EXACT CLASSICAL SOLUTIONS OF THE TWO DIMENSIONAL SIGMA MODEL☆

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Received 30 July 1976

We construct several new exact static solutions to the coupled, non-linear classical field equations of the sigma model, both with and without pions, in two space-time dimensions. We compare these solutions with previously known exact results and discuss briefly their implications for states in the quantum field theory.

The development of semi-classical approximation methods [1] in quantum field theories has stimulated substantial interest in localized solutions to the non-linear differential equations corresponding to the "classical" limits of the quantized equations of motion. In some instances, these classical solutions, coupled with the approximation schemes, provide very deep insight into the full theories, including, in the case of the two-dimensional sine-Gordon equation, what is apparently the exact bound state spectrum of the quantized theory [2, 3].

In the present note we derive non-trivial static solutions to the classical equations of motion which follow from the two-dimensional, chiral invariant sigma model Lagrangian with iso-scalar pions,

$$\mathcal{L} = \bar{\psi}(\mathrm{i}\gamma^{\mu}\partial_{\mu} - g(\sigma + \mathrm{i}\pi\gamma_5))\,\psi + \tfrac{1}{2}((\partial_{\mu}\sigma)(\partial^{\mu}\sigma) + (\partial_{\mu}\pi)(\partial^{\mu}\pi)) - \lambda/4(\sigma^2 + \pi^2 - f^2)^2. \tag{1}$$

The four dimensional version of this theory has recently been studied in two very interesting contexts: (1) in the SLAC "bag" calculations [4], as a model for the confinement of quarks; and (2) in the Lee-Wick [5] and pion condensation [6] calculations, as a model which suggests the existence of "abnormal" states of nuclear matter at high baryon density. In addition, in the two dimensional theory, two exact classical solutions are already known. In the model with only a σ field, there is a "shell" solution [4], which consists of a fermion in a zero energy bound state superimposed on the familiar ϕ^4 kink [7]. This solution has classical energy $E_{\mathrm{S}} = (2/3)\sqrt{2\lambda}\,f^3$. In the full chiral model, one has the "chiral confinement" solution [8], obtained by enforcing the restriction $\rho^2 \equiv \sigma^2 + \pi^2 = f^2$ for all x, with classical energy $E_{\mathrm{CC}} = 8gf^3 \sin(1/8f^2)$. The relations between these solutions and ours will provide further insight into the results of refs. [4] and [8].

One important aspect of our approach is that it is both systematic and constructive. Summarized very briefly [9] the method involves using the "trace identities" [10, 11] to replace the expression for the action in terms of the fields by an equivalent expression in terms of the (as yet undetermined) scattering data – the bound state energies and normalizations and the reflection coefficient – associated with the "potential" (a known expression in terms of the fields σ and π) in the Dirac equation. Varying the action then determines the scattering data, which, by solving the inverse problem, determine the "potential" and hence σ, π, and ψ.

Ignoring all quantum effects, we may approximate the action per unit time appropriate to the one fermion sector with time independent σ and π fields as [11]

$$S/T = \int_{-\infty}^{\infty} \mathrm{d}x\,(\mathcal{L}_{\mathrm{M}}(\sigma, \pi)) - \omega_{\mathrm{o}}\{\sigma, \pi\} \tag{2}$$

where $\mathcal{L}_{\mathrm{M}}$ is the purely mesonic part of $\mathcal{L}$ and ω_{o} is the energy of the one occupied fermion state in the potential generated by the σ and π fields.

When only σ and ψ are present, the two component Dirac equation can readily be converted into an equivalent Schrödinger problem [11]. By inspecting the trace identities [10, 11] for this problem, we observe

☆ Work performed under the auspices of the United States Energy Research and Development Administration.

that, if $\lambda = 2g^2$, the integral over $\mathcal{L}_M$ can be replaced by an expression involving the scattering data only. *We shall henceforth study only the case* $\lambda = 2g^2$ ‡1. Then making the replacement and minimizing over the reflection coefficient, with the result that $r(k) = 0$, we find that the action can be written as

$$S/T = -(8/3g^2)k_o^3 - \omega_o = -E \tag{3}$$

where E is the classical energy of the full field configuration and where the single bound state momentum k_o and energy ω_o are related by the condition $k_o = (g^2f^2 - \omega_o^2)^{1/2}$. Minimizing (3) with respect to k_o yields the constraint $\omega_o k_o = g^2/8$, or, in units of g (the only independent parameter with dimensions of mass),

$$k_o(f^2 - k_o^2)^{1/2} = 1/8. \tag{4}$$

This equation admits real solutions for k_o only for $f \geqslant 1/2$, and for $f > 1/2$ there are always two distinct solutions‡2. For large f, these solutions behave as $(k_o)_+ \sim f - 1/(128f^3)$ and $(k_o)_- \sim 1/(8f)$. We emphasize this large f limit because it is in this regime – since $f \gg 1$ corresponds to weak coupling [4, 7] – that the quantum corrections to the classical solutions should be of least importance. Since the potential is reflectionless and has only one bound state, it is straightforward to reconstruct its explicit form and hence to find σ and ψ [10]. We obtain‡3,4, with $\psi(x, t)$

$$= \exp(-i\omega_o t)\widetilde{\psi}(x),$$

$$\sigma(x) = f - (k_o^2/\omega_o)\operatorname{sech}(k_o x + x_o)\operatorname{sech}(k_o x - x_o) \tag{5a}$$

and

$$\widetilde{\psi}(x) = \left(\frac{k_o}{8}\right)^{1/2} \begin{pmatrix} \operatorname{sech}(k_o x + x_o) + \operatorname{sech}(k_o x - x_o) \\ -\operatorname{sech}(k_o x + x_o) + \operatorname{sech}(k_o x - x_o) \end{pmatrix} \tag{5b}$$

where (i) all dimensional quantities are measured in the appropriate powers of g, (ii) k_o is determined by f according to (4), and (iii) $\tanh x_o = (f - (f^2 - k_o^2)^{1/2})/k_o$. The interested reader can verify explicitly that these expressions do indeed solve the static field equations following from (1) when the π field is ignored‡5.

Considering the behavior of σ as shown by (5a), we see that it is natural to call the solution corresponding to $(k_o)_-$ the "shallow bag" and that corresponding to $(k_o)_+$ the "deep bag". As f decreases, the shallow bag deepens and the deep bag becomes shallower, until at $f = 1/2$ the solutions become the same. The classical energies of these states, given exactly by the last equality in (3), are for large f approximated by $E_{SB} = f - 1/(384f^3)$ and $E_{DB} = (8/3)f^3 + 1/(16f)$. The interpretation of the shallow bag state for large f is thus clear; it is the fundamental fermion of mass $m/g = f$, renormalized slightly by a small classical $\tilde{\sigma} \equiv \sigma - f$ field. Since the shallow bag has lower energy than the bare fermion, it is clearly stable at the classical level for large f. Referring to fig. 1, we see that a comparison of the SLAC "shell" [4] solution with the deep bag solution is very illuminating; for large f, the deep bag looks like a "shell- anti-shell" ($S\bar{S}$) state. Since the exact expression for the shell energy is, for $\lambda = 2g^2$, $E_S = (4/3)f^3$, for finite f the deep bag has slightly *greater* energy than a free shell- anti-shell configuration. Thus at the purely classical level, the deep bag state appears unstable to decay into (S + S). Further, at the classical level for large f, it also appears unstable to decay into a shallow bag plus fundamental meson excitations. However, to analyze meaningfully the stability of the corresponding states in the quantum theory as functions of f, one must calculate the quantum corrections – which are $O(f)$ [7, 13] and therefore larger

‡1 Since this restriction relates a fermion Yukawa coupling (g) to a purely boson coupling (λ), it might appear that when $\lambda = 2g^2$ some "supersymmetry" makes the model solvable. In fact, the interpretation of this restriction is much more mundane: $\lambda = 2g^2$ is simply a point at which the exact solutions can be written in terms of elementary functions, since at this point σ and ψ follow by algebraic manipulation from a one-dimensional Schrödinger potential which is reflectionless and has one bound state and is thus itself an elementary function. For $\lambda \neq 2g^2$ solutions similar to the deep and shallow bags will exist, but they will (probably) not be given by elementary functions.

‡2 The non-existence of solution for $f < 1/2$ is analogous to restrictions on coupling constants in other two dimensional field theories [e.g. 3, 8].

‡3 Our convention for the Dirac matrices is $\gamma_0 = \sigma_3$, $\gamma_1 = i\sigma_1$, and $\gamma_5 = \gamma_0\gamma_1 = -\sigma_2$, where the σ_j are the Pauli spin matrices.

‡4 The same analytic forms of σ and ψ, but with a different constraint on k_o, also solve the classical equations of the Gross-Neveu model. The similarity between our method and that of refs. [3, 8] explains this result. See refs. [3, 8] and Lee et al. [12].

‡5 By "static" solution we mean that the fields are time independent in their rest frames; clearly, the Lorentz-boosted solutions will satisfy the full field equations, including time derivatives.

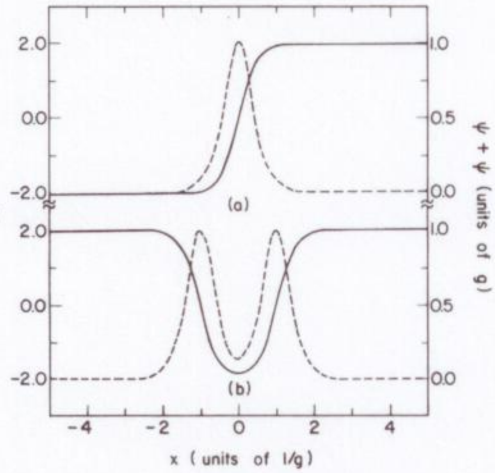

Fig. 1. Plot versus x of the σ field amplitude (solid lines, left scale) and the fermion density, $\psi^\dagger\psi$ (dashed lines, right scale) for $f = 2$ for (a) the SLAC "shell" (with $\lambda = 2g^2$) and (b) the "deep bag".

than the difference $E_{DB} - 2E_S$ for large f – to these classical solutions. One intriguing possibility is that the deep bag is a resonance in $(S\bar{S})$ scattering; we are currently studying this problem.

To study the case when both σ and π fields are present, one is essentially compelled to work directly with the Dirac "potential" rather than with an equivalent Schrödinger problem. Although the derivation of the trace identities is technically somewhat more complicated [9], the final results are almost the same as those for the Schrödinger equation. In particular, we find that the equation analogous to (3) becomes

$$S/T = -(4/3g^2)k_o^3 - \omega_o = -E \qquad (6)$$

which upon minimization over k_o leads to the constraint, in terms of scaled variables, $k_o(f^2 - k_o^2)^{1/2} = 1/4$. Again there exist two solutions, provided this time that $f > 1/\sqrt{2}$. In the large f limit, the two values of k_o become $(k_o)_+ \approx f - 1/(32f^3)$ and $(k_o)_- \approx 1/(4f)$. Since the Dirac potential is reflectionless and has only one bound state, using the methods of Frolov [14] it is straightforward to construct the σ and π fields and thence the field ψ. We obtain[‡6], with

[‡6] The same analytic forms, but with a different constraint on k_o, solve the chiral generalization of the Gross-Neveu model [15].

$$\psi(x, t) = \exp(-i\omega, t)\widetilde{\psi}(x),$$

$$\sigma = f - (k_o^2/f)(1 - \tanh k_o x) \qquad (7a)$$

$$\pi = -(1/4f)(1 - \tanh k_o x) \qquad (7b)$$

and

$$\widetilde{\psi} = \left(\frac{k_o(f-\omega_o)}{4f}\right)^{1/2} \operatorname{sech} k_o x \begin{pmatrix} k_o/(f-\omega_o) \\ 1 \end{pmatrix}. \qquad (7c)$$

As $x \to +\infty$, $\sigma \to f$ and $\pi \to 0$, whereas as $x \to -\infty$, $\sigma \to f - 2(k_o^2/f)$ and $\pi \to -1/(2f)$. But in each limit $\sigma^2 + \pi^2 \equiv \rho^2 \to f^2$ and thus the solutions merely connect chiral vacua corresponding to different values of the chiral angle $\theta \equiv \tan^{-1}(\pi/\sigma)$; as $x \to +\infty$, $\theta \to 0$, whereas for $x \to -\infty$, $\theta \to \tan^{-1}[1/(2(2k_o^2 - f^2))]$. The two solutions to the eigenvalue condition for k_o correspond for large f to a "shallow chiral bag" (SCB) of energy $E_{SCB} = f - 1/(96f^3)$ and a "deep chiral bag" (DCB) of energy $E_{DCB} = (4/3)f^3 + 1/(8f)$. Observing that in the large f limit the "chiral confinement" [8] solution has energy $E_{CC}/g \approx f - 1/(384f^3)$, we see that both the SCB and CC solutions correspond to the fundamental fermion renormalized by small (for large f) $\hat{\sigma}$ and π fields. Variational arguments can explain the result $E_{SCB} < E_{CC}$, since in the CC solution the constraint $\rho = f$ is enforced for all x. The stability of the SCB solution is assured by its being, for large f, the lowest-lying fermion state. To interpret the DCB solution we note that as $x \to -\infty$, $\sigma \to f - 2(k_o)_+^2/f \approx -f$ for large f and π is everywhere $O(1/f) \approx 0$. Further, the bound state energy of the fermion, $\omega_o \approx 1/(4f)$, approaches zero for large f, while the total DCB energy becomes $E_{DCB} \approx (4/3)f^3 + 1/(8f)$. Thus the deep chiral bag corresponds in the large f limit precisely to a chiral generalization of the "shell" state! This suggests that, at the classical level, the DCB solution is stable. It seems quite interesting that despite the manifest differences between the two theories – and indeed between the shell and DCB solution for finite f – the two states approach each other as $f \to \infty$.

We close with several remarks on various aspects of our results. First, the generalization of these solutions to theories in which fermions have an SU(N) internal symmetry is straightforward and can be used to study the behavior of states – toy "nuclei" – with $n_o < N$ fermions as a function of n_o. This may be of interest to the question of abnormal nuclear matter [5, 6]. Second, the qualitative similarities between states in different theories – for example, the similarity among the DCB and shell solutions and the soliton solution to the sine-Gordon equation [2, 3] – suggests that a persistence of "types" of states in different theories, at least in two dimensions. This leads to the idea of "perturbing" around states of an exactly solvable model – for example, the sine-Gordon equation – as a means of studying states in models which are not exactly solvable [2]. Finally, although the method of solution is unfortunately at present not extendable to (3 + 1) dimensions, since the analogues of the trace identities are not known, it may well prove very useful in other two dimensional field theories of interest in particle physics or statistical mechanics. Further, the exact solutions should be valuable as ways of testing various approximation schemes which can be applied in higher dimensions.

I am very grateful for discussions with and comments from M. Feigenbaum, A. Neveu, A. Patrascioiu, and particularly R. Dashen and S.-S. Shei.

Note added: Since the submission of this article we have studied in detail the stability of the "deep bag" and "deep chiral bag". We find that both of these field configurations are unstable and hence that they will not correspond to states in the quantum field theory. The details of this analysis will appear shortly [9].

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