavefunctions $\phi(x)$ and $\psi(x)$. (Hint: Let $\Psi = \phi + \lambda\psi$ in (1) and use the fact that λ is an arbitrary complex number.)

III.

1. Show that if $\psi(x)$ is a normalized wave function, and U is a unitary operator, then the function

$$\phi(x) = U\psi(x)$$

is also normalized to unity.

2. Consider a complete set of orthogonal, normalized eigenfunctions of some operator A denoted by $u_a(x)$. Given a unitary operator U we may construct the set $v_a(x)$ defined by $v_a(x) = Uu_a(x)$. Show that the new set of eigenfunctions is also orthonormal; that is,

$$\int_{-\infty}^{\infty} dx v_a^*(x) v_b(x) = \delta_{ab}$$

IV.

Consider eq. (5-41) of Gasiorowicz (3rd edition). Find the linear combinations of $u_a^{(1)}$, $u_a^{(2)}$ that lead to the form (5-42), and find the eigenvalues b_1 and b_2.

V.

1. An electron in an oscillating electric field is described by the Hamiltonian operator

$$H = \frac{p^2}{2m} - (eE_0 cos\omega t)x.$$

Calculate expressions for the time dependence of $\langle x \rangle$, $\langle p \rangle$ and $\langle H \rangle$.

2. Solve the equations of motion you obtained in 1. Write your solutions in terms of $\langle x \rangle_0$ and $\langle p \rangle_0$, the expectation values at time t=0.

Quantum Mechanics I, Spring 2015

Sheet 8

April 22, 2015 (EP, Auditoire Stuckelberg)

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Tutorial: J. Lavoie (jonathan.lavoie@unige.ch)

I. HERMITIAN OPERATOR

1. Show that the eigenvalues of a hermitian operator A (for which $A = A^\dagger$) are real.
2. Prove that $(AB)^\dagger = B^\dagger A^\dagger$.

II. TRACE

The trace of an operator is defined as follows:

$$\mathrm{Tr}A = \sum_n \langle n|A|n\rangle \tag{1}$$

where the sum is over a complete set of states. Prove that $\mathrm{Tr}AB$=$\mathrm{Tr}BA$.

III. HARMONIC OSCILLATOR

1. Consider the harmonic oscillator. Prove that

$$A|n\rangle = \sqrt{n}|n-1\rangle \tag{1}$$

2. Calculate $\langle m|x|n\rangle$ and show that it vanishes unless $n = m \pm 1$.
3. Calculate $\langle m|p|n\rangle$.
4. Use the previous results to calculate $\langle m|px|n\rangle$ and $\langle m|xp|n\rangle$. (Hint: use the completeness relation)

In problems 2-4, it might be useful to express x and p in terms of A and $A^\dagger$.

IV. COHERENT STATE

A state $|\alpha\rangle$ that obeys the equation $A|\alpha\rangle = \alpha|\alpha\rangle$, where A is the harmonic oscillator annihilation operator, is called a coherent state.

(a) Show that the state $|\alpha\rangle$ may be written in the form

$$|\alpha\rangle = Ce^{\alpha A^\dagger}|0\rangle \tag{1}$$

(b) Use the following result (that you can prove if you want) to obtain C: If $f(A^\dagger)$ is any polynomial in $A^\dagger$, then

$$Af(A^\dagger)|0\rangle = \frac{df(A^\dagger)}{dA^\dagger}|0\rangle. \tag{2}$$

(c) Expand the state $|\alpha\rangle$ in a series of eigenstates of the operator $A^\dagger A, |n\rangle$, and use this to find the probability that the coherent state contains n quanta. The distribution is called a Poisson distribution.
(d) Calculate $\langle\alpha|N|\alpha\rangle$, the average number of quanta in the coherent state, where $N = A^\dagger A$.

V. TIME DEPENDENCE OF OPERATOR

Use the general operator equation of motion (6-67 in Gasiorowicz) to solve for the time dependence of the operator $x(t)$ given that

$$H = \frac{p^2(t)}{2m} + mgx(t). \tag{1}$$

Quantum Mechanics I, Spring 2015

Sheet 9

May 6, 2015 (EP, Auditoire Stuckelberg)

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I. RAISING AND LOWERING OPERATORS

Consider the raising and lowering operators $L_\pm = L_x \pm iL_y$.
(a) Find the commutation relations $[L_z, L_\pm]$ and $[L^2, L_\pm]$.
(b) What properties of the eigenvalues of L^2 and L_z follow from these commutators?

II. SPHERICAL HARMONICS

In spherical coordinates, L^2 and L_z take the form

$$L^2 = -\hbar^2 \left(\frac{\partial^2}{\partial\theta^2} + \cot\theta \frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \right) \quad , \qquad L_z = -i\hbar\frac{\partial}{\partial\phi}$$

and the few first spherical harmonics

$$Y_{0,0} = \sqrt{\frac{1}{4\pi}}, \quad Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta, \quad Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi}.$$

(a) Show that these spherical harmonics are normalized and orthogonal to one another.
(b) Show that these spherical harmonics are eigenfunctions of both L^2 and L_z and compute the corresponding eigenvalues.
(c) Construct $Y_{3,1}$ from the given harmonics using the raising/lowering operators.

III. UNCERTAINTY

Consider a quantum system in a normalized eigenstate of L^2 and L_z, $\Psi \propto Y_{lm}$ and $\langle\Psi|\Psi\rangle = 1$.
(a) Show that in this case, $\langle L_x\rangle = \langle L_y\rangle = 0$.
(b) Show that $\langle L_y^2\rangle = \langle L_x^2\rangle = \frac{\hbar^2}{2}[l(l+1) - m^2]$.
(c) Using the results found in (a) and (b), verify the uncertainty relation $\Delta L_x \Delta L_y \geq \frac{\hbar}{2}|\langle L_z\rangle|$.

IV. DEGENERACY FOR AN AXIALLY SYMMETRIC ROTATOR

Consider a rotator with moment of inertia $I_x = I_y = I_z = I$. This system is described by the Hamiltonian

$$H = \frac{\mathbf{L}^2}{2I}$$

(a) What are the energy eigenstates and eigenvalues for this system?
(b) What is the degeneracy of the n^{th} energy eigenvalue?
Now suppose that the moment of inertia in the z direction becomes $I_z = (1+\epsilon)I$, with the other two moments unchanged.
(c) What are the energy eigenstates and eigenvalues of the new Halmitonian?
(d) Sketch the spectrum of energy eigenvalues as a function of ϵ.
(e) What is the degeneracy of the n^{th} energy eigenvalue?

Quantum Mechanics I, Sheet 10, Spring 2015

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I. CENTRAL POTENTIAL IN 2D

Chapter 8 of the Gasiorowicz book accounts in details the treatment of the central potential in 3D. Consider now the stationary Schrödinger equation for a central potential in two dimensions (2D),

$$-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Psi(x,y)+V(r)\Psi(x,y)=E\Psi(x,y),$$

where the potential $V(r)$ depends only on the distrance from the origin, $r=\sqrt{x^2+y^2}$.

1. Rewrite the equation in polar coordinates r, ϕ,

$$x = r\cos\phi,$$
$$y = r\sin\phi.$$

   [*Hint*: Express r, ϕ in terms of x and y and use the chain rule for partial derivatives.]
2. Write the obtained equation in terms of the z-component of the angular momentum operator L_z.
3. Separate variables r and ϕ in the equation by introducing $\Psi(r,\phi)=R(r)A(\phi)$ and derive an equation for the radial part $R(r)$.
   [*Hint*: Take into account the periodicity condition, $A(\phi+2\pi)=A(\phi)$, for the angular part.]

Now, consider a particular potential, namely the Coulomb potential, $V(r)=-\alpha/r$. With the appropriate choice of units, the radial equation for bound states $E<0$ can be written as

$$R''+\frac{1}{\rho}R'-\frac{m^2}{\rho^2}R+\left(\frac{\lambda}{\rho}-\frac{1}{4}\right)R=0,$$

where ρ is the rescaled radial coordinate r, m integer, $m\in\mathbb{Z}$, and λ the rescaled energy.

4. (*Optional) Try to derive this rescaled form of the radial equation (with the Coulomb potential) from the equation you obtained in the previous problems.
5. Investigate the behavior of the equation at $\rho\to\infty$, i.e. obtain the asymptotic solution $R^\infty(\rho)$.
6. Perform a substitution $R(\rho)=R^\infty(\rho)G(\rho)$ and obtain the equation for $G(\rho)$
7. Investigate the behvaior of $G(\rho)$ at samll ρ.

[*General hint*: In all the problems above you may find it useful to follow the derivations in Chapter 8 of the Gasiorowicz book. But bare in mind that the resulting equations for 2D are slightly different from those in 3D.]

II. SELECTION RULES

The structure of atomic levels can be studied by means of the absorption spectroscopy. A hydrogen-like atom adsorbs light only at certain frequencies corresponding to the energies of transitions between discrete atomic levels. The strongest in intensity transitions are the dipole transitions related to the dipole operator $\mathbf{d} = b\mathbf{r}$, where $\mathbf{r}$ is the position operator and b constant. Specifically, for a linearly polarized light the amplitude of the transition between states $|nlm\rangle$ and $|n'l'm'\rangle$ of the electron is proportional to

$$A_{\text{dip}} \sim \langle n'l'm'|\mathbf{r}|nlm\rangle = \int \Psi^*_{n'l'm'}(\mathbf{r})\,\mathbf{r}\,\Psi_{nlm}(\mathbf{r})\,d\mathbf{r},$$

where n, l, m and n', l', m' are the quantum numbers of the initial and final states, respectively. A transition $\langle n'l'm'|\mathbf{r}|nlm\rangle$ is said to be forbidden if A_{dip} is identically zero, independently of the radial part of the wavefunctions.

1. Use the transformation properties of wavefunctions $\Psi_{nlm}(\mathbf{r})$ under inversion symmetry, $\mathbf{r} \to -\mathbf{r}$, to find which combinations of l and l' are definitely forbidden by this symmetry, i.e. $A_{\text{dip}} = 0$.
2. (*Optional) For which combinations of m, m' dipole transitions are allowed?
   [*Hint:* Evaluate the matrix elements of the following commutators,

   $$[L_z, z], \qquad [L_z, x+iy], \qquad [L_z, x-iy],$$

   in two different ways: directly and using commutator relations.]

Quantum Mechanics I, Sheet 11, Spring 2015

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I. $l - 3/2$ ANGULAR MOMENTUM OPERATOR

For a given quantum number l the matrix elements of the angular momentum operator are given by the formulae

$$\begin{aligned}\langle lm'|L_z|lm\rangle &= \hbar\, m\, \delta_{mm'}\\ \langle lm'|L_\pm|lm\rangle &= \hbar\, \sqrt{l(l+1) - m(m\pm 1)}\, \delta_{m',m\pm 1}\end{aligned}$$

Take $l = 3/2$ (in this case m can take values $-3/2, -1/2, 1/2, 3/2$) and evaluate the matrices for operators L_z, L_y.

II. MATRIX DIAGONALIZATION

Consider the matrix

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Find its eigenvalues and the unitary matrix U that diagonalizes the matrix.

III. BASIS TRANSFORMATION

The standard basis for describing a particle with angular momentum $L = 1$ consists of states $|1, m\rangle$, where $m = -1, 0, 1$. Let us introduce a new basis consisting of states

$$\begin{aligned}|1\rangle &= \cos\theta|1,-1\rangle + \sin\theta|1,0\rangle,\\ |2\rangle &= -\cos\theta|1,1\rangle + \sin\theta|1,0\rangle,\end{aligned}$$

where θ is some arbitrary angle.

1. Use the formulae from Problem I to find L_z and L_x in the new basis, i.e. evaluate matrix elements $\langle\alpha|L_z|\beta\rangle$ and $\langle\alpha|L_x|\beta\rangle$, with $\alpha, \beta = 1, 2$.
2. We know that it is not possible to measure L_z and L_x simultaneously because these operators do not commute. How can you reconcile the result of the previous subproblem with this statement?