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Semiclassical analysis of bound states in the two-dimensional σ model*

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Using semiclassical methods, we study the bound-state spectrum in the chiral-symmetric σ model, with SU(N) fermions and both with and without pions, in one space plus one time dimension. In the σ-only case, we find a rich spectrum composed of n ($\leq N$)-particle bound states which can be termed "normal nuclei." In a specific limit, these states approach exactly those found by Dashen, Hasslacher, and Neveu in the Gross-Neveu model, and, in general, we argue that the σ model can be viewed as a low-energy, "effective field theory" for the Gross-Neveu model. In addition to the "normal nuclei," the σ model contains (previously known) "shell" states—a ϕ^4 kink plus n ($\leq N$) fermions—which are analogs of the "abnormal" nuclei suggested by Lee and Wick. We discuss the relations among these "normal" and "abnormal" states; the solvability of our model allows us to examine approximation schemes used in recent field-theoretic studies of "normal" and "abnormal" nuclear matter. In the ($\sigma + \pi$) case we also find a rich spectrum of "normal nuclei." There are no kinklike states. In a specific limit the "normal nuclei" approach exactly those found by Shei in the chiral Gross-Neveu model. However, since our semiclassical results predict in this case the spontaneous breakdown of a continuous symmetry in two dimensions, their interpretation remains unclear. We speculate on the implications of our predictions for theories with small explicit symmetry breaking and on the mechanism by which the semiclassical results could be altered by quantum-fluctuations corrections.

I. INTRODUCTION AND SURVEY OF RESULTS

The past two years have seen remarkable progress in the use of semiclassical approximation methods to study relativistic quantum field theories.[1-12] These methods, which transcend standard perturbation theory, have revealed unexpectedly rich bound-state spectra in a variety of model theories. Among these theories is at least one—the quantum sine-Gordon theory in two dimensions—in which the semiclassical methods[2] are now known to predict the *exact* bound-state spectrum.[13]

In the present article we apply semiclassical techniques to find bound states in the two-dimensional version of the familiar σ model.[14] More precisely, we study that variant of the model with isoscalar pions and with SU(N)-symmetric fermions; this is described by the (unrenormalized) Lagrangian

$$\begin{aligned}\mathcal{L} &= \bar{\psi}^{(\alpha)}(i\gamma^\mu\partial_\mu - g(\sigma + i\pi\gamma_5))\psi^{(\alpha)} \\ &\quad + \tfrac{1}{2}[(\partial_\mu\sigma)(\partial^\mu\sigma) + (\partial_\mu\pi)(\partial^\mu\pi)] \\ &\quad - \tfrac{1}{4}\lambda(\sigma^2 + \pi^2 - f^2)^2 \\ &\equiv \bar{\psi}^{(\alpha)}[i\gamma^\mu\partial_\mu - g(\sigma + i\pi\gamma_5)]\psi^{(\alpha)} + \mathcal{L}_M\,, \qquad (1.1)\end{aligned}$$

where the index α is the internal [SU(N)] symmetry label of the fermions. In one space and one time dimension the theory is superrenormalizable; the coupling constants g and λ have dimensions $(\text{mass})^1$ and $(\text{mass})^2$, respectively. The constant f is a dimensionless parameter which measures the intrinsic strength of the meson coupling ($\lambda/4$) relative to the meson mass squared ($m_\sigma^2|_{\text{tree}} = 2\lambda f^2$); thus $f^2 \gg 1$ is the *weak* (meson) coupling limit, and it is in this regime that we expect our semiclassical analysis to be valid.

A second dimensionless parameter in the theory, $a \equiv \lambda/g^2$, which measures the relative strength of meson-meson to fermion-meson couplings, will in all our calculations be fixed at $a = 2$. This restriction is inextricably tied to our method of solving the model, as the discussions of Secs. III and IV will show. A third dimensionless parameter characterizing the theory is, of course, N, the index of SU(N). As we indicate below and demonstrate in Sec. IV, the interplay of this parameter with f^2 can have important consequences—particularly for the ($\sigma + \pi$) case—for the spectrum of bound states.

When only the σ field is present, $\mathcal{L}$ is invariant under the discrete chiral transformation

$$\begin{aligned}&\sigma \to \sigma' = -\sigma\,, \\ &\psi \to \psi' = \gamma_5\psi\,.\end{aligned} \qquad (1.2)$$

Further, as in the Gross-Neveu model,[15] the apparent SU(N) symmetry is in fact an O($2N$) symmetry.[3] In the tree approximation, the fundamental field excitations have masses $m_\psi \equiv m = gf$ and $m_\sigma = \sqrt{2\lambda}f$. When both σ and π are present, $\mathcal{L}$ is invariant under the continuous chiral transformations

$$\begin{aligned}&\sigma \to \sigma' = \sigma\cos\theta + \pi\sin\theta\,, \\ &\pi \to \pi' = \pi\cos\theta - \sigma\sin\theta\,, \\ &\psi \to \psi' = e^{i\gamma_5\theta/2}\psi\end{aligned} \qquad (1.3)$$

as well as under the SU(N) internal symmetry of the fermions. The fundamental field excitations have masses, in the tree approximation, $m_\psi \equiv m = gf$, $m_\sigma = \sqrt{2\lambda} f$, and $m_\pi = 0$.[16,17]

Both the fundamental distinction between discrete and continuous symmetries (particularly in two dimensions)[16,17] and certain technical aspects of our approach dictate that in the ensuing sections we treat the σ-only and $(\sigma+\pi)$ cases separately.

Apart from the obvious general interest in any model field theory which can be studied analytically in a nonperturbative approximation, the σ model possesses specific interest in view of the variety of contexts in which the four-dimensional version has recently been studied. In the SLAC "bag" calculations,[7] the σ model without pions has been used to study questions regarding quark confinement. In this analysis, several calculations were motivated and interpretations clarified by an exact classical solution to the two-dimensional version of the theory.[7] In nuclear physics, the σ model has recently been used, in the Lee-Wick[18] and pion-condensation[19] calculations, to suggest the existence of "abnormal" states of nuclear matter at very high baryon density. Further, it has been analyzed as a possible phenomenological field theory of normal nuclei.[20] To study these nuclear physics applications, we can interpret the SU(N) symmetry as allowing us to construct bound states with varying numbers of "nucleons"; hence we can study the behavior of these "nuclei" as the number of constituents, n_0, varies. Here we shall find some very interesting results, particularly in comparing the various levels of approximation in which the theory has been studied.

To describe our results clearly, it is necessary to indicate briefly the nature of the semiclassical approximation in the σ model; a detailed discussion of the method appears in Sec. II. In essence, our approach is identical to that used by Dashen, Hasslacher, and Neveu (DHN) in their study of the Gross-Neveu model.[3,21] The bound-state energies[22] are determined by evaluating a functional integral over the meson and fermion fields in which the integrand is the exponential of the action. Since the fermion fields enter the action bilinearly, they can be integrated out *exactly*,[3,23] leaving an effective action in terms of the meson fields only. The functional integral over the meson fields is then *approximated* by the method of stationary phase. Using certain features of the elegant inverse scattering method,[24,25] we are able to find exact, analytic solutions to the stationary-phase condition—which in terms of the meson fields yields a set of coupled, nonlinear, partial differential equations—subject to the two restrictions that (1) *the meson fields are time independent* and (2) as noted above, $\lambda = 2g^2$. The details of the manipulations and the origins of these restrictions are discussed in Secs. III and IV. For time-independent meson fields the semiclassical approximation to the quantum bound-state energies is simply[1]

$$E \simeq -\bar{S}_{\text{eff}}(T)/T + \cdots, \tag{1.4}$$

where $\bar{S}_{\text{eff}}(T)$ is the effective action evaluated at the stationary-phase point. Actually, a "complete" semiclassical analysis would require, in addition to $\bar{S}_{\text{eff}}$, evaluation of (at least) the Gaussian functional integral corresponding to small fluctuations about the stationary-phase point. Although we have been unable to evaluate the appropriate functional integral explicitly, in Sec. V we discuss certain important qualitative aspects of these quantum fluctuations.

One final point on the nature of our approximations is essential to an understanding of some of our results. As DHN have shown[3]—and as we shall see explicitly in Sec. II—the contributions of the fermions to the effective action can be separated into two parts which, roughly speaking, correspond to (1) the "occupied positive-energy states" and (2) a "negative-energy sea." In the true semiclassical analysis, both contributions must (and can) be included; this is done in Sec. IV. But we have also found it extremely instructive to isolate a further, "classical" approximation, in which only the effects of the occupied states are included. Our primary reason for treating this classical approximation separately in Sec. III is that it corresponds precisely to an approximation frequently used in field-theoretic calculations of nuclear matter.[18-20] Comparison of the semiclassical and classical results thus allows us to test this approximation in a solvable model.

With these preliminaries aside, we can now describe our results, beginning with the σ-only case. From the semiclassical analysis of Sec. IV we deduce the spectrum indicated schematically in Fig. 1. The bound states in the leading tower correspond to time-independent solutions to the stationary-phase condition and are found explicitly by our analysis. They are bound states of n_0 fermions: the "nuclei" of the two-dimensional σ model. The other states shown in Fig. 1, which would follow from time-dependent solutions, are not found explicitly, but the arguments advanced in Sec. IV suggest their existence: the state labeled "A," for example, is clearly present, since it is just the elementary σ of mass $m_\sigma|_{\text{tree}} = \sqrt{2\lambda} f = 2gf \equiv 2m$, for $\lambda = 2g^2$. The spectrum shown in Fig. 1 is strikingly similar[26] to that found by DHN[3] for the Gross-Neveu model. Actually, it differs slightly from these DHN results in that the details of the spectrum—as well as the numerical values

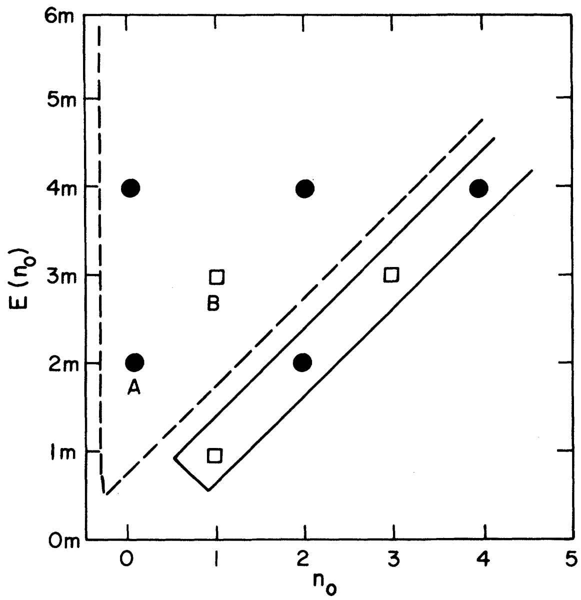

FIG. 1. A schematic illustration of the first few ($n_0 \leq 4$, $E \leq 4m$) bound-state levels in the σ model for large N. The equal spacing is approximately valid in this limit. The states in the "leading tower" (boxed area) are found explicitly by our time-dependent methods. The existence of states in the triangular region is strongly suggested by the correspondence between the σ and Gross-Neveu models as discussed in Sec. IV. The degeneracy of each dot (boson) or box (fermion) is $2N!/(2N-n_0)!n_0!$.

of the energy levels—depend on the parameter $\alpha \equiv 4f^2/N$, which is, roughly speaking, a measure of the relative importance of meson-to-fermion contributions to the bound-state energies. Nonetheless, we argue in Sec. IV that despite obvious differences in the theories, the σ model can be thought of as an "effective field theory" for the Gross-Neveu model in the same spirit that a field theory of pions and nucleons would represent a low-energy, "effective" version of an underlying theory of quarks and gluons. This connection offers the exciting opportunity to study the connections between underlying and effective theories in solvable models.

In addition to the states in Fig. 1 there are in the σ-only case the previously known SLAC "shell" states,[7] which consist of n_0 fermions "trapped" in zero-energy states on a ϕ^4 "kink." In Secs. III and IV we indicate that, in contrast to the "normal nuclei" discussed above, these shell states are the two-dimensional analogs of finite "abnormal nuclei"—"density isomers"[20]—first suggested by Lee and Wick[18] and subsequently studied by many authors.[27-29] Perhaps the central result of our comparison of the classical and semiclassical approximations is that while the former predicts that large normal nuclei can fission into two "abnormal states" $(S+\bar{S})$, in the latter analysis this possibility essentially disappears. This result confirms recent assertions of the importance of including vacuum fluctuation effects in comparative studies of normal and abnormal matter.[27-29]

In the $(\sigma+\pi)$ case the semiclassical predictions for the bound-state spectra depend strongly on the parameter $\alpha \equiv 4f^2/N$. For $\alpha \gg 1$, there exist N bound states with energies

$$E(n_0) \simeq n_0 m - \frac{n_0^3 m}{6N^2\alpha^2} \simeq n_0 m - \frac{n_0^3 m}{96 f^4}\,, \qquad (1.5)$$

where $1 \leq n_0 \leq N$. These states are thus normal nuclei, weakly bound in this limit. Note that there are no obvious analogs of abnormal nuclei in the $(\sigma+\pi)$ model because a kinklike state is *not* topologically stable in this theory. As α is decreased, the higher-lying bound states disappear, until for $\alpha \ll 1$, the spectrum stabilizes at precisely (for $\alpha \to 0$) that found by Shei in the chiral Gross-Neveu model, namely, a total of $N/2$ bound states with energies

$$E(n_0) = \frac{Nm}{\pi}\sin\frac{\pi n_0}{N}\,, \quad 1 \leq n_0 \leq N/2\,. \qquad (1.6)$$

Although in view of our previous remarks it is tempting to view the $(\sigma+\pi)$ model for small α as an effective field theory of the chiral Gross-Neveu model, there are at least two problems with this interpretation. First, there are no known analogs in either model of the time-dependent states shown in the triangular region of Fig. 1. Thus the similarity of the spectra here is by no means as certain as in the σ-only case. Second, our semiclassical analysis predicts the spontaneous breakdown of a continuous symmetry in two dimensions, in apparent violation of Coleman's theorem.[16,17] In the chiral Gross-Neveu model, this violation is evaded by a decoupling of the putative Goldstone boson.[11] In Sec. V we argue that a similar decoupling does not appear likely in the $(\sigma+\pi)$ model (for $\alpha \neq 0$); hence, the nature of the connection between the two models remains unresolved.

To conclude these introductory remarks we summarize the contents of the remaining sections. In Sec. II the details of the semiclassical approximation in the σ model are presented. Section III treats the classical approximation to the stationary-phase conditions and bound-state spectra in both the σ-only and $(\sigma+\pi)$ cases. The full semiclassical analysis of these cases is carried out in Sec. IV. Qualitative considerations regarding quantum fluctuation corrections to the semiclassical analysis and spontaneous breakdown of continuous symmetries are discussed in Sec. V. Finally, two appendixes provide supplementary technical details. Appendix A derives the trace identities for the Dirac equation, which are needed in the calculations of Secs. III and IV. Appendix B details the explicit reconstruction of the fields which solve the stationary-phase conditions.

II. THE SEMICLASSICAL APPROXIMATION IN THE σ MODEL

To study the bound-state spectrum in the σ model we use the semiclassical techniques developed in a series of papers by Dashen, Hasslacher, and Neveu.[1-3] In particular, our analysis relies heavily on their treatment[3] of the Gross-Neveu model,[15] and we shall henceforth refere to this specific paper as DHN.

Following DHN, we begin with the expression

$$\operatorname{tr} e^{-iHT} = \int [d\sigma][d\pi][d\bar{\psi}][d\psi]\exp\left\{i\int_0^T dt\int_{-\infty}^{\infty} dx[\mathcal{L}_M + \bar{\psi}(i\gamma\cdot\partial - g(\sigma + i\pi\gamma_5))\psi]\right\}, \tag{2.1}$$

where the meson fields satisfy periodic boundary conditions and the fermion fields antiperiodic boundary conditions.[3] Since the fermions enter the action bilinearly, the functional integral over $\{\bar{\psi}, \psi\}$ is Gaussian and can be computed exactly. Formally, this procedure yields

$$\operatorname{tr} e^{-iHT} = \int [d\sigma][d\pi]\exp\{iS_{\text{eff}}([\sigma],[\pi],T)\}, \tag{2.2}$$

where σ and π are periodic functions of time, and where the effective action has the form[30]

$$S_{\text{eff}} = \int_0^T dt\int_{-\infty}^{\infty} dx\{\mathcal{L}_M[\sigma,\pi]\} + \operatorname{tr}\ln\{[i\gamma\cdot\partial - g(\sigma + i\pi\gamma_5)]\}. \tag{2.3}$$

Thus far the manipulations are exact. The semiclassical approximation will consist of evaluating the functional integral in (2.2) by the method of stationary phase. Clearly a full semiclassical analysis of (2.2) must include all σ and π field configurations satisfying the stationary-phase conditions,

$$\left.\frac{\delta S_{\text{eff}}}{\delta\sigma}\right|_{\sigma_{\text{sp}},\pi_{\text{sp}}} = 0 = \left.\frac{\delta S_{\text{eff}}}{\delta\pi}\right|_{\sigma_{\text{sp}},\pi_{\text{sp}}}, \tag{2.4}$$

which are generalizations—because of the additional "tr ln" term in S_{eff}—of the classical—that is, c-number—equations of motion for σ and π. Each distinct stationary-phase point will correspond, loosely speaking, to a state in the quantum field theory. Included in the set of stationary-phase points are σ and π fields having nontrivial time dependence. However, in practice we have been able to solve (2.4) only for time-independent σ and π. Hence we expect to find only a subset of the complete semiclassical bound-state spectrum. For time-independent σ and π fields, S_{eff} has simple time dependence:

$$S_{\text{eff}}([\sigma],[\pi],T) = -E([\sigma],[\pi])T, \tag{2.5}$$

where E is the energy of the full field configuration. Thus denoting the action at the stationary-phase point by S_{eff}, and ignoring momentarily the corrections to the lowest-order stationary-phase approximation, we see that in the semiclassical approximation the energy of the quantum "bound state" corresponding to a stationary-phase point $[\sigma_{\text{sp}}, \pi_{\text{sp}}]$ will be

$$E([\sigma_{\text{sp}}],[\pi_{\text{sp}}]) \simeq -\frac{S_{\text{eff}}([\sigma_{\text{sp}}],[\pi_{\text{sp}}],T)}{T} + \cdots, \tag{2.6}$$

To make these formal manipulations more precise, let us begin by restricting our considerations to time-independent σ and π. Then the interpretation of the tr ln term in S_{eff} is intuitively clear: It represents the fermion loop[31] (see Fig. 2) in the presence of external fields $\sigma(x)$ and $\pi(x)$, and its contribution to S_{eff} can be expressed in terms of the eigenvalues ω_j of the Dirac equation

$$\left[\gamma_0\omega_j + i\gamma_1\frac{d}{dx} - g(\sigma + i\pi\gamma_5)\right]\psi_j(x) = 0. \tag{2.7}$$

Notice that the ψ's here are simply two-compo-

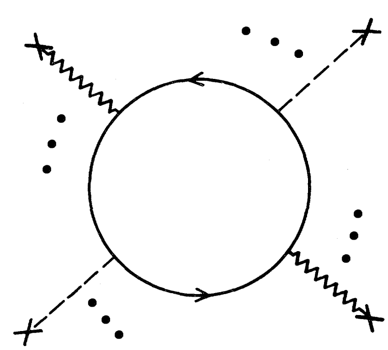

FIG. 2. The fermion loop in the presence of external σ (wiggly lines) and π (dashed lines) fields. As discussed in Sec. II this loop contains contributions from both "occupied states" and the "negative-energy sea."

nent, c-number spinors satisfying (2.7); in particular, they are *not* anticommuting c numbers (elements of a Grassmann algebra).

The precise form of the tr ln contribution follows most directly from a result of DHN[3,32]: For time-independent σ and π,

$$\prod_{\alpha=1}^{N}\int[d\bar\psi^{(\alpha)}][d\psi^{(\alpha)}]\exp\left(i\int_0^T dt\int_{-\infty}^{\infty}dx\{\bar\psi^{(\alpha)}[i\gamma\cdot\partial-g(\sigma+i\pi\gamma_5)]\psi^{(\alpha)}\}\right)$$
$$=\exp\left(iTN\sum_j\frac{|\omega_j|}{2}\right)\prod_j(1+e^{-iT|\omega_j|})^N$$
$$=\exp\left(iNT\sum_j\frac{|\omega_j|}{2}\right)\sum_{\{n_i\}}C(N,\{n_i\})\exp\left(-iT\sum_j n_j|\omega_j|\right), \tag{2.8}$$

where the sums over ω_j are over positive and negative eigenvalues of (2.7). Thus the ω_j are themselves functionals of σ and π. Further.

$$C(N,\{n_i\})=\prod_i\frac{N!}{(N-n_i)!\,n_i!}, \tag{2.9}$$

and the sum over $\{n_i\}$ is over all finite sets of integers such that $0\le n_i\le N$. Using (2.8) we can write (2.1) in the form

$$\mathrm{tr}e^{-iHT}=\sum_{\{n_i\}}C(N,\{n_i\})\int[d\sigma][d\pi]\exp\{iS_{\text{eff}}([\sigma],[\pi],\{n_i\})\}, \tag{2.10}$$

where

$$S_{\text{eff}}([\sigma],[\pi],\{n_i\})=T\int_{-\infty}^{\infty}\mathcal{L}_M(\sigma,\pi)\,dx+\frac{TN}{2}\sum_j|\omega_j([\sigma],[\pi])|-T\sum_i n_i|\omega_i([\sigma],[\pi])|. \tag{2.11}$$

We can interpret the three terms in (2.11) as representing contributions to the action from, respectively, the meson field configuration, a fermionic "sea" with each state filled by N fermions, and, finally, fermion states occupied by $n_i<N$ fermions. Further, $C(N,\{n_i\})$ in (2.13) represents the degeneracy of a configuration labeled by "occupation number" $\{n_i\}$. At this point we should digress briefly to mention an important distinction between the $(\sigma+\pi)$ and σ-only cases. Since the pion field is odd under charge conjugation,[3] in the general $(\sigma+\pi)$ case there is no relation between the positive and the negative ω_i. However, in the σ-only case these energy eigenvalues must come in charge-conjugate pairs, $-\omega_j^{(-)}=\omega_j^{(+)}>0$. In this case, the degeneracy of the bound-state spectrum is clarified by rewriting (2.8) as[3]

$$\prod_{\alpha=1}^{N}\int[d\bar\psi^{(\alpha)}][d\psi^{(\alpha)}]\exp[i\bar\psi^{(\alpha)}(i\gamma\cdot\partial-g\sigma)\psi^{(\alpha)}]=\exp\left(-iNT\sum_{\omega_j<0}\omega_j\right)\sum_{\{n_i\}}C(2N,\{n_i\})\exp\left(-iT\sum_{\omega_i>0}n_i\omega_i\right), \tag{2.12}$$

where in the sum over $\{n_i\}$, $0\le n_i\le 2N$. In (2.12) we have used the symmetry $\omega_j^{(-)}=-\omega_j^{(+)}$ to sum over only negative-energy states in the filled sea and over only positive-energy occupied states. Note that the degeneracy of a configuration labeled by occupation numbers $\{n_i\}$ is now $C(2N,\{n_i\})$. Thus in the σ-only case, the analogs of (2.10) and (2.11) are

$$\mathrm{tr}e^{-iHT}=\sum_{\{n_i\}}C(2N,\{n_i\})\int[d\sigma]\exp\{iS_{\text{eff}}([\sigma],\{n_i\})\} \tag{2.13}$$

and

$$S_{\text{eff}}([\sigma],\{n_i\})=T\int_{-\infty}^{\infty}\mathcal{L}_M(\sigma)dx-NT\sum_{\omega_i<0}\omega_i([\sigma])-T\sum_{\omega_i>0}n_i\omega_i([\sigma]). \tag{2.14}$$

The expressions (2.10) and (2.13) are *exact* in their respective cases, subject only to our restriction to time-independent meson fields. To proceed with our semiclassical analysis, we approximate these exact functional integrals by the method of stationary phase. Restricting our comments to the σ-only case for notational simplicity, we see that, formally, this yields

$$\mathrm{tr}\,e^{-iHT} \simeq \sum_{\{n_i\}} C(2N,\{n_i\})\exp(i\bar{S}_{\rm eff})\int[d\sigma]\exp\left[i\,\frac{\delta^2 S_{\rm eff}}{\delta\sigma^2}\bigg|_{\sigma_{\rm sp}}\frac{(\sigma-\sigma_{\rm sp})^2}{2}+\cdots\right]$$

$$\simeq \sum_{\{n_i\}} C(2N,\{n_i\})\exp(i\bar{S}_{\rm eff})\left[\det\left(\frac{\delta^2 S_{\rm eff}}{\delta\sigma^2}\bigg|_{\sigma_{\rm sp}}\right)\right]^{-1/2}. \qquad (2.15)$$

Here $\sigma_{\rm sp}$ is the solution to the stationary-phase condition, which is formally

$$\frac{\delta S_{\rm eff}}{\delta\sigma}([\sigma],\{n_i\})\bigg|_{\sigma_{\rm sp}}=0,$$

and $\bar{S}_{\rm eff}\equiv S_{\rm eff}([\sigma_{\rm sp}],\{n_i\})$. Further, the final approximate equality comes from evaluating the Gaussian integral over $\sigma_q\equiv\sigma-\sigma_{\rm sp}$ that results from neglecting higher than second-order terms in the expansion of $S_{\rm eff}$ around $\sigma_{\rm sp}$. From (2.15) using (2.14) and (2.6) we can write the semiclassical approximation to the (unrenormalized) energy of a quantum bound state labeled by $\{n_i\}$ as

$$E^u(\{n_i\})=\left\{-\int_{-\infty}^{\infty}\mathcal{L}_M(\sigma_{\rm sp})dx+\sum_{\omega_i>0}n_i\,\omega_i([\sigma_{\rm sp}])+N\sum_{\omega_i<0}\omega_i([\sigma_{\rm sp}])\right\}+\frac{1}{T}\,\frac{1}{2}\,\mathrm{tr}\ln\left(\frac{\delta^2 S_{\rm eff}}{\delta\sigma^2}\bigg|_{\sigma_{\rm sp}}\right)+\cdots. \qquad (2.16)$$

This result requires several clarifying and interpretive comments. First, we remark that the final term, which represents the Gaussian approximation to quantum fluctuations about the stationary-phase point, $\sigma_{\rm sp}$, will be discussed—though not explicitly evaluated—in Sec. V. Second, we reiterate that the terms in braces, which come from $\bar{S}_{\rm eff}$, include contributions from occupied positive-energy states ($\sum_{\omega_i>0}n_i\,\omega_i$) as well as from a field negative-energy sea ($N\sum_{\omega_i<0}\omega_i$). In a classical approximation, one would be tempted to include only the contributions of the occupied states and to ignore the effects of the negative-energy sea (or, at best, to include these as a perturbation). Strictly speaking, of course, this does not treat the fermion effects consistently in orders of $\hbar$, since both contributions come from the fermion loop in Fig. 2. However, this separation of occupied states and negative-energy sea is often intuitively reasonable: In field-theoretic treatments of low-energy nuclear physics,[20] for example, the approximation of ignoring negative-energy sea effects is often valid. In other circumstances—as we shall later see—the two contributions can be comparable, and it is essential that both be kept. To illustrate both circumstances we discuss separately in Sec. III the classical approximation—that is, the additional approximation to (2.16) in which the negative-energy sea effects are ignored—before proceeding to the full semiclassical analysis in Sec. IV. We emphasize that these two calculations are quite distinct: The classical approximation solves the stationary-phase condition including *only* occupied fermion states,

$$\frac{\delta}{\delta\sigma}\left(\int\mathcal{L}_M-\sum_{\omega_i>0}n_i\,\omega_i\right)=0, \qquad (2.17)$$

whereas the full semiclassical approximation requires that

$$\frac{\delta}{\delta\sigma}\left(\int\mathcal{L}_M-\sum_{\omega_i>0}n_i\,\omega_i-N\sum_{\omega_i<0}\omega_i\right)=0. \qquad (2.18)$$

Finally, we should remark on the method by which we solve the stationary-phase conditions (2.17) and (2.18) and their generalizations when pions are present. As we have indicated, in terms of σ and π these conditions become c-number field equations which are in fact coupled, nonlinear, partial—or, in the time-independent case, ordinary—differential equations. In general, it is not possible to find exact solutions to these equations directly. However, it will prove possible—as in several previous analyses of similar models[3,9-11]—to solve the equations indirectly, using certain aspects of the powerful inverse scattering method.[24] The details of this approach are presented in Secs. III and IV.

III. THE CLASSICAL APPROXIMATION[33]

In the classical approximation only the contributions of the occupied fermion states are included in $S_{\rm eff}$. Further, to determine the spectrum of bound states, only those occupied states corresponding to discrete eigenvalues of the Dirac equation need to be considered. In fact, it suffices to consider only a single discrete eigenvalue, ω_0, since additional contributions would enter additively in S_{eff} and thus, by (2.6), would lead to no new bound states in the field theory.[3]

In the sector in which n_0 fermions occupy a single discrete state ω_0, the classical approximation to the effective action per unit time can be written as

$$S_{eff}/T=\int_{-\infty}^{\infty} dx[\mathcal{L}_M(\sigma,\pi)]-n_0\omega_0([\sigma],[\pi]). \qquad (3.1)$$

Since there are no infinities in (3.1), no renormalizations are required; thus the classical approximation illustrates the calculational scheme with a minimum of complication. To proceed with the detailed calculations based on (3.1), we must treat the σ-only and $(\sigma+\pi)$ cases separately.

A. The σ-only case

Applying the stationary-phase condition to S_{eff} as given in (3.1) yields precisely the classical field equations, which for the (time-independent) σ-only case are[34]

$$\left[\omega_0\gamma_0+i\gamma_1\frac{d}{dx}-g\sigma(x)\right]\psi_0(x)=0 \qquad (3.2a)$$

and

$$-\frac{d^2\sigma}{dx^2}+\lambda\sigma(\sigma^2-f^2)=-gn_0\bar{\psi}_0\psi_0. \qquad (3.2b)$$

From the original form of S, in which the fermion fields enter explicitly, these equations appear obvious. That they in fact follow from $\delta S_{eff}/\delta\sigma=0$ is made clear by noting first that $\omega_0([\sigma])$ must be determined by (3.2a) and then by observing that, from first-order perturbation theory,

$$\delta\omega_0/\delta\sigma(x)=-g\bar{\psi}_0(x)\psi_0(x), \qquad (3.3)$$

so that varying S_{eff} explicitly with respect to σ yields (3.2b). In (3.2b) and (3.3) we have assumed that

$$\int_{-\infty}^{\infty}\psi_0^\dagger(x)\psi_0(x)dx=1, \qquad (3.4)$$

and thus the factor of n_0 appears in (3.2b).[34]

Three remarks should be made concerning (3.2). First, we reiterate that ψ_0 is an ordinary, two-component, c-number wave function, satisfying the Dirac equation with energy eigenvalue ω_0. Second, (3.2a) and (3.2b) are clearly "self-consistent" equations for $\sigma(x)$ and $\psi_0(x)$, and indeed they correspond precisely to the Hartree approximation familiar from field-theoretic calculations of nuclear matter.[20] This observation indicates why recent studies[8] of analogous self-consistent calculations in similar two-dimensional field theories have led to the same results as the semiclassical methods. Further, since a self-consistent calculation can be valid in a strong-coupling limit,[35] this equivalence offers hope that the semiclassical results may in some instances be correct even for strong coupling. Third, there exists a known solution to the coupled equations (3.2). It is the two-dimensional "SLAC shell,"[7] which consists of a ϕ^4 "kink" with n_0 fermions occupying an $\omega_0=0$ discrete state. Explicitly, with the conventions $\gamma_0=\sigma_3$ and $\gamma_1=i\sigma_1$,

$$\sigma(x)=f\tanh[(\lambda/2)^{1/2}fx] \qquad (3.5a)$$

and

$$\psi_0(x)=N[\cosh(\lambda/2)^{1/2}fx]^{-g/(\lambda/2)^{1/2}}\begin{pmatrix}1\\1\end{pmatrix}, \qquad (3.5b)$$

where N is a normalization factor. Since $\bar{\psi}_0(x)\psi_0(x)=0$, there is no fermion feedback in (3.2b), and hence the kink in (3.5a) remains a solution. Further, since $\omega_0=0$, the energy of the shell is independent of n_0, and is given, in the classical approximation, by

$$E_S=\tfrac{2}{3}\sqrt{2\lambda}f^3. \qquad (3.6)$$

The σ field and fermion density $\rho\equiv\psi_0^\dagger\psi_0$ are plotted in Fig. 3; the derivation of the forms of σ and ψ_0 in (3.5) using our techniques is discussed in Appendix B.

To solve (3.2) systematically we begin by returning to the explicit form of (3.1) in the time-

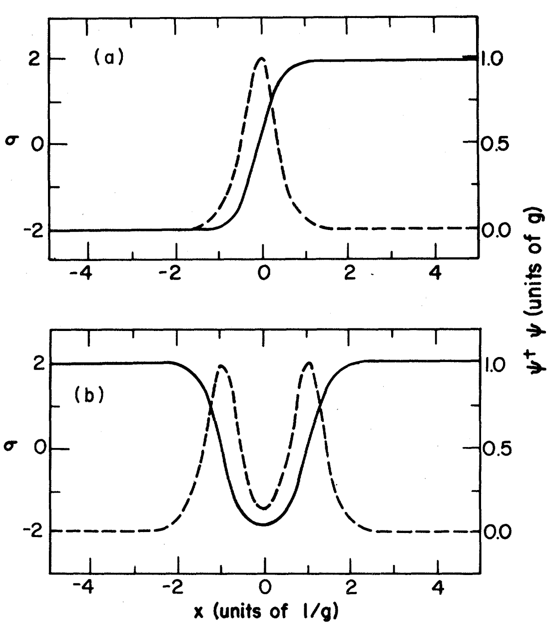

FIG. 3. Plot versus x of the σ field amplitude (solid lines, left scale) and the fermion density, $\psi^\dagger\psi$ (dashed lines, right scale) with $f=2$ for (a) the SLAC "shell" (with $\lambda=2g^2$), and (b) the unstable "deep-bag" configuration for $n_0=1$.

independent, σ-only case:

$$S_{\text{eff}}/T = -\int dx\left[\frac{1}{2}\left(\frac{d\sigma}{dx}\right)^2 + \tfrac{1}{4}\lambda(\sigma^2 - f^2)^2\right] - n_0\omega_0([\sigma]). \tag{3.7}$$

Summarized very briefly, our approach will be to replace this expression for S_{eff} by an equivalent expression in terms of the (as yet undetermined) scattering data—the discrete energy eigenvalues and normalizations and the reflection coefficient—associated with the "potential" [here, a known function of $\sigma(x)$] in the Dirac equation (3.2a). Varying the action with respect to the scattering data (instead of the fields) will yield algebraic equations which determine these data. Then by solving the inverse problem for the Dirac equation by the methods of Frolov,[36] we can construct explicitly $\sigma(x)$ and $\psi_0(x)$. That this "change of variables" is a natural one is suggested by the appearance of ω_0—the discrete energy eigenvalue of (3.2a)—in (3.7). That it is possible follows from the "trace identities" for the Dirac equation which we derive in Appendix A.[37] Referring to this appendix, we observe that the second trace identity becomes, in the σ-only case,[38]

$$\int_{-\infty}^{\infty}\left[\frac{1}{2}\left(\frac{d\sigma}{dx}\right)^2 + \frac{g^2}{2}(\sigma^2 - f^2)^2\right]dx = \frac{8\kappa_0^3}{3g^2} - \frac{1}{\pi g^2}\int_{-\infty}^{\infty} q^2 dq\,\{\ln[1 - |s_{12}^{(+)}(q)|^2] + \ln[1 - |s_{12}^{(-)}(q)|^2]\}$$
$$\equiv \frac{8\kappa_0^3}{3g^2} - F([s_{12}]). \tag{3.8}$$

Here $\kappa_0^2 \equiv g^2f^2 - \omega_0^2$, $s_{12}(q)$ is the reflection coefficient, and to obtain (3.8) from (A17b) we have used the charge-conjugation symmetry requirement, $\omega_0^{(+)} = |\omega_0^{(-)}| = \omega_0$, which must hold in the σ-only case. Comparing (3.8) with (3.7), we observe that *if* $\lambda = 2g^2$, we can write S_{eff} completely in terms of scattering data:

$$S_{\text{eff}}/T = -\frac{8\kappa_0^3}{3g^2} - n_0\omega_0 + F([s_{12}]). \tag{3.9}$$

Thus the origin of the restriction $\lambda = 2g^2$ is clear[39]: It is a sufficient (and probably also necessary) condition for making the change of variables which permits a simple analytic solution to the stationary-phase condition.

The independence and completeness of the scattering data[40] allow us to vary S_{eff} separately with respect to s_{12} and κ_0. The variation with respect to the reflection coefficient implies that $s_{12}^{(+)} = s_{12}^{(-)} = 0$; the potentials are reflectionless. Varying S_{eff} with respect to κ_0, and recalling that $\omega_0^2 = g^2f^2 - \kappa_0^2$, leads to the algebraic constraint

$$\omega_0\kappa_0 = n_0 g^2/8, \tag{3.10}$$

which quantizes the energy levels as functions of n_0. Defining

$$\omega_0 = gf\cos\phi \tag{3.11a}$$

and

$$\kappa_0 = gf\sin\phi \tag{3.11b}$$

allows us to rewrite (3.10) as

$$\sin 2\phi(n_0) = \frac{n_0}{4f^2}. \tag{3.12}$$

This equation determines $\phi(n_0)$ and, as can be seen from Fig. 4(a), admits two solutions for $\phi(n_0)$ whenever $(n_0/4f^2) < 1$. The energy—that is, the mass of the bound state[22]—corresponding to a solution $\phi(n_0)$ is

$$E(\phi(n_0)) = -S_{\text{eff}}/T\,|_{\phi(n_0)}$$
$$= \tfrac{8}{3}gf^3\sin^3\phi(n_0) + n_0 gf\cos\phi(n_0) \tag{3.13a}$$
$$= \tfrac{8}{3}gf^3\left[\sin\phi(n_0) + \frac{n_0}{4f^2}\cos\phi(n_0)\right], \tag{3.13b}$$

where (3.13b) follows by using (3.12).

The interpretation of these results is aided by reconstructing the explicit forms of $\sigma(x)$ and $\psi_0(x)$. Since the Dirac potential is reflectionless and has only one bound state, using the methods of Frolov[36] this reconstruction is straightforward. Referring to Appendix B for the details, we obtain[41]

$$\sigma(x) = f - \frac{\kappa_0^2}{g\omega_0}\operatorname{sech}(\kappa_0(x+x_0))\operatorname{sech}(\kappa_0(x-x_0)) \tag{3.14a}$$

and

$$\psi_0(x,t) = \left(\frac{\kappa_0}{8}\right)^{1/2} e^{-i\omega_0 t} \times \begin{pmatrix} \operatorname{sech}(\kappa_0(x+x_0)) + \operatorname{sech}(\kappa_0(x-x_0)) \\ -\operatorname{sech}(\kappa_0(x+x_0)) + \operatorname{sech}(\kappa_0(x-x_0)) \end{pmatrix}, \tag{3.14b}$$

where κ_0 and ω_0 are determined in terms of n_0 and f by (3.10) and where $\tanh\kappa_0 x_0 = (gf - \omega_0)/\kappa_0$. The dedicated reader can verify that these expressions do indeed solve the time-independent classical field equations, (3.2), when (3.10) is invoked. For $n_0/4f^2 \ll 1$—the "weak" meson coupling limit—the two solutions to (3.10) behave as $(\kappa_0(n_0))_+ \simeq gf - gn_0^2/128f^3$ and $(\kappa_0(n_0))_- \simeq gn_0/8f$. Considering

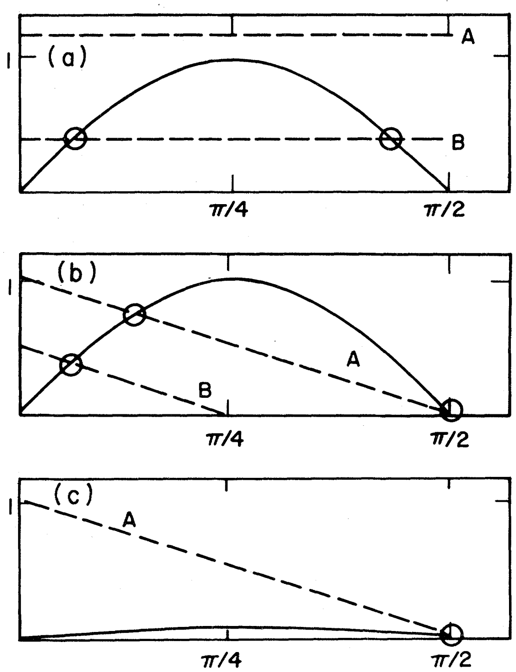

FIG. 4. Graphical solutions to the quantization condition in the σ-only case. Solid lines represent the left-hand side and dashed lines the right-hand side of the quantization equation. The cases shown are as follows: (a) The classical approximation [Eq. (3.12)]. For $n_0/4f^2 > 1$ (dashed line A), there are no solutions. For $n_0/4f^2 < 1$ (dashed line B), there are two solutions, corresponding to a stable "shallow bag" [$\phi(n_0) < \pi/4$] and an unstable "deep bag" [$\phi(n_0) > \pi/4$]. (b) The semiclassical approximation [Eq. (4.12)] for $4f^2/N = 1 > 1/\pi$. Dashed line A shows the two solutions for $n_0 = N$: a "shallow bag" ($\phi < \pi/4$) and the $(S\bar{S})$ state ($\phi = \pi/2$). Dashed line B shows a unique solution for $n_0 < N$. (c) The semiclassical approximation [Eq. (4.12)] for $4f^2/N = 0.1 < 1/\pi$. Dashed line A shows the unique solution for $n_0 = N$: the $(S\bar{S})$ state at $\phi = \pi/2$. Clearly all $n_0 < N$ also have unique solutions, which correspond to "shallow bags."

the behavior of σ as shown by (3.14), we see that it is natural to call the solutions corresponding to $(\kappa_0(n_0))_-$ "shallow bags" and those corresponding to $(\kappa_0(n_0))_+$ "deep bags." As $n_0/4f^2 \to 1$, the shallow- and deep-bag solutions become equal, and for $n_0/4f^2 > 1$ there are no solutions. Thus at fixed f there exists a maximum n_0, and at fixed n_0, a minimum f.

From these results we can determine the classical predictions for the bound-state spectrum in the σ model. The shallow bags should correspond to a set of bound states with energies, for $n_0/4f^2 \ll 1$,

$$E_{\text{SB}}(n_0) \simeq n_0 gf - g\frac{n_0^3}{384f^3} + \cdots . \tag{3.15}$$

In a picturesque terminology which we shall shortly find useful, these states are a sequence of loosely bound (for small $n_0/4f^2$) "normal nuclei" containing n_0 "nucleons." The degeneracy of a given "nucleus" is shown by (2.15) to be $(2N!)/[(2N-n_0)!n_0!]$. The deep bags appear to form a sequence of states with energies, for $n_0/4f^2 \ll 1$,

$$E_{\text{DB}}(n_0) \simeq \tfrac{8}{3}gf^3 + \frac{gn_0^2}{16f} + \cdots . \tag{3.16}$$

But recalling that the SLAC shell ("S") solution has energy, for $\lambda = 2g^2$, $E_S = \frac{4}{3}gf^3$, we see that the deep bags also appear energetically unstable to decay into an $(S+\bar{S})$ state; the similarity between the deep bag and $(S+\bar{S})$ field configurations is indicated in Fig. 3.

This stability analysis can be made more precise by noting that a *necessary* condition for stability is

$$d^2E/d\phi^2\big|_{\phi(n_0)} > 0.$$

From (3.13a) we see that

$$\left.\frac{d^2E}{d\phi^2}\right|_{\phi(n_0)} = 8gf^3\sin\phi(2-3\sin^2\phi) - n_0 gf\cos\phi\Big|_{\phi(n_0)}$$
$$= 8gf^3\sin\phi(n_0)\cos 2\phi(n_0), \tag{3.17}$$

where the final equality follows by using (3.12). Thus $d^2E/d\phi^2$ is positive only for $0 \le \phi < \pi/4$, and hence *all* the deep bags are indeed unstable.

For the shallow bags, the question of stability is more subtle. Clearly they all satisfy the necessary criterion for stability. Further, we can verify directly from the explicit form of E_{SB} as a function of n_0,

$$E_{\text{SB}}(n_0) = \tfrac{8}{3}gf^3[\tfrac{1}{2}(1+n_0/4f^2)^{3/2} - \tfrac{1}{2}(1-n_0/4f^2)^{3/2}]$$
$$\equiv \tfrac{8}{3}gf^3 B(n_0/4f^2), \tag{3.18}$$

that the shallow bags are stable against the possible decay

$$\text{SB}(n_1+n_2) \to \text{SB}(n_1) + \text{SB}(n_2). \tag{3.19}$$

But from (3.18) we also observe that for $n_0/4f^2$ in the range $0.7 \lesssim n_0/4f^2 \le 1$, $B(n_0/4f^2) \ge 1$, so that $E_{\text{SB}}(n_0) > 2E_S$; thus it appears that, in the classical approximation, the shallow-bag states for "large" $n_0/4f^2$ are energetically unstable to decay in $S+\bar{S}$. Further, these transitions are topologically allowed. Readers familiar with the abnormal nuclear matter states first suggested by Lee and Wick[18] and subsequently analyzed by numerous authors[20,27-29] will recognize this possible decay as the two-dimensional analog of the fission of a normal nucleus (shallow bag) into two abnormal nuclei $(S+\bar{S})$.[42] Of course, ours is only a toy model of nuclei, since it is two-dimensional and since we need the SU(N) internal symmetry (all the fermions are in the same spatial state) to construct multifermion states. Nonetheless, it is very interesting to see that, at the classical level, the model does predict a normal → abnormal transition. In view of the recent interest in studying the effects of quantum corrections on the existence of abnormal nuclei,[20,27–29] it will be perhaps even more interesting to see what happens to these transitions when the effects of the negative-energy sea are included.

B. The $\sigma + \pi$ case

In the case with σ and π the stationary-phase condition in the classical approximation yields the equations

$$\left[\omega_0\gamma_0 + i\gamma_1\frac{d}{dx} - g(\sigma + i\pi\gamma_5)\right]\psi_0(x) = 0, \tag{3.20a}$$

$$-\frac{d^2\sigma}{dx^2} + \lambda\sigma(\sigma^2+\pi^2-f^2) = -gn_0\bar{\psi}_0\psi_0, \tag{3.20b}$$

$$-\frac{d^2\pi}{dx^2} + \lambda\pi(\sigma^2+\pi^2-f^2) = -ign_0\bar{\psi}_0\gamma_5\psi_0. \tag{3.20c}$$

Again the correspondence with a self-consistent approach[8] is manifest. Further, as in the σ-only case, there exists a previously known solution to (3.20): This is the "chiral-confinement" result,[43] which solves (3.20) subject to the constraint $(\sigma^2 + \pi^2) \equiv \rho^2 = f^2$, independent of x. For consistency, this constraint requires that $\lambda/g^2 \to \infty$, and thus the chiral-confinement solution probes an *a priori* quite different limit of the theory than do our solutions, which are found for $\lambda/g^2 = 2$. The explicit form of the chiral-confinement solution is not necessary for our purposes; we note only that, in the sector with n_0 fermions, the energy of this solution is[43]

$$E_{cc} = 8gf^3\sin\left(\frac{n_0}{8f^2}\right). \tag{3.21}$$

Since the solution to (3.20) proceeds almost as in the σ-only case, we can be brief. From (A17b), we observe that

$$S_{\rm eff}/T = -\int dx\left\{\frac{1}{2}\left[\left(\frac{d\sigma}{dx}\right)^2 + \left(\frac{d\pi}{dx}\right)^2\right] + \tfrac{1}{4}\lambda(\sigma^2+\pi^2-f^2)^2\right\} - n_0\omega_0([\sigma],[\pi]) \tag{3.22}$$

can, *for* $\lambda = 2g^2$, be written entirely in terms of the scattering data only:

$$S_{\rm eff}/T = -\frac{4}{3g^2}\kappa_0^3 - n_0\omega_0([\sigma],[\pi]) + \frac{1}{\pi g^2}\int_{-\infty}^{\infty} q^2dq\{\ln[1-|s_{12}^{(+)}(q)|^2] + \ln[1-|s_{12}^{(-)}(q)|^2]\}. \tag{3.23}$$

Here, in view of our earlier remarks, we have included only one occupied positive-energy state $\omega_0^{(+)} \equiv \omega_0 > 0$ in (A17b). Varying (3.23) with respect to the reflection coefficients establishes $s_{12}^{(\pm)} = 0$. Varying $S_{\rm eff}$ with respect to κ_0 leads to the quantization condition $\kappa_0\omega_0 = n_0g^2/4$ or, in terms of $\phi \equiv \tan^{-1}(\kappa_0/\omega_0)$,

$$\sin 2\phi = \frac{n_0}{2f^2}. \tag{3.24}$$

As before there are two solutions, $\phi(n_0)$, provided in this case that $n_0/2f^2 < 1$. These solutions have energies—the putative bound-state masses[22]—

$$E(\phi(n_0)) = -S_{\rm eff}/T|_{\phi(n_0)}$$
$$= \tfrac{4}{3}gf^3\sin^3\phi(n_0) + n_0gf\cos\phi(n_0) \tag{3.25a}$$
$$= \tfrac{4}{3}gf^3\left[\sin\phi(n_0) + \frac{n_0}{2f^2}\cos\phi(n_0)\right]. \tag{3.25b}$$

To interpret these results we first reconstruct $\sigma(x)$, $\pi(x)$, and $\psi_0(x)$. From Appendix B we obtain[44]

$$\sigma(x) = f - (\kappa_0^2/g^2f)(1-\tanh\kappa_0x), \tag{3.26a}$$

$$\pi(x) = -\frac{\kappa_0\omega_0}{g^2f}(1-\tanh\kappa_0x), \tag{3.26b}$$

and

$$\psi_0 = \left[\frac{\kappa_0(gf-\omega_0)}{4gf}\right]^{1/2} e^{-i\omega_0t}\,{\rm sech}\kappa_0x\begin{pmatrix}\dfrac{\kappa_0}{gf-\omega_0}\\ 1\end{pmatrix}. \tag{3.26c}$$

These expressions solve Eqs. (3.20) exactly. As $x \to +\infty$, $\sigma \to f$ and $\pi \to 0$, whereas for $x \to -\infty$, $\sigma \to f - 2(\kappa_0^2/g^2f)$ and $\pi \to -2\omega_0\kappa_0/g^2f$. But in each limit, $\sigma^2 + \pi^2 = \rho^2 \to f^2$, and thus the solutions merely connect chiral vacua corresponding to different values of the chiral angle $\theta \equiv \tan^{-1}(\pi/\sigma)$; as $x \to +\infty$, $\theta \to 0$, whereas for $x \to -\infty$,

$$\theta = \tan^{-1}\left(\frac{2\omega_0\kappa_0}{\kappa_0^2 - \omega_0^2}\right) = -2\phi. \tag{3.27}$$

Thus, since ϕ is quantized by (3.24), the change in the chiral angle θ is also quantized.[11]

For $n_0/2f^2 \ll 1$, the two solutions for κ_0 are $(\kappa_0(n_0))_- \simeq gn_0/4f$ and $(\kappa_0(n_0))_+ \simeq gf - gn_0^2/32f^3$. Thus the classical prediction for the bound-state spectrum includes a sequence of "shallow chiral bags"

(SCB's) with energies

$$E_{\rm SCB}(n_0) \simeq n_0 g f - g n_0^3/96 f^3 + \cdots \tag{3.28}$$

and "deep chiral bags" (DCB's) with energies

$$E_{\rm DCB}(n_0) \simeq \tfrac{4}{3} g f^3 + \frac{g n_0^2}{8f} + \cdots . \tag{3.29}$$

From (2.10), we deduce that the degeneracy of the states with n_0 fermions is given by $N!/[(N-n_0)!n_0!]$. The shallow chiral bags correspond to loosely bound (for $n_0/2f^2 \ll 1$) nuclei; in this regard, it is interesting to observe that (for $n_0/2f^2 \ll 1$), the chiral-confinement solutions have energies

$$E_{cc} \simeq n_0 g f - \frac{g n_0^3}{384 f^3} + \cdots , \tag{3.30}$$

and thus also correspond to loosely bound (for $n_0/2f^2 \ll 1$) nuclei. The apparent persistence[45] of this type of state in these two *a priori* very different limits of the theory—$\lambda/g^2 = 2$ in our case, $\lambda/g^2 \to \infty$ in the chiral-confinement model—is itself quite interesting.

For $n_0/2f^2 \ll 1$, the similarity of the DCB states and the SLAC shell solution to the σ-only case is striking. Not only are the energies roughly equal equal—$E_S = \frac{4}{3} g f^3$ versus $E_{\rm DCB} \simeq \frac{4}{3} g f^3 + g n_0^2/8f$—but, from (3.26) we see that π is everywhere $O(n_0/f) \sim 0$ and further that as $x \to -\infty$, $\sigma \to f - 2(\kappa_0)_+{}^2/g^2 f \simeq -f$. Thus it appears that the deep chiral bags correspond to chiral generalizations of the SLAC shell. However, the necessary stability criterion, $d^2E/d\phi^2 > 0$, is easily seen to be violated by the DCB states; hence these field configurations are unstable and do not correspond to states in the quantum theory. Physically, this result is understood by noting that although $\sigma \to -f$ as $x \to -\infty$, $\rho \equiv (\sigma^2 + \pi^2)^{1/2} \to +f$ for both $x \to \pm\infty$. The continuous symmetry of the Lagrangian permits us to change $\theta \equiv \tan^{-1}(\pi/\sigma)$ *smoothly* from $-\pi$ to 0 in the region of large negative x, and thus the DCB states are not topologically stable. This is the first illustration in our analysis of the essential difference between discrete and continuous symmetries in two dimensions. A more serious consequence of this difference is that, since the *continuous* chiral symmetry of this two-dimensional theory with $\sigma + \pi$ seems spontaneously broken, the results of the present classical analysis are in apparent contradiction with Coleman's theorem.[17] As we shall see in the following section, this difficulty is *not* resolved by the inclusion of the negative-energy sea effects. To avoid a necessarily long digression at this juncture, we postpone until Sec. V a detailed discussion of this vital question.

IV. RENORMALIZATION AND EFFECTS OF THE NEGATIVE-ENERGY SEA

The classical analysis of the preceding section solved only an approximate version—(2.17)—of the true stationary-phase condition, (2.18). We chose to distinguish this approximation because (1) its similarity to approximation schemes used in recent field theory descriptions of nuclear physics[20] allows us to scrutinize these schemes in a solvable model, and (2) it allowed introduction of our application of inverse scattering techniques with a minimum of complications. In the present section, we show that these same techniques can be used to solve the full stationary-phase condition. Once again, for clarity of presentation we treat separately the σ-only and $(\sigma + \pi)$ cases.

A. The σ-only case

When the negative-energy sea effects are included, the stationary-phase condition contains infinities which must be removed by renormalization. Following DHN,[3] we perform this renormalization directly in $S_{\rm eff}$ by (1) subtracting the vacuum value of the negative-energy sea and (2) adding a meson mass counterterm to cancel the divergent self-energy graph in Fig. 5. The expression for the renormalized effective action then becomes

$$S_{\rm eff}/T = -\int_{-\infty}^{\infty} dx\left[\frac{1}{2}\left(\frac{d\sigma}{dx}\right)^2 + \tfrac{1}{4}\lambda(\sigma^2 - f^2)^2\right] - n_0\omega_0([\sigma]) + \frac{N}{2}\sum_i \left[\{\omega_i^{(+)}([\sigma]) - \omega_i^{(+)}(f)\} - \{\omega_i^{(-)}([\sigma]) - \omega_i^{(-)}(f)\}\right] - \tfrac{1}{2}\delta m^2 \int_{-\infty}^{\infty} dx(\sigma^2 - f^2). \tag{4.1}$$

Here the renormalizations are chosen such that the vacuum expectation value of σ is f and the vacuum energy ($\sigma = f$, $n_0 = 0$) is zero. Further, for later clarity we have explicitly indicated how the positive- ($\omega_i^{(+)} > 0$) and negative- ($\omega_i^{(-)} < 0$) energy eigenvalues enter in the negative-energy sea contribution. The explicit value of δm^2 is, with a cutoff at $|k| = \Lambda$,

$$\delta m^2(\Lambda) = \frac{N g^2}{\pi}\int_0^{\Lambda} \frac{dk}{(k^2 + g^2 f^2)^{1/2}} . \tag{4.2}$$

In Sec. III we established that the first two terms in (4.1) can be expressed in terms of scattering data *provided that* $\lambda = 2g^2$. Using the first trace

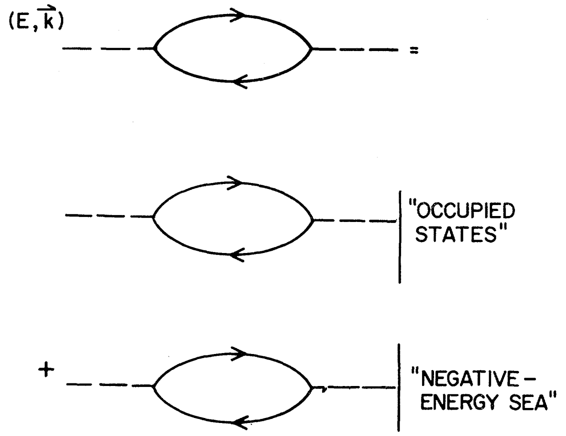

FIG. 5. The fermion contribution to the meson self-energy. The term labeled "negative-energy sea" diverges and must be renormalized.

identity, (A17a), we can also express the mass counterterm in (4.1) in terms of the scattering data. We now show that the negative-energy sea contribution to (4.1) can be written in terms of the scattering data and, further, that it contains an infinite (as $\Lambda\to\infty$) term which cancels the infinity (for $\Lambda\to\infty$) from the mass counterterm. First we write

$$\frac{N}{2}\sum_i [\{\omega_i^{(+)}([\sigma])-\omega_i^{(+)}(f)\}-\{\omega_i^{(-)}([\sigma])-\omega_i^{(-)}(f)\}]$$
$$\equiv N\{\omega_0([\sigma])-gf\}+\sum_c \qquad (4.3)$$

which separates the continuum contribution, $\sum_c$, from that of the bound states at $\omega_0^{(+)}=|\omega_0^{(-)}|\equiv\omega_0>0$. Note that the "bound-state" energy in the vacuum is just $\omega_0(f)=gf$. Putting the system in a box of length L,[1,3] we have

$$\omega_n^{(\pm)}([\sigma])=\pm(\tilde{k}_n^{(\pm)2}+g^2f^2)^{1/2} \qquad (4.4a)$$

and

$$\omega_n^{(\pm)}(f)=\pm(k_n^{(\pm)2}+g^2f^2)^{1/2}, \qquad (4.4b)$$

where

$$\tilde{k}_n^{(\pm)}L+\delta^{(\pm)}=2n\pi \qquad (4.5a)$$

and

$$k_n^{(\pm)}L=2n\pi, \qquad (4.5b)$$

with $\delta^{(+)}$ and $\delta^{(-)}$ as the phase shifts for positive and negative energies, respectively. Expanding the differences $\{\omega_n^{(\pm)}([\sigma])-\omega_n^{(\pm)}(f)\}$ for large L, making the replacement

$$\sum_n \to L\int_{-\Lambda}^{\Lambda} dk/2\pi\,,$$

and ignoring terms that vanish in the $L\to\infty$ limit, we see that

$$\sum_c=-\frac{N}{2}\frac{1}{2\pi}\int_{-\Lambda}^{\Lambda}\frac{k\,dk(\delta^{(+)}+\delta^{(-)})}{(k^2+g^2f^2)^{1/2}}\,. \qquad (4.6)$$

To reexpress (4.6) in terms of reflection coefficients and bound-state energies, we can use the dispersion relation for $\ln s_{11}$ given by (A2):

$$\delta^{(+)}+\delta^{(-)}\equiv\text{Im}(\ln s_{11}^{(+)}+\ln s_{11}^{(-)})=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dq\{\ln[1-|s_{12}^{(+)}(q)|^2]+\ln[1-|s_{12}^{(-)}(q)|^2]\}}{q-k}+4\tan^{-1}(\kappa_0/k)\,. \qquad (4.7)$$

Here, as usual, $\kappa_0{}^2=g^2f^2-\omega_0{}^2$. Combining (4.1)–(4.7) and (A17), we can write S_{eff} entirely in terms of the scattering data κ_0 and $|s_{12}^{(\pm)}|$. From the form of the reflection coefficient contribution to S_{eff}, it is easy to see that minimizing with respect to this parameter yields (as before) $s_{12}^{(\pm)}=0$. With this simplification, we can write

$$S_{\text{eff}}/T=-\frac{8\kappa_0{}^3}{3g^2}+2\delta m^2\frac{\kappa_0}{g^2}-n_0\omega_0+N(\omega_0-gf)$$
$$-\frac{2N}{\pi}\int_0^{\Lambda}\frac{k\,dk\tan^{-1}(\kappa_0/k)}{(k^2+g^2f^2)^{1/2}}\,. \qquad (4.8)$$

Using the result

$$\tan^{-1}(\kappa_0/k)=k\int_0^{\kappa_0}\frac{dy}{y^2+k^2} \qquad (4.9)$$

and changing the order of integration, we find that

$$\lim_{\Lambda\to\infty}\int_0^{\Lambda}\frac{k\,dk\tan^{-1}(\kappa_0/k)}{(k^2+g^2f^2)^{1/2}}=\lim_{\Lambda\to\infty}\kappa_0\int_0^{\Lambda}\frac{dk}{(k^2+g^2f^2)^{1/2}}$$
$$+\omega_0\tan^{-1}(\omega_0/\kappa_0)-\frac{\pi}{2}gf+\kappa_0\,. \qquad (4.10)$$

Thus

$$S_{\text{eff}}/T=-\frac{8\kappa_0{}^3}{3g^2}+(N-n_0)\omega_0-\frac{2N\kappa_0}{\pi}$$
$$-\frac{2N}{\pi}\omega_0\tan^{-1}(\omega_0/\kappa_0)\,. \qquad (4.11)$$

As before, we define $\omega_0=gf\cos\phi$, $\kappa_0=gf\sin\phi$. Then the minimization requirement, $\partial S_{\text{eff}}/\partial\kappa_0=0$, becomes in terms of ϕ, the "quantization" condition[46]

$$\frac{4f^2}{N}\sin 2\phi(n_0)=\frac{n_0}{N}-\frac{2\phi(n_0)}{\pi}\,. \tag{4.12}$$

The energy—i.e., the mass of the bound state[22]— corresponding to a solution $\phi(n_0)$ is

$$E(\phi(n_0))=\left[\tfrac{8}{3}gf^3\sin^3\phi+\left(n_0-\frac{2N\phi}{\pi}\right)gf\cos\phi+\frac{2Ngf}{\pi}\sin\phi\right]\bigg|_{\phi(n_0)}$$
$$=\left(8gf^3+\frac{2gfN}{\pi}\right)\sin\phi(n_0)-\frac{16gf^3}{3}\sin^3\phi(n_0)\,. \tag{4.13}$$

Since the Dirac "potential" remains reflectionless and with only one bound state, the functional form of σ is exactly as shown in (3.14a). Now, however, κ_0 is determined by (4.12) instead of (3.12). In Fig. 4 the full semiclassical quantization condition is contrasted with the classical condition. Note that for each $n_0<N$, there is only one solution to (4.12). From Fig. 4, it is clear that these solutions correspond to the shallow-bag states of Sec. III. Thus, for $n_0<N$, all deep-bag field configurations have disappeared.

For $n_0=N$, the situation is more complicated. If $\alpha=(4f^2/N)>1/\pi$ there exist two solutions, as indicated in Fig. 4(b).[47] As $\alpha\to 1/\pi$, the shallow-bag solution with $\phi(n_0=N)=\phi_{(1)}<\pi/2$ approaches the deep-bag solution at $\phi(n_0=N)=\phi_{(2)}=\pi/2$; for $\alpha<1/\pi$, the $n_0=N$ shallow bag disappears, leaving only the $\phi(n_0=N)=\pi/2$ solution; this is shown in Fig. 4(c). The uniqueness of this last remnant of the deep-bag solution suggests that we discuss if first. Since $\phi=\pi/2$, $\omega_0\equiv gf\cos\phi=0$, and thus

$$E_{\rm DB}(n_0=N)=\tfrac{8}{3}gf^3+\frac{2Ngf}{\pi}\,. \tag{4.14}$$

Observing that the SLAC shell corresponds to a single $\omega_0=0$ discrete state—as opposed to a charge-conjugate pair, $\omega_0^{(+)}=\omega_0^{(-)}=0$—we see that, including the negative-energy sea effects, the SLAC shell has energy[48]

$$E_S=\tfrac{4}{3}gf^3+\frac{Ngf}{\pi}\,. \tag{4.15}$$

As before, this energy is independent of the number of fermions ($0\le n_0\le N$) in the shell. Further, from the functional form of σ in (3.14a), we see that when $\omega_0\to 0$, $\tanh x_0\to 1$ and hence $x_0\to\infty$. Thus this deep-bag field configuration corresponds *precisely* to a noninteracting $(S+\bar{S})$ state and does not represent a new bound state in the theory.

That the inclusion of the negative-energy sea should have such a profound effect on the deep-bag field configurations is in fact easy to motivate physically, at least for $f^2\gg 1$. For the deep bags of Sec. III, $\omega_0(n_0)\sim n_0/f\simeq 0$ for large f^2; hence the negative-energy discrete state at $\omega_0^{(-)}<0$ lies very close to $\omega_0^{(+)}>0$, and thus the effects of this $\omega_0^{(-)}$ state should be roughly comparable to those of the occupied states. Again an analogy to nuclear matter calculations is useful: In a dispersion relation calculation of, say, the meson self-energy (see Fig. 5) the contribution of particle-hole pairs in the Fermi sea has a threshold at $E=0$, whereas the particle (antiparticle) contribution begins at $E=2m$, and is thus relatively damped, *except* in the case $m\to 0$, where it is *a priori* as important as the particle-hole contribution.[20,29]

Returning to the shallow-bag states, we see that for $f^2/N\gg 1$ (or $n_0/N\ll 1$) the energy of the state corresponding to $\phi(n_0)$ is approximately

$$E_{\rm SB}(n_0)\simeq n_0 gf-\frac{n_0^3}{N^2}\frac{gf}{24}\frac{1}{(4f^2/N+1/\pi)^2}+\cdots\,. \tag{4.16}$$

Thus the shallow bags remain a sequence of loosely bound (in this limit) nuclei; the degeneracies are still as given by (2.15).[49] The stability of these states can be studied as in Sec. III. The necessary condition becomes

$$\frac{d^2E}{d\phi^2}\bigg|_{\phi(n_0)}=2Ngf\sin\phi(n_0)\left[\frac{1}{\pi}-\frac{4f^2}{N}+\frac{8f^2}{N}\cos^2\phi(n_0)\right]$$
$$>0\,, \tag{4.17}$$

which shows that for stability we must have $0\le\phi(n_0)<\phi_0$, where[50]

$$\cos^2\phi_0=\frac{N}{8f^2}\left(\frac{4f^2}{N}-\frac{1}{\pi}\right). \tag{4.18}$$

Notice that $d^2E/d\phi^2=0$ at $\phi=\pi/2$; this reflects the "neutral" stability of the noninteracting $(S+\bar{S})$ state.

It is relatively straightforward to show that the shallow-bag states all satisfy (4.17). Further, one can show that the shallow-bag states cannot decay into each other, since

$$\frac{d^2E}{dn_0^{\,2}}=\frac{-gf\sin\phi(n_0)}{2N[1/\pi-4f^2/N+(8f^2/N)\cos^2\phi(n_0)]} \tag{4.19}$$

is less than zero for all $\phi<\phi_0$ as determined by (4.18). However, the most interesting aspect of shallow-bag stability is the essential disappearance of the transition SB$\to S+\bar{S}$ suggested by the classical results of Sec. III. By rewriting (4.13) in the form

$$E(\phi(n_0)) = 2E_S\left[\sin\phi(n_0) + \frac{(4f^2/N)}{4f^2/N + 3/\pi}\sin 2\phi(n_0)\cos\phi(n_0)\right], \tag{4.20}$$

we can show that for *all* $n_0 < N$, $E(\phi(n_0))$ is always less than $2E_S$. Phrased in the picturesque language of Sec. III, although the Lee-Wick abnormal states (SLAC shells, S) exist in the theory, normal nuclei (shallow bags for $n_0 < N$) do *not* become unstable to decay into abnormal states as f^2 decreases or as n_0 increases. Only that specific nucleus with $n_0 = N$ (and then only as $\alpha \to 1/\pi$) can fission into two abnormal states. The substantial difference between these semiclassical predictions and the classical results confirms the importance of understanding negative-energy sea effects in comparative studies of normal and abnormal matter.[20,27-29]

Finally, we observe that in the limit $\alpha \to 0$, both the quantization condition (4.12) and the energy levels [see (4.20)] become exactly equal to the corresponding expressions found by DHN from the time-independent solutions of the Gross-Neveu model.[3] Further, for $0 < \alpha \leq 1/\pi$, our solutions remain in one-to-one correspondence with the time-independent Gross-Neveu solutions; for $\alpha > 1/\pi$, we find one additional state—the $n_0 = N$ shallow bag. In Fig. 1 we have plotted schematically the bound-state energies in the σ model as predicted by our semiclassical analysis. This figure is deliberately patterned after DHN, who in the Gross-Neveu model found in addition to the "leading tower" of particles—which correspond to time-independent meson fields—a number of additional states inside the triangle indicated in Fig. 1. In the Gross-Neveu model, these states correspond to time-dependent solutions to the stationary-phase condition. Since we have been unable to find such solutions explicitly in the σ model and in view of the similarities in the models, it is of particular interest to ask whether the missing states in this triangle should exist here. The answer is clearly "yes," as the following observations establish. Consider the state labeled "A" in Fig. 1, which should be an O(2N) singlet with mass $E \simeq 2gf$ in the $n_0/N \ll 1$ or $f^2 \gg 1$ limits. In the Gross-Neveu model, this is a boson generated dynamically from vacuum fluctuations. Recalling that $m_\sigma|_{\text{tree}} = \sqrt{2\lambda}f = 2gf$ (for $\lambda = 2g^2$), we see that in the σ model this state is simply the elementary σ field. Similarly, the state labeled "B" would correspond in the σ model to a "σ + fermion" bound state with mass roughly $m_{\text{BS}} \simeq m_\sigma + m \simeq 3gf$. Further, its anticipated degeneracy is just 2N—since each type of fermion or antifermion will have the same interactions with the σ—which agrees exactly with the DHN result.[51] Thus despite obvious and significant differences between the Gross-Neveu and σ models—in high-energy behavior (renormalizable vs superrenormalizable), in symmetry breaking (dynamically generated σ vs elementary σ) and in binding mechanism ("vacuum polarization" vs explicit meson exchange)—the two theories have very similar bound-state spectra. In essence, one can view the ("simpler") σ model as a low-energy, effective field theory of the ("more fundamental") Gross-Neveu model in the same spirit that a field theory of nucleons and pions would be a low-energy effective version of an underlying gauge theory of quarks and gluons. This perspective strongly suggests that the connection be further investigated, and we are currently doing so.[51]

B. The $(\sigma + \pi)$ case

To renormalize the effective action in the $(\sigma+\pi)$ case we again subtract the vacuum energy and add a chiral-symmetric mass counterterm. Thus

$$S_{\text{eff}}/T = -\int_{-\infty}^{\infty} dx\left[\frac{1}{2}\left(\frac{d\sigma}{dx}\right)^2 + \frac{1}{2}\left(\frac{d\pi}{dx}\right)^2 + \tfrac{1}{4}\lambda(\sigma^2+\pi^2-f^2)^2\right] - n_0\omega_0([\sigma],[\pi]) - \frac{\delta m^2}{2}\int_{-\infty}^{\infty} dx(\sigma^2+\pi^2-f^2)$$
$$+\frac{N}{2}\left(\sum_{\omega_i>0}\{\omega_i([\sigma],[\pi]) - \omega_i([f,0])\} + \sum_{\omega_i<0}\{|\omega_i([\sigma],[\pi])|\} - (|\omega_i(f,0)|)\right), \tag{4.21}$$

where δm^2 is given by (4.2). The steps involved in expressing S_{eff} in terms of scattering data are essentially identical to that in the σ-only case; in particular, the constraint $\lambda = 2g^2$ is still required. The only difference here is that, in using the trace identities (A17), we keep only a single, positive-energy discrete state, $\omega_0^{(+)} \equiv \omega_0$. With this change the calculation proceeds precisely as in the previous case. After minimizing S_{eff} with respect to the reflection coefficients, with the result $s_{12}^{(\pm)} = 0$, we obtain

$$S_{\text{eff}}/T = -\frac{4}{3g^2}\kappa_0{}^3 - \left(n_0 - \frac{N}{2}\right)\omega_0 - \frac{N}{\pi}\omega_0\tan^{-1}(\omega_0/\kappa_0) - \frac{N}{\pi}\kappa_0. \tag{4.22}$$

Thus the quantization condition $\partial S_{\text{eff}}/\partial\kappa_0 = 0$ becomes, in terms of $\phi = \tan^{-1}(\kappa_0/\omega_0)$,

$$\frac{2f^2}{N}\sin 2\phi(n_0)=\frac{n_0}{N}-\frac{\phi(n_0)}{\pi}, \tag{4.23}$$

and the "mass" of the bound state corresponding to $\phi(n_0)$ is

$$E(\phi(n_0))=\left[(4gf^3/3)\sin^3\phi+gf\cos\phi\left(n_0-\frac{N\phi}{\pi}\right)+\frac{gNf}{\pi}\sin\phi\right]\Bigg|_{\phi=\phi(n_0)}$$
$$=\left(4gf^3+\frac{Nfg}{\pi}\right)\sin\phi(n_0)-(8gf^3/3)\sin^3\phi(n_0). \tag{4.24}$$

The degeneracy of a state corresponding to $\phi(n_0)$ is shown by (2.10) to be $N!/(N-n_0)!n_0!$. Comparing (4.23) with (4.12) and (4.24) with (4.13) we find an exact relation between energy levels in the $(\sigma+\pi)$ and-σ only cases, namely,[52]

$$E^{(\sigma+\pi)}(n_0)=\tfrac{1}{2}E^{(\sigma)}(2n_0). \tag{4.25}$$

Since $E^{(\sigma)}(n)$ is defined only for $n<N$, (4.25) determines the energy levels in the $(\sigma+\pi)$ case only up to $n_0=N/2$.

The apparently slight difference between the quantization conditions in the σ-only and $(\sigma+\pi)$ cases implies a substantial difference in the behavior of the bound-state spectra as the parameter α is varied. Figure 6 illustrates the solutions to (4.23) in several cases. As (4.25) would suggest, for $n_0\leq N/2$, the situation is as in the σ-only case: That is, for each $n_0<N/2$, independent of f^2/N there is only one solution (a stable shallow chiral bag), whereas for $n_0=N/2$, there are two solutions (for $\alpha>1/\pi$) corresponding to a (stable) shallow chiral bag $[\phi_{\rm SLB}(n_0=N/2)<\pi/2]$ and a deep chiral bag $[\phi_{\rm DCB}(n_0=N/2)=\pi/2]$. Comparing (4.24) and (4.15) we note that this DCB state appears to have exactly the same energy as a *single* SLAC shell; however, the remarks of Sec. III and the stability analysis below establish that the DCB is *not* a stable state in the quantum theory. As $\alpha\to 1/\pi$, the two solutions to (4.23) for $n_0=N/2$ converge at $\phi=\pi/2$, and for $\alpha<1/\pi$, only the (unstable) solution at $\phi=\pi/2$ remains.

For $n_0>N/2$—and for sufficiently large f^2/N—Fig. 6(b) shows that there can be two solutions to (4.20), $\phi_{\rm SCB}(n_0)<\phi_{\rm DCB}(n_0)$, corresponding to possible SCB and DCB states, respectively. The by now standard stability analysis establishes that, for very large f^2/N, the SCB states are stable and the DCB states are unstable. As f^2/N is decreased, the SCB and DCB states for $n_0>N/2$ approach each other. For each $\tilde{n}_0$ in the range $N/2\leq\tilde{n}_0\leq N$, there exists a critical value of the parameter f^2/N—call it $f^2/N(\tilde{n}_0)$—such that the two solutions for $\tilde{n}_0$ are equal at $f^2/N=f^2/N(\tilde{n}_0)$ and cease to exist for $f^2/N<f^2/N(\tilde{n}_0)$.[53] Thus, in contrast to the σ-only case, the spectrum of bound states here changes substantially as f^2/N varies.

For $f^2/N\to 0$, the spectrum approaches that found by Shei[11] in the chiral generalization of the Gross-Neveu model. It is thus tempting to interpret the $(\sigma+\pi)$ σ model as an effective field theory of the chiral Gross-Neveu model, but there are difficulties with this interpretation. First, neither Shei's analysis[11] nor ours has found time-dependent solu-

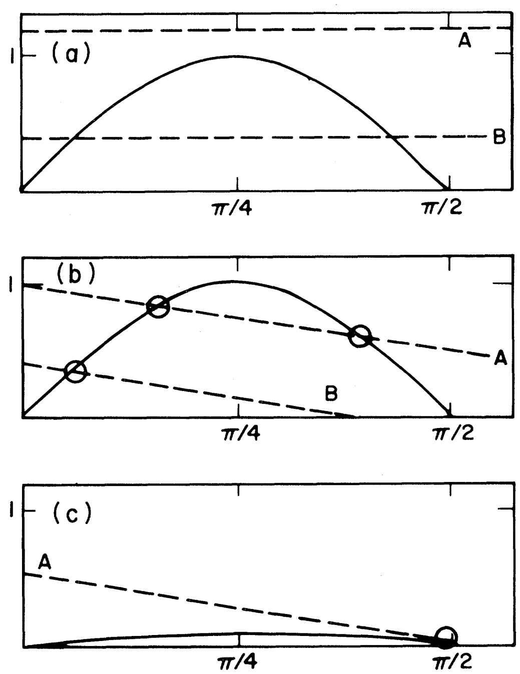

FIG. 6. Graphical solutions to the quantization condition in the $(\sigma+\pi)$ case. Solid lines represent the left-hand side and dashed lines the right-hand side of the quantization equation. The cases shown are as follows: (a) The classical approximation [Eq. (3.24)]. For $n_0/4f^2>\frac{1}{2}$—so $n_0/2f^2>1$, dashed line A—there are no solutions. For $n_0/4f^2<\frac{1}{2}$—so $n_0/2f^2<1$, dashed line B—there are two solutions, corresponding to a stable "shallow bag" $[\phi(n_0)<\pi/4]$ and an unstable "deep bag" $[\phi(n_0)>\pi/4]$. (b) The semiclassical approximation [Eq. (4.23)] for $2f^2/N=1>1/\pi$. Dashed line A shows that there are still two solutions for $n_0=N$. Actually for all $n_0\geq N/2$ there are two solutions: a stable "shallow bag" and an unstable "deep bag." Dashed line B shows the unique solution for $n_0/N=0.4<\frac{1}{2}$. (c) The semiclassical approximation [Eq. (4.23)] for $2f^2/N=0.1$ so that $4f^2/N<1/\pi$. Dashed line A shows the unique (unstable) solution at $\phi=\pi/2$ for $n_0=N/2$. Clearly all $n_0<N/2$ are also unique solutions, and there are no solutions for $n_0>N/2$.

tions; thus, a substantial segment of the bound-state spectrum remains unknown. In particular, we do not know if the chiral Gross-Neveu model contains analogs of the elementary σ—it probably does—and π—it probably does not[54]—mesons of our model. Second, our calculation stands in apparent contradiction with Coleman's theorem,[17] since we predict the spontaneous breakdown of a continuous symmetry in two dimensions. This is reflected by the pion's appearing in our semiclassical analysis as a Goldstone boson, $m_\pi = 0$. In the chiral Gross-Neveu model, the analogous putative Goldstone boson actually decouples from the theory,[11] and hence the theory avoids the conclusion of Coleman's theorem. In the next section, we show that a similar decoupling seems unlikely in the $(\sigma + \pi)$ model; thus, the relation of our semiclassical results to the actual bound-state spectrum in the $(\sigma + \pi)$ case must be examined carefully.

V. SPONTANEOUS SYMMETRY BREAKING AND QUANTUM FLUCTUATIONS

In this section, which raises more questions than it answers, we confront two important issues which have thus far been mentioned only briefly: (1) the problems associated with "spontaneous symmetry breaking" in two dimensions, and (2) the detailed nature of the quantum fluctuation corrections to our present results.

The existence of spontaneous symmetry breaking of the *discrete* chiral symmetry in the σ-only case poses no problems. Thus we anticipate that our semiclassical predictions for the bound-state spectrum are accurate in either the $f^2 \gg 1$ (see Ref. 7) *or* the $N \gg 1$ (see Ref. 3) limits. Given the equivalence in our model between the semiclassical and self-consistent methods,[8] it is also possible that our predictions remain qualitatively accurate outside these regimes.

However, in the $(\sigma + \pi)$ case the semiclassical analysis predicts the "spontaneous breakdown" of a continuous chiral symmetry, and here we run afoul of Coleman's theorem.[17,55] Briefly stated, the conclusion of this theorem is that, because the strong infrared singularities associated with two dimensions forbid the existence of a (coupled) Goldstone boson, the spontaneous breakdown (or "Goldstone realization") of a continuous symmetry *cannot* (in general) occur in two dimensions. In our calculation, this result would imply that $\langle\sigma\rangle = \langle\pi\rangle = 0$, instead of $\langle\sigma\rangle = f$, $\langle\pi\rangle = 0$. This suggests that the spectrum of bound states (if any) could be quite different from that predicted semiclassically. To discuss this point we first note that in the chiral Gross-Neveu model[11,23] and the chiral-confinement[43] solution—both of which also appear to contradict Coleman's theorem—the putative Goldstone bosons actually *decouple* from the remainder of the theory.[56] Hence the conclusion of Coleman's theorem is evaded, and the semiclassical predictions for the bound-state spectra should be valid.

Unfortunately, the arguments used to establish decoupling in these models can be shown not to apply to our case. Consider first the technique of employing Bose operators in the study of the fermion field,[57] which Shei[11] used to prove decoupling in the chiral Gross-Neveu model. It is intuitively clear that this technique will not work in our case since it involves transformations of the fermion fields only, and our calculation would predict spontaneous breakdown of the continuous chiral symmetry in the meson sector even if there were no fermion fields present in the theory. Explicit calculations verify that this intuition is correct and that the technique due to Halpern[57] does *not* indicate decoupling in the σ model.

The arguments used to establish decoupling in the chiral-confinement case[43] appear more promising but, as we shall see, also fail to establish decoupling in our model. If we introduce meson variables ρ and θ

$$\rho e^{i\theta/f} \equiv \sigma + i\pi \tag{5.1}$$

and a rotated fermion field

$$\psi' = e^{i\gamma_5\theta/2f}\psi, \tag{5.2}$$

then the Lagrangian (1.1) becomes[43]

$$\mathcal{L} = \bar{\psi}'\left[i\gamma^\mu\partial_\mu - g\rho - \frac{1}{2f}\gamma_5\gamma^\mu(\partial_\mu\theta)\right]\psi' + \tfrac{1}{2}(\partial_\mu\rho)^2 + \frac{1}{2}\left(\frac{\rho}{f}\right)^2(\partial_\mu\theta)^2 - \tfrac{1}{4}\lambda(\rho^2 - f^2)^2. \tag{5.3}$$

First, we observe that even if the fermions are ignored, the Goldstone boson θ is, in general, coupled to the field ρ. Thus there is in general no reason to anticipate that Coleman's theorem can be circumvented by a decoupling. However, in the special case that $\rho \equiv f$ independent of x, the meson part of the Lagrangian collapses to the single term $\frac{1}{2}(\partial_\mu\theta)^2$, corresponding to a free, massless scalar field. Recalling that $\rho = f$ is precisely the constraint imposed in the chiral-confinement solution, we can see immediately why—at least in the absence of fermions—decoupling occurs in this model. Further, we also see that the decoupling *cannot* be expected in the σ model. Precisely the same situation occurs when fermions are included. A close examination of the decoupling argument[43] in the chiral-confinement case reveals that it depends crucially on the constraint $\rho = f$ and hence cannot be used to establish decoupling in the σ model.

It thus seems likely that Coleman's theorem cannot be evaded and hence that the semiclassical analysis may not predict the actual bound-state spectrum of the $(\sigma+\pi)$ model. However, for a variety of reasons this result is perhaps not as disconcerting as it seems. First, if we view the two-dimensional theories as models to exhibit possible features of higher-dimensional theories—as, for example, the two-dimensional shell indicates the essential features of the four-dimensional SLAC "bag"—then we should remember that Coleman's theorem is in a sense a pathology of two dimensions and that our semiclassical results might well indicate features of the σ model in higher dimensions. Second, if we added a small chiral-symmetry-breaking term—$\epsilon\sigma$ or $\epsilon\bar{\psi}\psi$, for example—to the σ model, there would be no problem with Coleman's theorem. Further, various examples suggest[43,55] that the bound-state spectrum of this broken symmetry theory might be—at least as far as the "leading tower" of "nucleons" is concerned—very similar to the results found here.[58]

Finally, we note that understanding when and how the semiclassical method can fail is interesting in its own right. In the present case, it is clear that the semiclassical results follow from a mathematically consistent solution of an approximation to the quantum field theory. Thus the failure of these results should arise from effects beyond those treated in the approximation. It is easy to see where such effects might arise. Our present stationary-phase approximation includes only the effects of the classical field configuration and of the fermion loop. In particular, there are no meson-loop effects; in our treatment these would arise from the Gaussian integral around the stationary-phase point. But since $m_\pi=0$ in our approximation, including meson loops would introduce the infrared problems associated with massless scalar bosons in two dimensions. Thus we would expect the breakdown of the semiclassical approximation to be signaled by uncontrollable quantum fluctuations arising from infrared divergences in the Gaussian integrals around the stationary-phase point.

We now see that the problem with spontaneous breakdown of the continuous chiral symmetry, the possible failure of the semiclassical approximation, and the question of quantum fluctuations are all closely linked in our model. To close this section let us discuss this final question—quantum fluctuations—in more detail. Although we are unable to evaluate the Gaussian functional integral corresponding to the quantum fluctuations,[59] nonetheless the discussion should clarify certain aspects of our approach. For simplicity and clarity, we restrict our comments to the σ-only case.

From the expression for $S_{\rm eff}$ in (2.3), we see that the Gaussian approximation to the quantum fluctuations in (2.16) is[23,60]

$$i\frac{\delta^2 S_{\rm eff}}{\delta\sigma^2}\bigg|_{\sigma_{\rm sp}}\frac{(\sigma-\sigma_{\rm sp})^2}{2}=\frac{i}{2}\int d^2x\int d^2y\,\sigma(x)\{[-\partial^2-3\lambda\sigma_{\rm sp}{}^2(x)+\lambda f^2]\delta(x-y)-g^2\,{\rm tr}S_F(x,y;\sigma_{\rm sp})S_F(y,x;\sigma_{\rm sp})\}\sigma(y). \tag{5.4}$$

Here $\partial_t^2\equiv\partial^2-\partial_x^2$, the trace is over Dirac indices only, and $S_F(x,y;\sigma_{\rm sp})$ is the full fermion propagator—including contributions from the occupied states—in the presence of the field $\sigma_{\rm sp}(x)$. To evaluate the determinant resulting from this Gaussian integral we must find the eigenvalues of the integro-differential equation[60]

$$[-\omega^2-\partial_x{}^2+3\lambda\sigma_{\rm sp}{}^2(x)-\lambda f^2]\sigma(x)+g^2\,{\rm tr}\int S_F(x,y;\sigma_{\rm sp})S_F(y,x;\sigma_{\rm sp})\sigma(y)dy=0. \tag{5.5}$$

Given the complicated[61] forms of $\sigma_{\rm sp}(x)$ and of the eigenfunctions of the Dirac equation with $\sigma_{\rm sp}(x)$ as a potential, it is perhaps not surprising that we have been unable to solve (5.5).[59] Two further comments are in order. First, simply ignoring the term containing $S_F(x,y;\sigma_{\rm sp})$ would yield the boson-loop correction—illustrated in Fig. 7(a)—to our semiclassical results. With the terms in S_F, the quantum corrections include higher loops, as shown in Fig. 7(b).

Second, one might hope that replacing the fields by scattering data would as before simplify the calculation dramatically. Unfortunately, this attempt flounders on the difficulty that although at the stationary-phase point the fields can be time independent—and thus one can use standard trace identities—the fluctuations around $\sigma_{\rm sp}$ *cannot* be assumed time independent. Thus one would need trace identities for *time-dependent* potentials, and these are presently unknown.

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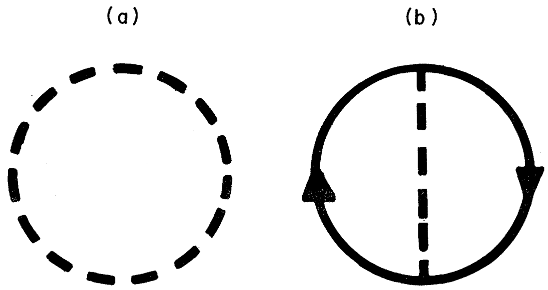

FIG. 7. (a) The boson-loop correction. The heavy lines indicate that the meson propagator is evaluated in the presence of an external field $\sigma_{sp}^2(x)$. (b) A two-loop correction generated by including the terms in S_F in (5.5). The fermion propagators are evaluated in an external field $\sigma_{sp}(x)$, and the meson propagator is evaluated in $\sigma_{sp}^2(x)$.

and our colleagues at Los Alamos are gratefully acknowledged.

APPENDIX A: TRACE IDENTITIES FOR THE DIRAC EQUATION

To motivate our derivation of the trace identities for the Dirac equation let us recall the two essential ingredients used by Zakharov and Faddeev[62] in the case of the Schrödinger equation. First, one needs a dispersion relation relating the transmission coefficient—which we shall call $s_{11}(k)$—as a function of momentum k to an expression involving the modulus of the reflection coefficient—$s_{12}(k)$—and the bound-state poles at $k=i\kappa_l$. Second, one introduces an auxiliary function—call it $\chi(x)$—constructed from a fundamental solution to the Schrödinger equation in a manner such that

$$\int_{-\infty}^{\infty}\chi(x)dx=\ln s_{11}(k). \tag{A1}$$

From the Schrödinger equation one deduces the differential equation satisfied by χ. Then comparing the asymptotic expansion (in $1/k$) of the solution to this equation with that of the dispersion relation, one obtains the trace identities.

In the case of the Dirac equation, both steps involve some additional technical complications. First, the analytic structure of $s_{11}(k)$ as a function of k is two-sheeted,[36,63] with a branch cut running from $k=+im$ to $i\infty$ connecting the sheets. On the first sheet, the energy satisfies $E=+(k^2+m^2)^{1/2}$, whereas on the second sheet, $E=-(k^2+m^2)^{1/2}$. However, since the branch point is of the second order linear combinations of the function on the first and second sheets can be chosen so as to remove the integral along this cut from the dispersion relation. In particular, one can show that the following representation holds:

$$\ln s_{11}^{(+)}(k)+\ln s_{11}^{(-)}(k)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}dq\,\frac{\ln[1-|s_{12}^{(+)}(k)|^2]+\ln[1-|s_{12}^{(-)}(k)|^2]}{q-k}+\sum_{l=1}^{N_+}\ln\left(\frac{k+i\kappa_l^{(+)}}{k-i\kappa_l^{(+)}}\right)+\sum_{l=1}^{N_-}\ln\left(\frac{k+i\kappa_l^{(-)}}{k-i\kappa_l^{(-)}}\right). \tag{A2}$$

Here the superscripts $(\pm)$ refer to quantities on the first $[E=+(k^2+m^2)^{1/2}>0]$ and second $[E=-(k^2+m^2)^{1/2}<0]$ Riemann sheets and the sums are over the $N^+(N^-)$ bound-state poles located at $0\le\kappa_l\le m$ on the first (second) Riemann sheet. To clarify the form of (A2) we observe that for a function analytic in the upper half plane (UHF) and suitably behaved at $(z)\to\infty$, there exists an integral representation for the function in terms of its real part on the real axis[64]:

$$f(z)=\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{du\,\mathrm{Re}f(u)}{u-z},\quad u\in\mathrm{R},\quad \mathrm{Im}z>0. \tag{A3}$$

Since the $\mathrm{Re}\ln s_{11}(k)=\ln|s_{11}(k)|$, and since unitarity implies[36] for real k that $|s_{11}(k)|^2=1-|s_{12}(k)|^2$, if we momentarily ignore the singularities of $s_{11}(k)$ in the UHP, we see that using (A3) with $f(z)=\ln s_{11}(k)$ yields the first term in (A2). To include the effects of the singularities in $s_{11}(k)$ for $\mathrm{Im}k>0$, we note that because the function is two-sheeted, the integral along the branch cut $im\le k<i\infty$ enters with opposite signs for $\ln s_{11}^{(+)}$ and $\ln s_{11}^{(-)}$ and thus cancels in the sum. Further, the explicit form of the contribution from the bound-state poles follows by observing that for real k, unitarity requires their contributions to the real part of (A2) to vanish, and hence the expressions in the logarithms must be unimodular for real k, that is, of the form $(k+i\kappa)/(k-i\kappa)$. For large k, the asymptotic expansion of (A2) in powers of $(1/k)$ is

$$\ln s_{11}^{(+)}(k)+\ln s_{11}^{(-)}(k)=\sum_{n=1}^{\infty}\frac{c_n}{k^n}, \tag{A4}$$

where $c_{2n}=0$,

$$c_{2n+1}=-\frac{1}{2\pi i}\int_{-\infty}^{\infty}q^{2n}dq\{\ln[1-|s_{12}^{(+)}(q)|^2]+\ln[1-|s_{12}^{(-)}(q)|^2]\}+\frac{2}{2n+1}\left[\sum_{l=1}^{N_+}(i\kappa_l^{(+)})^{2n+1}+\sum_{l=1}^{N_-}(i\kappa_l^{(-)})^{2n+1}\right]. \tag{A5}$$

To obtain the second expression for $\ln s_{11}(k)$, we resort directly to the Dirac equation. We need study only the equation in which both σ and π are present, since the results for σ-only can also be obtained from this case. In Sec. III and Appendix B we establish that $(\sigma,\pi)|_{x\to+\infty}$ differs from $(\sigma,\pi)|_{x\to-\infty}$, that is, the solutions approach physically equivalent but distinct chiral vacua at $x\to\pm\infty$. In terms of ψ, this simply corresponds to a finite rotation of the mass matrix; for both $x\to\pm\infty$, we have a free Dirac particle of mass m, but the rotation of the mass matrix introduces a phase between the wave functions $\psi|_{x\to+\infty}$ and $\psi|_{x\to-\infty}$. Since this phase does not correspond to a "phase shift" arising from the interaction, it complicates the extraction of the true phase shift. To avoid this difficulty, we shall change meson field variables from (σ,π) to (ρ,θ), where

$$\sigma+i\pi=\rho e^{i\theta/f}, \tag{A6}$$

and introduce

$$\psi'=e^{i\gamma_5\theta/2f}\psi. \tag{A7}$$

For time-independent σ and π the Dirac equation for $\tilde\psi'$ defined by $\psi'(x,t)=e^{-i\omega t}\tilde\psi'(x)$ then becomes

$$\left[\omega\gamma_0+i\gamma_1\frac{d}{dx}-m-g\hat\rho(x)-\gamma_5\gamma_1 s(x)\right]\tilde\psi'(x)=0, \tag{A8}$$

where $m=gf$, $\hat\rho(x)=\rho(x)-f$, and $s(x)\equiv(1/2f)d\theta/dx$. Since $\hat\rho(x)$ and $s(x)$ must both approach zero as $x\to\pm\infty$ for localized solutions, the phase difference between the wave functions at $x\to\pm\infty$ yields directly the true phase shift.

Using our standard representation for the γ matrices, $\gamma_0=\sigma_3$, $\gamma_1=i\sigma_1$, $\gamma_5=\gamma_0\gamma_1=-\sigma_2$, we can write (A8) in terms of the two components of $\tilde\psi'$ as

$$\frac{d\psi_2}{dx}+a_-(x)\psi_1(x)=0 \tag{A9a}$$

and

$$\frac{d\psi_1}{dx}+a_+(x)\psi_2(x)=0, \tag{A9b}$$

where $\tilde\psi'\equiv\binom{\psi_1}{\psi_2}$ and

$$a_\pm(x)\equiv m+g\hat\rho(x)\pm[\omega+s(x)]. \tag{A10}$$

Consider the specific solution $f\equiv\binom{f_1}{f_2}$ to (A9) which satisfies

$$\ln f_1(x,k)\underset{x\to+\infty}{\simeq} ikx+c(k),$$

where $c(k)$ is a constant depending on the normalization of f. Frolov's results[36] establish that, for $\operatorname{Im}k>0$,

$$\ln f_1(x,k)\underset{x\to-\infty}{\simeq} ikx-\ln s_{11}(k)+c(k).$$

Thus if one introduces

$$\chi(x)\equiv\frac{d}{dx}[\ln f_1(x,k)]-ik, \tag{A11}$$

it follows that

$$\int_{-\infty}^{\infty}\chi(x)dx=\ln s_{11}(k). \tag{A12}$$

Further, from the Dirac equation in the form (A9) one can derive an equation for χ. Differentiating (A9b) with respect to x and substituting from (A9a) we obtain

$$\frac{d^2f_1}{dx^2}-\frac{1}{a_+}\left(\frac{da_+}{dx}\right)\frac{df_1}{dx}-a_+a_-f_1=0. \tag{A13}$$

Using (A11) this can be written, after some algebra, in terms of χ as

$$\frac{d\chi}{dx}+\chi^2+\alpha(x)\chi+\beta(x)=0, \tag{A14}$$

where

$$\alpha(x)=2ik-(\omega+m+g\hat\rho+s)^{-1}\left(g\frac{d\hat\rho}{dx}+\frac{ds}{dx}\right)$$

and

$$\beta(x)=2\omega s+s^2-2gf\hat\rho-g^2\hat\rho^2$$
$$-ik\left(g\frac{d\hat\rho}{dx}+\frac{ds}{dx}\right)(\omega+m+g\hat\rho+s)^{-1}.$$

Up to now we have not specified which Riemann sheet we are considering. Indeed it is clear that if we consider $\chi^{(\pm)}$, defined by (A11) in terms of $f_1^{(\pm)}$, both satisfy (A14) with the only difference being that for $\chi^{(+)}$, $\omega=+(k^2+m^2)^{1/2}$, whereas for $\chi^{(-)}$, $\omega=-(k^2+m^2)^{1/2}$. This change of sign leads to important cancellations in the sum, $\chi^{(+)}+\chi^{(-)}$, whose integral satisfies

$$\int_{-\infty}^{\infty}[\chi^{(+)}(x)+\chi^{(-)}(x)]dx=\ln s_{11}^{(+)}(k)+\ln s_{11}^{(-)}(k). \tag{A15}$$

The remaining calculations are straightforward but tedious. Expanding

$$\chi^{(\pm)}(x)=\sum_{n=0}^{\infty}\frac{\chi_n^{(\pm)}(x)}{(2ik)^n}$$

and equating powers of $(1/k)$ in (A14) leads eventually to the following expressions for the $\chi_n^{(+)}+\chi_n^{(-)}$:

$$\chi_0^{(+)}+\chi_0^{(-)}=0, \tag{A16a}$$

$$\chi_1^{(+)}+\chi_1^{(-)}=2g^2(\rho^2-f^2), \tag{A16b}$$

$$\chi_2^{(+)}+\chi_2^{(-)}=4g\frac{d}{dx}(\rho s), \tag{A16c}$$

and

$$\chi_3^{(+)}+\chi_3^{(-)}=-2g^4(\rho^2-f^2)^2-2g^2\left(\frac{d\rho}{dx}\right)^2-8g^2\rho^2s^2, \qquad \text{(A16d)}$$

where in the last equation we have dropped certain total derivative terms which vanish in (A15). Note that $\chi_2^{(+)}+\chi_2^{(-)}$ is a total derivative and thus there is no $O(1/k^2)$ term in (A15). This is of course necessary for consistency with (A4), since $c_{2n}=0$.

Comparing the expressions in (A16) with those in (A5), we arrive at the first two nontrivial trace identities:

$$\int_{-\infty}^{\infty}dx(\sigma^2+\pi^2-f^2)=\int_{-\infty}^{\infty}dx(\rho^2-f^2)$$
$$=-\frac{1}{2\pi g^2}\int_{-\infty}^{\infty}dq\{\ln[1-|s_{12}^{(+)}(q)|^2]+\ln[1-|s_{12}^{(-)}(q)|^2]\}-\frac{2}{g^2}\left[\sum_{l=1}^{N_+}\kappa_l^{(+)}+\sum_{l=1}^{N_-}\kappa_l^{(-)}\right], \qquad \text{(A17a)}$$

$$\int_{-\infty}^{\infty}\left\{\frac{1}{2}\left[\left(\frac{d\sigma}{dx}\right)^2+\left(\frac{d\pi}{dx}\right)^2\right]+\frac{g^2}{2}(\sigma^2+\pi^2-f^2)^2\right\}dx=\int_{-\infty}^{\infty}\left\{\frac{1}{2}\left[\left(\frac{d\rho}{dx}\right)^2+2\rho^2s^2\right]+\frac{g^2}{2}(\rho^2-f^2)^2\right\}dx$$
$$=-\frac{1}{\pi g^2}\int_{-\infty}^{\infty}q^2dq\{\ln[1-|s_{12}^{(+)}(q)|^2]+\ln[1-|s_{12}^{(-)}(q)|^2]\}$$
$$+\frac{4}{3g^2}\left[\sum_{l=1}^{N_+}(\kappa_l^{(+)})^3+\sum_{l=1}^{N_-}(\kappa_l^{(-)})^3\right]. \qquad \text{(A17b)}$$

These are the forms used in Secs. III and IV to express the action entirely in terms of the scattering data.

APPENDIX B: RECONSTRUCTION OF THE MESON FIELDS AND FERMION BOUND-STATE WAVE FUNCTIONS

To reconstruct the meson fields and Dirac wave functions we use the techniques developed by Frolov,[36] who has established that a matrix generalization of the Gelfand-Levitan[65]-Marchenko[66] formalism is applicable to the inverse problem for the Dirac equation. The procedure can be simply summarized. From the scattering data one forms a matrix kernel, $F(x,y)=F_S(x,y)+F_{BS}(x,y)$; for reflectionless potentials—$s_{12}(k)=0$—the scattering contribution, F_S, vanishes and only the bound-state contribution, F_{BS}, remains. In our standard representation for the Dirac γ matrices

$$F_{BS}(x,y)=\sum_{l=1}^{N_+,N_-}c_l e^{-\kappa_l(x+y)}M_l \qquad \text{(B1)}$$

when the matrices M_l are given by[67]

$$M_l=\begin{bmatrix}\dfrac{m+(m^2-\kappa_l^2)^{1/2}}{m-(m^2-\kappa_l^2)^{1/2}} & \dfrac{m+(m^2-\kappa_l^2)^{1/2}}{\kappa_l}\\[2ex] \dfrac{m+(m^2-\kappa_l^2)^{1/2}}{\kappa_l} & 1\end{bmatrix} \qquad \text{(B2a)}$$

for $\omega_l>0$, and by

$$M=\begin{bmatrix}1 & \dfrac{m+(m^2-\kappa_l^2)^{1/2}}{\kappa_l}\\[2ex] \dfrac{m+(m^2-\kappa_l^2)^{1/2}}{\kappa_l} & \dfrac{m+(m^2-\kappa_l^2)^{1/2}}{m-(m^2-\kappa_l^2)^{1/2}}\end{bmatrix} \qquad \text{(B2b)}$$

for $\omega_l<0$. The c_l are normalization constants, and the sum is over all positive- and negative-energy bound states. From the kernel F one constructs the transformation operator $K(x,y)$ by solving the equation

$$K(x,y)+F(x,y)+\int_x^{\infty}K(x,t)F(t,y)dt=0. \qquad \text{(B3)}$$

Then the "potential"

$$V(x)\equiv\gamma_0 g\hat{\sigma}+i\gamma_0\gamma_5\pi=\begin{bmatrix}g\hat{\sigma} & -g\pi\\ -g\pi & -g\hat{\sigma}\end{bmatrix} \qquad \text{(B4)}$$

is given by the commutator

$$V(x)=[-i\gamma_5,K(x,x)]. \qquad \text{(B5)}$$

Further, the Dirac wave functions are given by

$$f(x,k)=e(x,k)+\int_x^{\infty}K(x,y)e(y,k)dy, \qquad \text{(B6)}$$

where $e(x,k)$ is a plane-wave solution to the free Dirac equation

$$e(x,k)=\begin{bmatrix}\dfrac{ik}{(k^2+m^2)^{1/2}-m}\\ 1\end{bmatrix}e^{+ikx} \qquad \text{(B7a)}$$

for $\omega>0$ and

$$e(x,k)=\begin{bmatrix}1\\ \dfrac{ik}{(k^2+m^2)^{1/2}-m}\end{bmatrix}e^{+ikx},\quad \omega<0. \qquad \text{(B7b)}$$

To proceed we distinguish three cases: (1) σ-only, "baglike" structure ($\omega_0^{(+)}=|\omega_0^{(-)}|\neq 0$), (2) σ-only, "kinklike" structure ($\omega_0=0$), and (3) $\sigma+\pi$. For the first case, the requirement of charge symmetry for the σ-only potential is satisfied by having one positive- and one negative-energy bound state with $\omega_0^{(+)}=|\omega_0^{(-)}|=\omega_0$ and with identical normalizations. Thus, with $\omega_0\equiv(m^2-\kappa_0{}^2)^{1/2}$

$$F(x,y)=c_0e^{-\kappa_0(x+y)}\begin{bmatrix}\dfrac{2m}{m-\omega_0} & \dfrac{2(m+\omega_0)}{\kappa_0}\\ \dfrac{2(m+\omega_0)}{\kappa_0} & \dfrac{2m}{m-\omega_0}\end{bmatrix}$$

$$\equiv c_0e^{-\kappa_0(x+y)}\begin{bmatrix}\alpha & \beta\\ \beta & \alpha\end{bmatrix}. \qquad \text{(B8)}$$

The assumption $K(x,y)=\hat{K}(x)e^{-\kappa_0 y}$ separates the integral equation (B3), leaving an algebraic matrix equation whose explicit solution is

$$\hat{K}(x)=\frac{-c_0e^{-\kappa_0 x}}{[1+2\alpha z+z^2(\alpha^2-\beta^2)]}\times\begin{bmatrix}\alpha+z(\alpha^2-\beta^2) & \beta\\ \beta & \alpha+z(\alpha^2-\beta^2)\end{bmatrix}, \qquad \text{(B9)}$$

where $z\equiv c_0e^{-2\kappa_0 x}/2\kappa_0$. Thus using (B5) we find that

$$V(x)\equiv\begin{bmatrix}g\hat{\sigma} & 0\\ 0 & -g\hat{\sigma}\end{bmatrix}, \qquad \text{(B10)}$$

where

$$g\hat{\sigma}=\frac{-4\beta\kappa_0 z}{[1+(\alpha+\beta)z][1+(\alpha-\beta)z]}. \qquad \text{(B11)}$$

After some algebra, (B11) can be cast into the more familiar form quoted (for $\delta=0$) in the text:

$$g\hat{\sigma}=\frac{-\kappa_0{}^2}{\omega_0}\operatorname{sech}(\kappa_0(x+x_0)+\delta)\times\operatorname{sech}(\kappa_0(x-x_0)+\delta), \qquad \text{(B12)}$$

where

$$\kappa_0x_0=\tanh^{-1}\left(\frac{m-\omega_0}{\kappa_0}\right)$$

and

$$\tanh\delta=\frac{\kappa_0(m-\omega_0)-c_0\omega_0}{\kappa_0(m-\omega_0)+c_0\omega_0}.$$

Reconstructing the positive-energy bound-state wave function from (B6) we obtain

$$\psi_0(x)=\frac{e^{-\kappa_0 x}}{1+2\alpha z+z^2(\alpha^2-\beta^2)}\begin{pmatrix}\beta z(\alpha/2-1)+\beta/2\\ 1+z(\alpha-\beta^2/2)\end{pmatrix}. \qquad \text{(B13)}$$

After a substantial amount of algebra, we can write this in the form

$$\psi_0(x)=\left(\frac{\kappa_0}{8}\right)^{1/2}\times\begin{pmatrix}\operatorname{sech}(\kappa_0(x+x_0)+\delta)+\operatorname{sech}(\kappa_0(x-x_0)+\delta)\\ -\operatorname{sech}(\kappa_0(x+x_0)+\delta)+\operatorname{sech}(\kappa_0(x-x_0)+\delta)\end{pmatrix}, \qquad \text{(B14)}$$

where we have normalized ψ_0 in (B14) to 1 and where the parameters are all as previously defined. Of course, the full time-dependent bound-state wave function is

$$\psi_0(x,t)=e^{-i\omega_0 t}\psi_0(x).$$

In the case of the kink, there is only one bound state, and it has $\omega_0=0$. For $\omega_0=0$ the matrix F is trivial,

$$F(x,y)=c_0e^{-m(x+y)}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}, \qquad \text{(B15)}$$

where $m=\kappa_0=gf$. Solving (B3) as before we find that

$$K(x,y)=\frac{-c_0e^{-m(x+y)}}{1+2z}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}, \qquad \text{(B16)}$$

where $z=c_0e^{-2mx}/2m$. From (B16) using (B10) and simplifying we find that

$$\sigma=\hat{\sigma}+f=f\tanh(gfx+\alpha), \qquad \text{(B17)}$$

where $\tanh\alpha=(gf-c_0)/(gf+c_0)$.

The Dirac wave function, reconstructed from (B6) using $K(x,y)$ as in (B16) is

$$\psi_0=\left(\frac{gf}{4}\right)^{1/2}\operatorname{sech}(gfx+\alpha)\begin{pmatrix}1\\ 1\end{pmatrix}, \qquad \text{(B18)}$$

where we have normalized ψ_0 to 1. For $\lambda=2g^2$, these forms of σ and ψ_0 are exactly equal to those in (3.5).

Finally consider the reconstruction problem when both σ and π are present. As noted in the text, we need study only the case of a single positive-energy bound state, ω_0. Thus

$$F(x,y)=c_0e^{-\kappa_0(x+y)}\begin{bmatrix}\dfrac{m+\omega_0}{m-\omega_0} & \dfrac{m+\omega_0}{\kappa_0}\\ \dfrac{m+\omega_0}{\kappa_0} & 1\end{bmatrix}. \tag{B19}$$

Solving (B3) as before yields

$$K(x,y)=\frac{-c_0e^{-\kappa_0(x+y)}}{1+z(\frac{1}{4}\beta^2+1)}\begin{bmatrix}\frac{1}{4}\beta^2 & \frac{1}{2}\beta\\ \frac{1}{2}\beta & 1\end{bmatrix}, \tag{B20}$$

where

$$\beta=2\left(\frac{m+\omega_0}{\kappa_0}\right)$$

and

$$z=\frac{c_0e^{-2\kappa_0 x}}{2\kappa_0}.$$

Using (B5) we find that

$$g\hat{\sigma}=\frac{-2\beta\kappa_0 z}{1+(\frac{1}{4}\beta^2+1)z}, \tag{B21a}$$

$$g\pi=\frac{-(\frac{1}{4}\beta^2-1)2\kappa_0 z}{1+(\frac{1}{4}\beta^2+1)z}, \tag{B21b}$$

or that

$$g\hat{\sigma}=\frac{-\kappa_0^2}{gf}[1-\tanh(\kappa_0x+\gamma)], \tag{B22a}$$

$$g\pi=\frac{-\omega_0\kappa_0}{gf}[1-\tanh(\kappa_0x+\gamma)], \tag{B22b}$$

where

$$\tanh\gamma=\frac{\kappa_0^3/m(m+\omega_0)-c_0}{\kappa_0^3/m(m+\omega_0)+c_0}. \tag{B23}$$

Finally, using (B6) with (B20) and simplifying, we find the normalized bound-state wave function to be

$$\psi_0=\left[\frac{\kappa_0(m-\omega_0)}{4m}\right]^{1/2}\operatorname{sech}(\kappa_0x+\gamma)\begin{bmatrix}\dfrac{\kappa_0}{m-\omega_0}\\ 1\end{bmatrix}. \tag{B24}$$

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[1]R. F. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D 10, 4114 (1974); 10, 4130 (1974).

[2]R. F. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D 11, 3424 (1975).

[3]R. F. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D 12, 2443 (1975).

[4]L. D. Faddeev, IAS report, 1975 (unpublished).

[5]J. Goldstone and R. Jackiw, Phys. Rev. D 11, 1486 (1975).

[6]J. L. Gervais and B. Sakita, Phys. Rev. D 11, 2943 (1975); J. L. Gervais, A. Jevicki, and B. Sakita, *ibid.* 12, 1038 (1975); C. Callan and D. Gross, Nucl. Phys. B93, 29 (1975); M. Creutz, Phys. Rev. D 10, 1749 (1974); 12, 3126 (1975); M. Creutz and K. S. Soh, *ibid.* 12, 443 (1975); E. Tomboulis, *ibid.* 12, 1678 (1975); R. Jackiw and G. Woo, Phys. Rev. D 12, 1643 (1975); N. M. Christ and T. D. Lee, Phys. Rev. D 12, 1606 (1975).

[7]W. Bardeen, M. Chanowitz, S. Drell, M. Weinstein, and T.-M. Yan, Phys. Rev. D 11, 1094 (1975).

[8]A. Klein and F. Krejs, Phys. Rev. D 12, 3112 (1975); 13, 3295 (1976); A. Klein, *ibid.* 14, 558 (1976).

[9]S. Orfanidis, Phys. Rev. D 14, 472 (1976).

[10]L. F. Abbott, Phys. Rev. D 14, 552 (1976).

[11]S.-S. Shei, Phys. Rev. D 14, 535 (1976).

[12]For a review of the early stages of these developments, see R. Rajaraman, Phys. Rep. 21C, 229 (1975). For a collection of relevant individual papers, see Phys. Rep. 23C, 240 (1976).

[13]A. Luther, Nordita Report No. 76/8, 1976 (unpublished).

[14]For a review of the four-dimensional version of this model, see B. W. Lee, *Chiral Dynamics* (Gordon and Breach, New York, 1972).

[15]D. J. Gross and A. Neveu, Phys. Rev. D 10, 3235 (1974).

[16]Thus, in the tree approximation, it appears that the continuous chiral symmetry of $\mathcal{L}$ is spontaneously broken and that the (massless) pion is the associated Goldstone boson. Indeed, we shall see that the full semiclassical analysis also suggests the existence of spontaneous breakdown of the continuous chiral symmetry. However, Coleman (Ref. 17) has shown (we shall call this result Coleman's theorem) that *in general* such spontaneous symmetry breaking *cannot* occur in two dimensions because of the infrared singularities arising from the associated Goldstone boson. The caveat "in general" must be added in view of the two known examples (see Refs. 11 and 43) in which Coleman's theorem is (deviously) evaded by a decoupling of the Goldstone boson. These examples and other considerations germane to the important question of reconciling the semiclassical results for the $(\sigma+\pi)$ case with Coleman's theorem are discussed in Sec. V.

[17]S. Coleman, Commun. Math. Phys. 31, 259 (1973). This result was originally shown, in statistical mechanics, by N. D. Mermin and H. Wagner, Phys. Rev. Lett. 17, 1133 (1966) and by P. C. Hohenberg, Phys. Rev. 158, 383 (1967).

[18]T. D. Lee and G. C. Wick, Phys. Rev. D 9, 2291 (1974).

[19]D. Campbell, R. Dashen, and J. Manassah, Phys. Rev. D 12, 979 (1975); G. Baym, D. Campbell, R. Dashen, and J. Manassah, Phys. Lett. 58B, 304 (1975).

[20]A. Kerman and L. D. Miller, Second High Energy Heavy Ion Summer Study, 1974, Report No. LBL-3675 (unpublished).

[21]Similar techniques are also used in Refs. 10 and 11.

[22]Throughout the article we shall study the bound states in their rest frames; hence the energies are the bound-state masses. The proper way to include the "center-of-mass" motion of the bound state is discussed in Ref. 1.

[23]R. G. Root, Phys. Rev. D 11, 831 (1975).

[24]For a review of, and for original references to, this method, see A. Scott, F. Chu, and D. McLaughlin, Proc. IEEE 61, 1443 (1973).

[25]This technique has also proved useful in the studies of Refs. 3, 9, 10, and 11.

[26]At this level, the spectra appear identical in structure. However, the energy levels of the σ model are—for $4f^2/N \neq 0$—numerically different from those of the Gross-Neveu model and, possibly more importantly, preliminary calculations suggest that the Gross-Neveu model's extensive degeneracy [beyond that required by the O($2N$) symmetry] is *not* found in the σ model.

[27]T. D. Lee and M. Margulies, Phys. Rev. D 11, 1591 (1975).

[28]C.-G. Källman and S. A. Moszkowski, Phys. Lett. 57B, 183 (1975).

[29]E. Nyman and M. Rho, Nucl. Phys. B (to be published).

[30]The expression in (2.3) is formal in that it depends on the normalization of the functional integral. The precise form of the result appears in (2.8).

[31]Note further that the value of this fermion loop depends on the sector of the theory, that is, on the nature of the occupied fermion states. This is familiar from field-theoretic calculations in nuclear physics, where the nucleons in the Fermi sea contribute to the fermion-loop integrals.

[32]Actually, DHN proved that this expression is valid for σ and π with general periodic-time dependence with the replacement $\omega_k T \rightarrow \alpha_k$, the Floquet index. For clarity of exposition, we have quoted only the result for time-independent σ and π.

[33]Many of these results have appeared, in condensed form, in D. Campbell, Phys. Lett. (to be published). Further, these classical solutions can also be obtained by careful analysis of the appropriate Hamilton-Jacobi equations (C. Sommerfield, private communication).

[34]In (3.2) we have deleted the SU(N) index on the fermions to emphasize that they are all in the same spatial state, $\psi_0(x)$, that is, $\psi^{(\alpha)}(x) = \eta^\alpha \psi_0(x)$, where η^α is a unit vector in the SU(N) space. Note further that at this classical level one can solve (3.2) for any normalization of the fermion wave function. However, our interpretation of the "$\eta_0 \omega_0$" term as the occupied-state contribution tells us to choose the condition (3.4).

[35]S.-J. Chang, Phys. Rev. D 12, 1071 (1975).

[36]I. S. Frolov, Dokl, Akad. Nauk. SSSR 207, 44 (1972) [Sov. Math. Dokl. 13, 1468 (1972)].

[37]These trace identities have been derived independently by S.-S. Shei in Ref. 11.

[38]In the O(N) ϕ^4 model in two dimensions, a related form of this second trace identity can also be used to write the action in terms of the scattering data. See Ref. 10.

[39]This relation between fermion (g) and boson (λ) coupling constants is reminiscent of supersymmetry relations. Indeed, the Wess-Zumino model [J. Wess and B. Zumino, Phys. Lett. 49B, 52 (1974)], written in terms of the fields ψ, A, and B, both (1) looks formally quite similar to our ($\sigma + \pi$) case and (2) contains precisely the constraint $\lambda = 2g^2$. We have not yet investigated the interesting possible relation between supersymmetry and the solvability of the σ model where $\lambda = 2g^2$. (We thank T. F. Wong and M. Gell-Mann for calling this point to our attention.)

[40]Establishing the independence and completeness of the scattering data is not as straightforward for the Dirac equation as for the Schrödinger equation. See Ref. 11 for an example in which these properties do *not* hold. However, in our case the correctness of our solution provides *a posteriori* justification for assuming these properties here.

[41]The same analytic forms of σ and ψ_0, but with a different "quantization condition" for the κ_0, also solve the classical equations of the Gross-Neveu model. See Ref. 3 and S. Lee, T. Kuo, and A. Gavrielides, Phys. Rev. D 12, 2249 (1975).

[42]To see that the "shell" states are the finite analogs of "abnormal" nuclear matter, recall that the essential feature of this matter is that the nucleons "lose" their mass in an abnormal σ field. This corresponds precisely to the trapping of fermions in an $\omega_0 = 0$ bound state on the kink in the "shell" solution.

[43]S.-J. Chang, B. Lee, and S. D. Ellis, Phys. Rev. D 11, 3572 (1975); S.-J. Chang, Phys. Rep. 23C, 259 (1976).

[44]These same analytic forms of σ, π, and ψ_0 but with a different "quantization condition" solve the classical equations of the chiral Gross-Neveu model. See Ref. 11.

[45]As our later comments will indicate, in the present ($\sigma + \pi$) case, the actual existence in the quantum field theory of these "nuclei" predicted by our semiclassical analysis remains moot.

[46]This condition is similar to that found in the O(N) ϕ^4 model studied in Ref. 10.

[47]Recall that α is a rough measure of the relative importance of the meson contribution to energy levels to the negative-energy sea contribution. This remark is most clearly illustrated by (4.12).

[48]As indicated, this result follows directly from our formalism. As a check, we note that S.-J. Chang and T.-M. Yan [Phys. Rev. D 12, 3225 (1975)] have also calculated the fermion-loop corrections to the SLAC "shell" mass. Their result (specialized to $\lambda = 2g^2$) agrees precisely with ours (specialized to $N = 1$). However, note that it is *only* because the minimum corresponding to the shell occurs at $\omega_0 = 0$ in *both* the classical and semiclassical analyses that their calculation, which treats the negative-energy sea at a perturbation on the classical solution, agrees with ours, which includes the negative-energy sea in the stationary-phase condition. In general—for the shallow-bag states, for example—these two approaches would yield different results.

[49]The smallness of the effect of the negative-energy sea on the shallow-bag states can be motivated physically by observing that (for $\alpha \gg 1$) the σ field of the shallow bag is only slightly perturbed from its vacuum value, and hence the differences in (4.3) are small.

[50]Of course this constraint applies only for $\alpha \equiv 4f^2/N$

$>1/\pi$. As $\alpha \to 1/\pi$, $\phi_0 \to \pi/2$, and as (4.17) shows, remains at $\pi/2$ for all smaller values of α.

[51]To study these states in detail requires knowledge of (at least approximate) time-dependent solutions to the σ model. We have made some progress in this area, and very preliminary results suggest that the Gross-Neveu model's extensive degeneracy [beyond that required by the O($2N$) symmetry] is *not* found in the σ model.

[52]This exact result also relates the energy levels of the chiral—$(\sigma+\pi)$—and "ordinary"—σ— Gross-Neveu models. See Refs. 3 and 11.

[53]This sequence of critical values $f^2/N\,(\tilde{n}_0)$ can be determined numerically by solving for a given $\tilde{n}_0 > N/2$ Eqs. (4.20) and (4.15)—with $\phi(\tilde{n}_0) = \phi_0$—simultaneously. For $\tilde{n}_0 = N/2$, for example, this gives the expected result $f^2/N\,(N/2) = 1/4\pi$.

[54]This conclusion is based on the decoupling of the putative Goldstone boson—which we would expect to be the analog of our pion—in the chiral Gross-Neveu model.

[55]S.-K. Ma and R. Rajaraman, Phys. Rev. D 11, 1701 (1975). These authors provide a useful pedagogical discussion of spontaneous symmetry breaking in two dimensions.

[56]The questions of the actual existence of spontaneous symmetry breaking and of the decoupling of the Goldstone boson in the Gross-Neveu model remain partially unresolved. The $(1/N)$ expansion of Ref. 23 suggests that, although spontaneous symmetry occurs in leading order, higher corrections in $(1/N)$ appear to restore the symmetry. This apparently contradicts the (*a priori* more general) decoupling argument, based on the use of Bose operators in the fermion Lagrangian, of Ref. 11.

[57]M. B. Halpern, Phys. Rev. D 12, 1684 (1975).

[58]Unfortunately it is difficult to verify this suggestion in our model because (1) the trace identities involve chiral-symmetric combinations of σ and π and hence we cannot solve explicitly any broken symmetry theory, and (2) a perturbation expansion in ϵ would almost certainly fail because of the nonanalytic dependence on ϵ of certain quantities—e.g., m_π—in the theory.

[59]It may be possible to estimate these fluctuations by solving (5.5) in some approximation. We are currently studying this problem.

[60]We ignore renormalization in these remarks.

[61]The form of $\sigma_{\rm sp}(x)$ is substantially more complicated than that which arises in the SLAC "shell" calculations. In particular, our $\sigma_{\rm sp}(x)$ is such that, even in the absence of the fermion contributions, the resulting Schrödinger equation for $\sigma(x)$ is *not* immediately solvable analytically.

[62]V. E. Zakharov and L. D. Faddeev, Funkt. Anal. Ego Pril. 5, 18 (1971) [Funct. Anal. and its Applic. 5, 280 (1972)].

[63]In general, our notation will follow that of Ref. 36, to which readers are referred for proofs of the results on the analyticity of the S-matrix elements, the properties of the fundamental solutions to the Dirac equation, and the explicit solution to the Dirac inverse problem in one spatial dimension.

[64]This result, which follows directly from careful application of Cauchy's theorem, is derived in E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (Clarendon Press, Oxford, 1937), Chap. V.

[65]I. M. Gelfand and B. M. Levitan, Izvest. Akad. Nauk. SSSR Ser. Mat. 15, 309 (1951) [Am. Math. Soc. Trans. 1, 253 (1953)].

[66]V. A. Marchenko, Dokl. Akad. Nauk. SSR 72, 457 (1950); 104, 695 (1955); Z. S. Agranovich and V. A. Marchenko, *The Inverse Problem of Scattering Theory* (Gordon and Breach, New York, 1963).

[67]Here we differ slightly from Ref. 36 in that the normalization we use is the same, at given k and $|\omega|$, for positive- and negative-energy fundamental solutions.