THE ASTROPHYSICAL JOURNAL, 216:77–85, 1977 August 15

BETA DECAY OF PION CONDENSATES AS A COOLING MECHANISM FOR NEUTRON STARS*

O. MAXWELL AND G. E. BROWN
State University of New York, Stony Brook
D. K. CAMPBELL
Theoretical Division, Los Alamos Scientific Laboratory
AND
R. F. DASHEN AND J. T. MANASSAH
Institute for Advanced Study
Received 1977 January 10

ABSTRACT

Using recently developed dynamical models of pion condensation, we reconsider the analysis of Bahcall and Wolf, who showed that the existence of free pions in neutron star interiors could stimulate neutron β-decay and thus increase dramatically the rate at which these stars cool. Although the actual pion condensate is quite different from the free pions assumed by Bahcall and Wolf and although the stimulated β-decay process is also different, our final results for the luminosity arising from this mechanism are very similar to their earlier predictions. We indicate the manner in which the pion condensate enters into the complete scenario of neutron star cooling and discuss several questions which must be resolved before the full cooling process can be known quantitatively.

Subject headings: dense matter — nuclear reactions — stars: neutron

I. INTRODUCTION

In 1965 Bahcall and Wolf demonstrated that the existence of free pions in neutron star interiors could dramatically increase the neutron star cooling rate via neutrino emission over the rate predicted in the absence of pions (Bahcall and Wolf 1965*a*, *b*, *c*). Among other reactions, hey considered ordinary β-decay,

$$n \rightarrow p + e^- + \bar{\nu}_e , \qquad (1)$$

the "modified Urca process,"

$$n + n \rightarrow n + p + e^- + \bar{\nu}_e , \qquad (2)$$

and the pion-induced β-decay,

$$n + \pi^- \rightarrow n + e^- + \bar{\nu}_e , \qquad (3)$$

where the interactions of the pions in the medium were essentially ignored and the pions were assumed to have zero momentum. For neutron stars with temperatures in the range $T \sim 10^9$–10^{10} K, Bahcall and Wolf showed that reaction (1) cannot proceed essentially because of Fermi-Dirac statistical factors and that reaction (3) offers a substantially more efficient cooling mechanism than does (2). Specifically, for a neutron star of roughly 1 $M_\odot$ and having constant density equal to nuclear matter density, they found the luminosity due to (3) to be

$$L_\nu^{\pi n} \approx (10^{46}\ \mathrm{ergs\ s^{-1}})\left(\frac{n_\pi}{n_b}\right)\left(\frac{M_s}{M_\odot}\right)T_9{}^6 . \qquad (4)$$

Here n_π and n_b are the pion and baryon number densities in the neutron star interior and $T_9 \equiv T/10^9$ K. This result includes a factor of 2 to account for the reaction inverse to (3),

$$n + e^- \rightarrow n + \pi^- + \nu_e , \qquad (5)$$

and another factor of 2 to account for reactions involving muons instead of electrons.

The luminosity shown in (4) is $10^7/T_9{}^2$ times larger than that arising from the modified Urca process (2); thus neutron stars containing free pions would cool much more rapidly than those without pions. Indeed, Bahcall and Wolf showed that the effective surface temperature, T_e, of a 1 $M_\odot$ neutron star with $(n_\pi/n_b) \approx 1$ drops below 10^6 K in a matter of days as compared to 3×10^5 years for a star with $n_\pi = 0$.

At the time of their calculation, Bahcall and Wolf emphasized that there was no proof that pions would exist in neutron stars and further that the calculation based on assuming "quasi-free" pions should be regarded as indicative rather than definitive.

In light of recent work demonstrating the probable existence of condensates in neutron matter at neutron

* Work performed in part under the auspices of the U.S. Energy Research and Development Administration contracts E(11-1)-3001 and E(11-1)-2220.

star densities,[1] it is of considerable interest to reconsider Bahcall and Wolf's results on neutron star cooling. In the remaining sections of this article, we study in some detail the effects of realistic pion condensates on neutron star cooling. In § II we review briefly the salient features of ordinary and pion-condensed neutron star matter. Section III shows how the chiral symmetry approach, as used by Campbell, Dashen, and Manassah (1975*a*, *b*) and Baym *et al.* (1975) (hereafter referred to collectively as BCDM), provides a simple solution to the problem of β-decay in the presence of a pion condensate. Our results demonstrate that, although several aspects of Bahcall and Wolf's calculations are changed, their prediction for the luminosity is remarkably unaltered. In § IV we discuss the full problem of neutron star cooling and indicate several questions which must be resolved before the cooling process can be described completely. Although the article is generally self-contained, for comparative purposes it is deliberately patterned after the pioneering work of Bahcall and Wolf.

II. PION CONDENSATES IN NEUTRON STAR MATTER

In this section we review those features of ordinary and pion condensed neutron star matter which are essential to understanding possible neutrino cooling mechanisms.

The normal ground state ($T = 0$) of neutron star matter is presumed to be an electrically neutral mixture of neutrons, protons, and electrons[2] in equilibrium under the normal β-decay process (1). Since the average baryon density in neutron star matter is on the order of nuclear density ($\rho_{\text{nuc}} = 0.17$ nucleons fm^{-3}), both the nucleons and the electrons are highly degenerate. From the condition of β-equilibrium we see that the chemical potentials of the degenerate species satisfy[3]

$$\mu(n) = \mu(p) + \mu(e) . \tag{6}$$

Further, charge neutrality implies

$$n(e) = n(p) , \tag{7}$$

and hence the Fermi momenta of the electrons and protons are equal,

$$k_{\text{F}}(e) = k_{\text{F}}(p) . \tag{8}$$

Estimating the values of the Fermi momenta in the noninteracting gas model yields (Bahcall and Wolf 1965*a*, *b*, *c*)

$$k_{\text{F}}(e) = k_{\text{F}}(p) \approx 0.25 k_{\text{F}}(n) \approx 100 \text{ MeV}/c \tag{9}$$

for densities in the range (1–3)ρ_{nuc}. Thus the electrons are ultrarelativistic and the nucleons nonrelativistic.

The existence of neutron Fermi energies on the order of hadronic mass differences suggests that the actual ground state of neutron star matter might contain hadrons other than neutrons and protons. In particular, the large difference

$$\mu(n) - \mu(p) = \frac{k_{\text{F}}(n)^2}{2m} - \frac{k_{\text{F}}(p)^2}{2m} \approx 100 \text{ MeV} \approx \tfrac{3}{4} m_\pi \tag{10}$$

at $\rho = \rho_{\text{nuc}}$ indicates that at slightly higher densities negatively charged pions might appear in the ground state via, for example, the reaction

$$n \rightarrow p + \pi^- . \tag{11}$$

If they did appear, such pions would condense (at $T = 0$) in the lowest available energy state.

Precisely these considerations led Bahcall and Wolf to consider the effects of a condensate of free pions on neutron star cooling. However, as Bahcall and Wolf were careful to emphasize, a realistic study of the effects of pion condensation requires a detailed analysis of the strong interactions of the putative condensed pions with the neutron star matter. Recently, a number of authors, using a wide range of approaches, have studied this problem.[4] Their results support the qualitative conclusion that at densities in the range of those found in neutron stars, neutron matter undergoes a phase transition to a ground state containing a pion condensate. Assuming the pions to condense in a single mode with energy-momentum given by ($E = \mu$, k), where μ is the π^- chemical potential (Baym 1973), the quantitative results of these various authors are roughly consistent with[5]

$$(\rho)_{\text{crit}} \approx (1.5\text{–}3)\rho_{\text{nuc}} \sim 2\rho_{\text{nuc}} , \tag{12a}$$

$$(\mu)_{\text{crit}} \approx (1\text{–}3)m_\pi \sim 2m_\pi \approx 250 \text{ MeV} , \tag{12b}$$

and

$$(k)_{\text{crit}} \approx (2\text{–}4)m_\pi \sim 3m_\pi \approx 400 \text{ MeV}/c , \tag{12c}$$

where the subscript "crit" refers to the value at the threshold of the phase transition and the values in the last column are the values we use.[6]

[1] For example, Migdal 1973; Sawyer and Scalapino 1972; Kogut and Manassah 1972; Hartle, Sawyer, and Scalapino 1975; Migdal, Markin, and Mishustin 1974; Au and Baym 1974; Campbell, Dashen, and Manassah 1975*a*, *b*; Baym *et al.* 1975.

[2] If the chemical potential for negative charge is sufficiently large ($\mu > m_\mu \approx 100$ MeV), then muons will also be present. Although for simplicity of presentation we ignore muons in this discussion, we shall later indicate how to include their effects.

[3] This result assumes that the antineutrinos are non-degenerate.

[4] See n. 1.

[5] Differences of a factor of 2 here will not alter our qualitative conclusions.

[6] We take here a rough average of the values found in the references listed in n. 1. Hence these results differ slightly from those in the σ model discussed later. These differences are a reasonable indication of the theoretical uncertainties in the parameters.

To describe the pion condensed state more explicitly, we must adopt a specific formalism for treating the strong interactions in the neutron star matter. We shall follow an approach based on the observed approximate chiral SU(2) × SU(2) symmetry of low-energy πN interactions (BCDM), because, as we shall see, this formalism is particularly suited for calculating β-decay in the presence of a pion condensate. Although many features of this chiral symmetry approach can be formulated in a model-independent way (Campbell, Dashen, and Manassah 1975*a*, *b*), for simplicity we shall discuss the problem in terms of the linear model, which in addition to the nucleon $N = (p, n)$ and pion, contains a σ meson. It is important to note that the most significant phenomenological omissions of this simple model—(1) the exclusion of the $N^*(1238)$ and (2) the neglect of short-range repulsive nuclear correlations—produce effects which essentially cancel each other over the relevant range of parameters.[7] Thus the model containing only n, p, π, and σ is quite adequate for our purpose.

Let us briefly summarize the results of the model which are needed for the present calculation. The simplicity of the chiral symmetry approach hinges on the relation of the macroscopic pion condensate field, for which we take the *Ansatz*[8]

$$\begin{aligned}\langle\sigma\rangle &= f_\pi \cos\theta\,,\\ \langle\pi^\pm\rangle \equiv \langle\pi_1 \pm i\pi_2\rangle &= f_\pi \exp(\pm i\boldsymbol{k}\cdot\boldsymbol{x})\sin\theta\,,\\ \langle\pi_0\rangle \equiv \langle\pi_3\rangle &= 0 \end{aligned}\qquad(13)$$

to the normal vacuum state in the σ model, in which[9]

$$\langle\sigma\rangle = f_\pi\,,\qquad \langle\pi_i\rangle = 0 \qquad (i = 1, 2, 3)\,. \qquad (14)$$

Here $f_\pi = 94.5$ MeV is the pion decay constant. From the standard SU (2) × SU (2) transformation laws for σ and π^6, it is easy to see that the fields in (13) are simply chiral rotations of the fields in (14). Specifically, rotating the fields in (14) with

$$U \equiv \exp\left(-i\boldsymbol{k}\cdot\int \boldsymbol{x}d^3x V_3{}^0\right)\exp(+iQ_1{}^5\theta) \qquad (15)$$

yields (13) (BCDM). It follows that to study matrix elements of any operator, $\mathcal{O}$, involving strongly interacting particles in the presence of the macroscopic pion condensed field (13), one can transform the operator to $\mathcal{O}' = U^{-1}\mathcal{O}U$ and study matrix elements of the transformed operator $\mathcal{O}'$ in the simpler macroscopic field (14). We shall find this transformation particularly useful for simplifying the calculation of β-decay in the presence of the pion condensate.

In the σ model the pion condensed state of neutron star matter consists of the macroscopic σ and π fields in (13) and a filled Fermi sea of quasiparticles which are mixtures of nonrelativistic neutron and proton states. The quasiparticles are eigenstates of the effective strong interaction Hamiltonian, $H_{\text{eff}} = H + \mu\rho_Q$, where ρ_Q is the charge density operator; specifically, with the condensate momentum chosen such that $\boldsymbol{k} = |k|\hat{z}$, the filled quasiparticle states are[10]

$$u(\boldsymbol{p}, s = \pm) = \mp i\sin\phi\, p(\boldsymbol{p}, \pm) + \cos\phi\, n(\boldsymbol{p}, \pm) \qquad (16)$$

where + (−) refers to spin up (down),

$$\tan\phi = \frac{g_A k \sin\theta}{\mu\cos\theta + [\mu^2\cos^2\theta + g_A{}^2k^2\sin^2\theta]^{1/2}}\,, \qquad (17)$$

and $g_A = 1.36$ measures the πN coupling strength.[11] In this model the pion-condensed phase first appears at $\rho = \rho_{\text{crit}} = 2f^2m/(g_A(g_A{}^2 - 1)) \approx 1.5\rho_{\text{nuc}}$, and the parameters μ, k, and θ increase smoothly as ρ is increased above ρ_{crit} (BCDM). For later use we note that $\theta(\rho)$ can be determined from

$$\rho/\rho_c = \frac{1 + (g_A{}^{*2} - 1)\sin^2\theta}{(\cos\theta)^{1/2}}\,. \qquad (18)$$

The excitation spectrum of the pion condensed matter for $T \neq 0$ (Campbell, Dashen, and Manassah 1975*a*, *b*; Baym and Dashen 1976), an understanding of which is essential to any discussion of neutron star cooling, can be described succinctly by noting that there exist no low-lying ($\Delta E \approx k_BT \approx 1$ MeV) boson excitations and that, at least for $T \gtrsim 1$ MeV, the fermion excitations begin just above the quasiparticle Fermi surface.[12] Hence the obvious candidate for (anti)neutrino emission is essentially the process suggested by Bahcall and Wolf: namely, quasiparticle β-decay,

$$u(p) \to u(p') + e^- + \bar{\nu}_e\,, \qquad (19a)$$

in the presence of the condensed pion field, as indicated in Figure 1. The related process,

$$e^- + u(p) \to u(p') + \nu_e\,, \qquad (19b)$$

[7] More specifically, we note that including nuclear correlations of the Lorentz-Lorenz type with strength γ effectively reduces g_A to $g_A' = (1 - \gamma)^{1/2}g_A$, whereas including the $N^*(1238)$ effectively increases g_A to $g_A'' = (1 + \xi)g_A$. The net effect is $g_A \to g_A^* = (1 - \gamma)^{1/2}(1 + \xi)g_A$, and γ and ξ are such that $g_A \approx g_A^*$ (Brown and Weise 1975, 1976; Baym *et al.* 1975).

[8] We indicate only the spatial variation of the condensed pion field. The correct time dependence, $\langle\pi^\pm\rangle = e^{\pm i\mu t}$, follows automatically from our use of the effective Hamiltonian density, $H_{\text{eff}} = H + \mu\rho_Q$, where ρ_Q is the charge density operator.

[9] The amplitude of the pion condensed field is chosen so that the chiral invariant $\langle\sigma^2 + \pi^{-*}\pi^-\rangle$ is the same in the condensed and normal phases. This is clearly necessary for the two field configurations to be simply related by a chiral rotation (BCDM).

[10] The approximations involved in constructing this simple model are discussed in BCDM. The most important of these for our present discussion is the restriction to pure neutron matter: that is, in the limit $\theta \to 0$, $u \to n$ only, with no admixture of p. Since the admixture of protons is, by the results of § II, roughly 1–2%, this approximation would not be expected to cause significant errors in general and does not in our calculation.

[11] Explicitly, $g_A = f_\pi g/m$, where m is the nucleon mass and $g^2/4\pi = 14.4$.

[12] As we shall discuss in § IV, it appears likely that pairing forces between the quasiparticles may produce a gap in the fermion excitation spectrum for $T \lesssim 1$ MeV. This gap would clearly reduce our estimated cooling rates for $T \lesssim 1$ MeV.

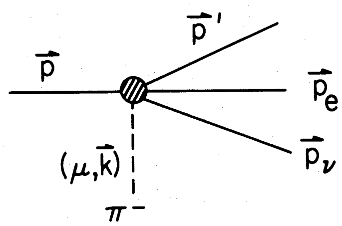

FIG. 1.—Quasiparticle β-decay. The quasiparticle of momentum $\boldsymbol{p}$ can be considered to scatter on the classical π^--field of energy-momentum $(\mu, \boldsymbol{k})$, then decaying into a quasiparticle of moment $\boldsymbol{p}'$, electron, and antineutrino.

is an expected source of neutrinos. In the next section, we shall examine these processes in detail.

III. β-DECAY IN THE PRESENCE OF THE PION CONDENSATE

The total neutrino luminosity due to process (19) is obtained in the usual way from Fermi's golden rule:

$$L_\nu^{\pi n} = (2\pi)\frac{\Omega^4}{(2\pi)^{12}} \times \sum_{\text{spins}} \int d^3p d^3p' d^3p_e d^3p_\nu S\delta(E_f - E_i)E_{\bar{\nu}} \times |\langle (p', e, \nu)|H_w|p\rangle|^2\,, \tag{20}$$

where Ω is the volume of the neutron star, S is the appropriate product of Fermi-Dirac distribution factors, and H_w is the weak-interaction Hamiltonian that describes the decay of quasiparticles. For simplicity we shall use natural units, $\hbar = 1 = c$. Making use of the fact that the massive fermions are all near their Fermi energies, this expression for $L_\nu^{\pi n}$ can be factored into a phase-space factor P and a factor Q containing the squared matrix element:

$$L_\nu^{\pi n} = (2\pi)\frac{\Omega^4}{(2\pi)^{12}} PQ\,. \tag{21}$$

The factors P and Q are given by

$$P = \int p^2dpp'^2dp'p_e^2dp_ep_\nu^2dp_\nu S\delta(E_f - E_i)E_{\bar{\nu}} \tag{22}$$

$$Q = \int d\Omega_pd\Omega_{p'}d\Omega_ed\Omega_\nu \sum_{\text{spins}} |\langle (p', e, \nu)|H_w|p\rangle|^2\,. \tag{23}$$

Performing the integral that occurs in the squared matrix element over coordinate space introduces a squared momentum conserving δ-function into Q. As we discuss later, the spin sum of the remaining part of the squared matrix element is then approximately angle-independent so that the factor Q can be further factored as follows:

$$Q = \left(\frac{2\pi}{\Omega}\right)^3 A|M|^2\,, \tag{24}$$

where $|M|^2$ is the spin sum of the squared matrix element after the squared momentum conserving δ-function and plane wave normalization factors (Ω^{-4}) have been factored out. The factor A is given by

$$A = \int d\Omega_pd\Omega_{p'}d\Omega_ed\Omega_\nu\delta^{(3)}(\boldsymbol{p} + \boldsymbol{k} - \boldsymbol{p}' - \boldsymbol{p}_e - \boldsymbol{p}_\nu)\,. \tag{25}$$

Evaluation of these factors for process (19) leads to an expression for $L_\nu^{\pi n}$ which is similar to that obtained by Bahcall and Wolf for process (3); however, these two processes differ in two important ways. First, since the pions are not "quasi-free," but are interacting strongly with the baryons, there are two dynamical effects producing differences between (19) and (3): (1) the decaying particles are not neutrons, but rather the quasiparticle eigenstates of H_{eff}; and (2) that part of the weak Hamiltonian involving strongly interacting particles is modified by the macroscopic pion field.

We shall term these "matrix element effects." Second, since the pions have nonzero momentum k, and energy $\mu \neq m_\pi$, there are also "kinematical effects" in the phase-space factors which can produce additional differences between the rates of (19) and (3).

Let us treat the matrix elements effects first. The weak interaction Hamiltonian describing β-decay of nonstrange hadrons is

$$H_w = \frac{G}{\sqrt{2}}(V_1{}^\mu + iV_2{}^\mu + A_1{}^\mu + iA_2{}^\mu)\bar{\nu}\gamma_\mu(1 + \gamma_5)e + \text{Hermitian conjugate}\,, \tag{26}$$

where V_i and A_i are the vector and axial vector hadronic isospin currents, respectively. To include the effects of the pion condensate on these hadronic currents, instead of working explicitly with the space-dependent macroscopic fields in (13) we can apply the chiral rotation U of (15) to H_w. Under this rotation the current

$$J_h{}^\mu \equiv V_1{}^\mu + iV_2{}^\mu + A_1{}^\mu + iA_2{}^\mu \tag{27}$$

transforms to

$$\begin{aligned}\tilde{J}_h{}^\mu &\equiv UJ_h{}^\mu U^{-1}\\ &= \exp(i\boldsymbol{k}\cdot\boldsymbol{x})\{V_1{}^\mu + A_1{}^\mu + i[\cos\theta(V_2{}^\mu + A_2{}^\mu)\\ &\qquad + \sin\theta(V_3{}^\mu + A_3{}^\mu)]\}\,.\end{aligned} \tag{28}$$

In terms of this rotated current, the square of the quasiparticle β-decay matrix element, summed over spins, becomes

$$\left\{\sum_{s,s'}\bar{u}(p', s')\tilde{J}_h{}^\mu u(p, s)\bar{u}(p, s)\tilde{J}_h{}^{\alpha\dagger}u(p', s')\right\}(x)L_{\mu\alpha} \equiv H^{\mu\alpha}L_{\mu\alpha}\,, \tag{29}$$

where the final equality defines the hadronic tensor, $H^{\mu\alpha}$, and where $L_{\mu\alpha}$ is the standard leptonic tensor[13]

$$L^{\mu\alpha} = 4\{p_e{}^{\mu}p_{\bar{\nu}}{}^{\alpha} + p_e{}^{\alpha}p_{\bar{\nu}}{}^{\mu} - (p_e{}' p_{\bar{\nu}})g^{\mu\alpha} - i\epsilon^{\mu\alpha\gamma\delta}(p_e)_{\gamma}(p_{\bar{\nu}})_{\delta}\}\,. \tag{30}$$

The repeated Lorentz indices are summed. In the nonrelativistic limit, one can ignore the momentum dependence of the hadronic matrix elements[14,15] and approximate the vector and axial vector matrix elements between nucleons by

$$\langle N'(p')|V_i{}^{\mu}|N(p)\rangle = \tfrac{1}{2}\langle N'|\tau_i|N\rangle\,, \qquad \text{for } \mu = 0 \tag{31a}$$

and

$$\langle N'(p')|A_i{}^{\mu}|N(p)\rangle = \tfrac{1}{2}g_A\langle N'|\tau_i\sigma_j|N\rangle\,, \qquad \text{for } \mu = j \quad [1, 2, 3]\,. \tag{31b}$$

Here τ_i are the Pauli matrices for isospin and σ_j those for spin. All other matrix elements are zero in this limit.

At this point it is useful to digress briefly to clarify physically why one must take the matrix element of the *rotated* weak current between *quasiparticle* states, since at first glance it might appear that this is somehow "double-counting" and that one should take, say, matrix elements of the *unrotated* weak current between quasiparticle states. To clarify this point, it is simplest to go to the very weak condensate limit, $\theta \to 0$, and to work to lowest order in θ. Then the *rotated* current is

$$\tilde{J}_h{}^{\mu} \approx \exp(i\boldsymbol{k}\cdot\boldsymbol{x})[(V_1{}^{\mu} + iV_2{}^{\mu}) + (A_1{}^{\mu} + iA_2{}^{\mu}) + \theta(V_3{}^{\mu} + A_3{}^{\mu})] + O(\theta^2)\,; \tag{32}$$

and the quasiparticle states, ignoring spin for simplicity, are

$$u \approx \phi|p\rangle + |n\rangle\,, \tag{33}$$

where

$$\phi \approx \frac{g_A k}{2\mu}\,\theta\,. \tag{34}$$

Then to $O(\theta)$, the matrix element will contain *two*

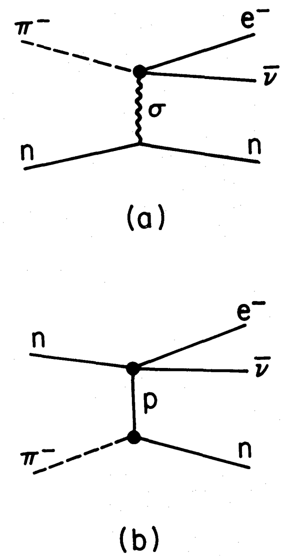

FIG. 2.—The two contributions to quasiparticle β-decay

nonvanishing terms,

$$\langle u|\tilde{J}_h{}^{\mu}|u\rangle \approx \theta\langle n|V_3{}^{\mu} + A_3{}^{\mu}|n\rangle + \frac{g_A k}{2\mu}\langle p|(V_1 + iV_2) + (A_1 + iA_2)|n\rangle\,. \tag{35}$$

Clearly, if one used the *unrotated* weak current, only the second term would be present. That in fact *both* terms must be included can be seen in two related ways. First, in the context of the sigma model described in § II, noting that for small θ the pion field is $\sim O(\theta)$, we see that the two terms in (35) are related[16] to the diagrams in Figures 2*a* and 2*b*, respectively.

More generally, we note that for a very small amplitude pion condensate in originally pure neutron matter, the β-decay process is determined completely by the matrix element

$$\mathscr{M} \equiv \langle \pi^- n|J_h{}^{\mu}|n\rangle\,. \tag{36}$$

Using standard current algebra techniques, including the LSZ reduction and PCAC, one can show that this matrix element contains a "pole" contribution—roughly corresponding to Figure 2*b*, albeit with pseudovector coupling at the ($p\pi^-$–n) vertex—and a "commutator" contribution, which (roughly) corresponds to Figure 2*a*. Thus to obtain a result consistent with chiral symmetry as reflected in the current algebra low-energy theorem, one must *both* rotate the weak current *and* use quasiparticle states to determine the full effect of the pion condensate on β-decay.

[13] See Lifshitz and Pitaevskii 1974. Note that these authors use a Lorentz covariant normalization, which includes factors of $(1/2E)$ in the phase space and yields canceling factors of $(2E)$ in the matrix element from the normalization of the spinor states. For consistency with the notation of Bahcall and Wolf, we use noncovariant normalization: eq. (52), which fixes the normalization of $|M|^2$ for the pion-condensed case to $|M|^2$ for ordinary β-decay, allows us readily to translate from one convention to the other.

[14] See n. 13.

[15] This is the "allowed" approximation of nuclear physics and is adequate for our purposes.

[16] Readers wishing to follow the full technical details underlying this discussion are referred to Adler and Dashen (1968) in general, and to Adler and Dothan (1966) for a particular calculation related to eq. (36).

For arbitrary $\mathscr{O}$ we can use (31) to perform the explicit spin sums indicated in (29). We obtain

$$H_{00} = \tfrac{1}{4}[\sin^2(\theta + 2\phi) + \sin^2(\theta - 2\phi)]\,, \tag{37a}$$

$$H_{33} = (g_A{}^2/4)[\sin^2(\theta + 2\phi) + \sin^2(\theta - 2\phi)]\,, \tag{37b}$$

$$H_{11} = H_{22} = (g_A{}^2/4)[2\sin^2 2\phi + 2\sin^2\theta]\,, \tag{37c}$$

$$H_{03} = H_{30}{}^* = (g_A/4)[\sin^2(\theta + 2\phi) - \sin^2(\theta - 2\phi)]\,, \tag{37d}$$

and

$$H_{12} = H_{21}{}^* = (ig_A{}^2/4)[4\sin\theta\sin 2\phi]\,, \tag{37e}$$

where θ and ϕ are as defined in § II. For ordinary, unpolarized neutron β-decay in this approximation, one would have $H_{00} = 2, H_{11} = H_{22} = H_{33} = (2)g_A{}^2$, with all other $H_{\mu\alpha} = 0$. Here two polarization-type effects arise in $H_{\mu\alpha}$ because the pion condensate momentum provides a preferred direction, which we have chosen to be $\hat{z} = \hat{3}$. First, there exist off-diagonal terms, H_{03} and H_{12}. Second, the decays parallel and perpendicular to the condensate momentum are different. For small θ, to order θ^2, $|M|^2$ is independent of spatial angle, and we shall carry out the integration over spatial angle neglecting any such dependence. For large θ, terms proportional to $\boldsymbol{p}_e(p_\nu)_0$ and $(p_e)_0\boldsymbol{p}_\nu$ appear, and these could give corrections of order $[k_{\mathrm{F}}(e)/k]^2$ to terms of order θ^3 and higher. In fact,

$$k_{\mathrm{F}}(e)/k \approx \mu/k \approx 0.67\,, \tag{38}$$

where the first approximate equality follows from thermal equilibrium and the ultrarelativistic nature of the electrons in the matter and the second follows from (12). These terms may not be negligible, but their inclusion would not change our conclusions.

Given the matrix element squared as determined by (29), (30), and (37), to calculate the total decay rate we must perform the phase-space integrals. It is here that the "kinematical effects" of the pion condensate appear. First, because of the factor exp $(i\boldsymbol{k}\cdot\boldsymbol{x})$ in (28) the momentum conservation equation for the quasiparticle β-decay is

$$\boldsymbol{k} + \boldsymbol{p} = \boldsymbol{p}' + \boldsymbol{p}_e + \boldsymbol{p}_{\bar{\nu}}\,. \tag{39}$$

Further, since the hadronic weak current decreases the hadronic charge by one unit, the correct energy balance equation includes the chemical potential for charge:

$$\mu + \frac{p^2}{2m} = \frac{p'^2}{2m} + E_e + |\boldsymbol{p}_{\bar{\nu}}|\,. \tag{40}$$

The physical interpretation of (39) and (40) is clear: a quasiparticle of momentum p picks up energy-momentum (μ, k) from a condensed pion and then decays into a second quasiparticle and the lepton pair. Symbolically,

$$\pi^- + u \to u' + e^- + \bar{\nu}_e\,.$$

The central role of the pion condensate in inducing the quasiparticle decay is further clarified by recalling why the ordinary β-decay reaction in (1) from excited states of the neutron star matter at $T \neq 0$ does *not* cool the star. Since by (9) $k_{\mathrm{F}}(n) \gg k_{\mathrm{F}}(e) = k_{\mathrm{F}}(p) \gg p_{\bar{\nu}} \approx k_{\mathrm{B}}T$, the reaction (1) cannot conserve momentum for particles near the Fermi surface. Hence the rate must be strongly suppressed by Fermi-Dirac statistical factors. By providing energy-momentum (μ, k), the condensed pion permits the reaction (19) to proceed for quasiparticles and electrons near their respective Fermi surfaces, provided only that $k_{\mathrm{F}}(u) \geq k/2$. Since this constraint is satisfied in our calculations, the quasiparticle decay in the presence of the pion condensate does indeed provide a rapid cooling mechanism for neutron stars.

We now discuss the phase space factors in more detail beginning with the factor P. Following Bahcall and Wolf's calculation, we first shift variables from momentum to energy and then remove the energy conserving δ-function by performing the integral over the neutrino energy. If we then set all fermion momentum magnitudes equal to their values on the respective Fermi surfaces, we obtain in place of (22):

$$P = k_{\mathrm{F}}{}^2(n)(m_n{}^*)^2 E_{\mathrm{F}}{}^2(e)\int dE_p dE_{p'} dE_e S E_{\bar{\nu}}{}^3\,, \tag{41}$$

where F subscripts denote Fermi surface values, $m_n{}^*$ is the neutron effective mass, and the value of $E_{\bar{\nu}}$ is now fixed by energy conservation. The integrals in (41) are evaluated through another shift in variables from energy to the dimensionless variables x_i defined by Bahcall and Wolf. The final result for P is

$$P = k_{\mathrm{F}}{}^2(n)m_n{}^2 m_\pi \mu\left(\frac{m_n{}^*}{m_n}\right)^2\left(\frac{\mu}{m_\pi}\right)(k_{\mathrm{B}}T)^6\cdot I\,, \tag{42}$$

where k_{B} is Boltzmann's constant, I is a three-dimensional analytic integral equal to $457/5040\pi^6$, and we have used the equilibrium result that $E_{\mathrm{F}}(e) \approx \mu$. Note in this expression for P the occurrence of the factors $(m_n{}^*/m_n)^2$ and (μ/m_π). Bahcall and Wolf set both these factors equal to one, in accordance with their approximations that the pions are free and have $k = 0$ and that $m_n{}^* \approx m_n$. As we have indicated, dynamical models of pion condensation yield roughly $\mu \approx 2m_\pi$ at threshold; we shall use this value. Further, the effective mass problem in neutron matter has been studied thoroughly by Bäckman, Källman, and Sjöberg (1973), who obtain $m_n{}^* \approx 0.8m_n$ at neutron star densities. Thus we will take $(m_n{}^*/m_n)^2 \approx 0.65$.

Now consider the angular integral given by (25). Using ordinary neutron star matter as a guide, Bahcall and Wolf noted that since $k_{\mathrm{F}}(n) \gg k_{\mathrm{F}}(e) \gg |p_{\bar{\nu}}| \approx k_{\mathrm{B}}T$, requiring the fermions in (3) to be near their respective Fermi surfaces to avoid suppression by statistical factors gives

$$|\boldsymbol{p}| \approx |\boldsymbol{p}'| \approx k_{\mathrm{F}}(n) \gg k_{\mathrm{F}}(e) \approx |\boldsymbol{p}_e| \gg k_{\mathrm{B}}T \approx |\boldsymbol{p}_{\bar{\nu}}|\,. \tag{43}$$

With the further assumption that $k = 0$, Bahcall and Wolf approximate the exact momentum conserving δ-function in (25) by

$$\delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}' - \boldsymbol{p}_e)\,, \tag{44}$$

which yields

$$A \approx \frac{(4\pi)^3}{2k_{\rm F}^2(n)k_{\rm F}(e)}\,. \tag{45}$$

The results for pion condensed matter reviewed in § II suggest that

$$|\boldsymbol{p}| \approx |\boldsymbol{p}'| \approx k_{\rm F}(u) \approx |\boldsymbol{k}| > k_{\rm F}(e) \approx |\boldsymbol{p}_e| \gg k_{\rm B}T \approx |\boldsymbol{p}_{\bar{\nu}}|\,. \tag{46}$$

Here $k_{\rm F}(u)$ is the Fermi momentum of the quasi-particles and k is the momentum of the pion condensate. Thus for pion condensed matter the vector $\boldsymbol{k}$ cannot be dropped from the momentum conserving δ-function as in (44). If one retains $\boldsymbol{k}$ but drops $\boldsymbol{p}_\nu$ from the δ-function, (25) becomes

$$A = 4\pi \int d\Omega_p d\Omega_{p'} d\Omega_e \delta^{(3)}(\boldsymbol{p} + \boldsymbol{k} - \boldsymbol{p}' - \boldsymbol{p}_e)\,. \tag{47}$$

This is evaluated by making use of the integral representation of the δ-function:

$$\delta^{(3)}(\boldsymbol{p} + \boldsymbol{k} - \boldsymbol{p}' - \boldsymbol{p}_e) = \frac{1}{(2\pi)^3} \int \exp\left[i(\boldsymbol{p}\cdot\boldsymbol{x} + \boldsymbol{k}\cdot\boldsymbol{x} - \boldsymbol{p}'\cdot\boldsymbol{x} - \boldsymbol{p}_e\cdot\boldsymbol{x})\right] d^3x\,. \tag{48}$$

Inserting this into (47) and performing all of the angular integrations then results in

$$A = \frac{8(4\pi)^2}{pp'p_e k} \times \int \frac{dx}{x^2} \sin(px)\sin(p'x)\sin(kx)\sin(p_e x)\,. \tag{49}$$

The final integral over x in this expression is evaluated by replacing $\sin(px)\sin(p'x)\sin(kx)$ by a sum of single sine functions and then observing that $|p \pm p' \pm k| > p_e$. Finally, replacing fermion momentum magnitudes by their Fermi surface values, we obtain

$$A = \frac{(4\pi)^3}{2k_{\rm F}^2(u)k}\,, \tag{50}$$

which differs from Bahcall and Wolf's result by the factor [since $k_{\rm F}(u) = k_{\rm F}(n)$] $k_{\rm F}(e)/k = 0.67$ from (38).

The expressions (29), (30), and (37), together with the kinematic results, allow us to calculate the total rate at any θ. However, we shall see that the effectiveness of the pion condensate in cooling the star is so great that estimating the rate near threshold—that is, for small θ^2—is sufficient for our purposes. For small θ, using (34), we have

$$H_{00} \approx \frac{1}{2}\left[1 + \left(\frac{g_A k}{\mu}\right)^2\right]\theta^2 \tag{51a}$$

and

$$H_{11} = H_{22} = H_{33} = (g_A{}^2/2)\left[1 + \left(\frac{g_A k}{\mu}\right)^2\right]\theta^2\,. \tag{51b}$$

Thus in this limit we have

$$\sum_{\rm spin} |M|^2{}_{\text{pion-cond}} = \frac{\theta^2}{4}\left[1 + \left(\frac{g_A k}{\mu}\right)^2\right] \sum_{\rm spins} |M|^2{}_{\rm norm}\,, \tag{52}$$

where the subscript "pion-cond" refers to the quasi-particle decay in the presence of the pion condensate and "norm" refers to ordinary β decay.

Combining all factors, we find that the luminosity per unit volume from reaction (19a) is

$$L_\nu{}^{\pi n}/\Omega = 7.6 \times 10^{25}\ {\rm ergs\ s^{-1}\ cm^{-3}} \times \left(\frac{\mu^2}{km_\pi}\right)\left(\frac{m_n{}^*}{m_n}\right)^2 \frac{\theta^2}{4}\left[1 + \left(\frac{g_A k}{\nu}\right)^2\right] \times (1 + 3g_A{}^2)T_9{}^6\,. \tag{53}$$

Using the values of μ and k from (12) and taking $g_A = 1.36$ and $(m_n{}^*/m_n)^2 = 0.65$, we estimate the total luminosity of a mass M_s of pion-condensed neutron star matter at uniform density ρ to be

$$L_\nu{}^{\pi n} \approx 1.5 \times 10^{46}\ {\rm ergs\ s^{-1}} \times \theta^2 T_9{}^6 \frac{M_s}{M_\odot}\left(\frac{\rho_{\rm nuc}}{\rho}\right)\,. \tag{54}$$

In (54), we have included a factor of 2 to account for the "inverse" reaction shown in (19b) and, since Fermi levels suggest that muons are also present, an additional factor of 2 to account for the reaction

$$u \to u + \mu^- + \bar{\nu}_\mu$$

and its inverse. Note that in (54), θ^2 replaces the factor (n_π/n_b) in the corresponding formula of Bahcall and Wolf.

IV. DISCUSSION AND CONCLUSIONS

A comparison of equations (4) and (54) reveals that the actual pion condensate increases $L_\nu{}^{\pi n}$ only slightly from its free pion value; clearly our results support the conclusion of Bahcall and Wolf concerning the importance of this cooling mechanism. Indeed, if we apply (54) directly to a 1 $M_\odot$ neutron star at twice nuclear density, we find, using $\theta^2 = 0.1$ (Brown and Weise 1975, 1976),[17] that the time to cool from

[17] In the sigma-model calculation, this value of θ^2 corresponds to an admixture of 6% protons and would occur at a density $\rho = 1.8\rho_{\rm nuc}$.

T_{9i} to T_{9f} is

$$(\Delta t) \approx (3 \text{ minutes})[T_{9f}^{-4} - T_{9i}^{-4}] . \qquad (55)$$

Thus if the internal temperature T and the effective surface temperature T_e are related by $T_e \approx 10^{-2}T$ (Tsuruta and Cameron 1965, 1966), the effective surface temperature would drop below 4.7×10^6 K, the Crab pulsar observability limit (Wolff *et al.* 1975) in a few tens of minutes.

There are several possible complications that can beset this simple estimate. First, if densities sufficiently large for pion condensation are formed in supernova explosions at temperatures of, say, $T_9 = 100 \approx 10$ MeV, then the luminosity predicted by (54) for a 1 $M_\odot$ star at twice nuclear density is

$$L_\nu^{\pi n} \approx 10^{58} \text{ ergs s}^{-1} , \qquad (56)$$

which is very large, since the solar mass is

$$M_\odot c^2 \approx 2 \times 10^{54} \text{ ergs} . \qquad (57)$$

Indeed, Fermi statistics restrict the number of final states which can be filled (see Bludman and Ruderman 1975) and thus must slow down these processes.

This effect should be distinguished from the high neutrino degeneracy which may accompany the original formation of the neutron star (Lamb and Pethick 1976). Although this degeneracy may slow down the neutronization process $p + e^- \rightarrow n + \nu$ by blocking possible states of the final neutrino, we do not believe that it will affect the pion condensate, since these neutrinos should mostly be gone before the pion condensate is formed. However, this question probably deserves more detailed consideration.

Second, as Sawyer and Soni (1977) have recently shown, if the neutrino (antineutrino) luminosities in the presence of a pion condensate are as large as predicted by (54), then the absorption of neutrinos via the processes

$$\nu + u(p) \rightarrow e^- + u(p') \qquad (58a)$$

and

$$\bar{\nu} + e^- + u(p) \rightarrow u(p') \qquad (58b)$$

will also be large. Hence the neutrino and antineutrino mean free paths will be short—typically only a fraction of the neutron star radius—so that the escape of neutrinos and antineutrinos will be substantively delayed. This suggests that the star may take a few hours, rather than minutes, to cool to $T = 10^9$ K.

Finally, our luminosity calculation is based on the assumption that there exists no gap in the quasi-particle excitation spectrum. However, a rough estimate suggests that at $T \sim 10^9$ K, the u-quasi-particles could go superfluid in the $^3P_2$ state as neutrons in dense neutron matter are predicted to do (Hofberg *et al.* 1970; Tamagaki 1970).

The new quasiparticles in the superfluid state will be linear combinations of u-quasiparticles and u-quasi-holes; i.e., the new quasiparticles will be linear combinations of neutron and proton particle *and* neutron and proton hole. Calculations within a strong-coupled theory are now proceeding at Stony Brook by J. Sauls and J. Serene; preliminary results show similarity with the neutron superfluidity of Hofberg *et al.* (1970) and Tamagaki (1970). The low effective masses of Bäckman, Källman, and Sjöberg (1973) lower the transition temperature from that found by Hofberg *et al.*, however. At about 10^9 K, therefore, it would appear that the onset of super-fluidity of the u-quasiparticles would change the scenario for cooling, in that the resultant energy gaps would severely decrease the participation of the quasi-particles in the cooling.

In summary, we can say that a pion condensate, if present in a neutron star, does provide an efficient mechanism for initial cooling of the star. More quantitative calculations, including the important question of cooling below $T = 10^9$ K, will require more detailed knowledge of the initial state in which the neutron star is formed, as well as consideration of neutrino transport and degeneracy and of possible gaps in the quasiparticle spectrum.

We thank John Bahcall, Gordon Baym, and Dan Freedman for stimulating suggestions and helpful criticism. We also thank Don Lamb and Chris Pethick for discussions of possible further implications of their results (1976). Also we are grateful to Ray Sawyer and A. Soni for informing us of their results prior to publication. Two of the authors (G. E. B. and D. K. C.) wish to acknowledge the hospitality of the Aspen Center for Physics, where part of this work was done.

REFERENCES

Adler, S. L., and Dashen, R. F. 1968, *Current Algebras* (New York: Benjamin), chap. 3.
Adler, S. L., and Dothan, Y. 1966, *Phys. Rev.*, **151**, 1267.
Au, C. K., and Baym, G. 1974, *Nucl. Phys.*, **A236**, 500.
Bäckman, S.-O., Källman, C. G., and Sjöberg, O. 1973, *Phys. Letters*, **43B**, 263.
Bahcall, J. N., and Wolf, R. A. 1965*a*, *Phys. Rev.*, **140B**, 1445.
———. 1965*b*, *Phys. Rev.*, **140B**, 1452.
———. 1965*c*, *Ap. J.*, **142**, 1254.
Baym, G. 1973, *Phys. Rev. Letters*, **30**, 1340.
Baym, G., Campbell, D., Dashen, R., and Manassah, J. 1975, *Phys. Letters*, **58B**, 304 (BCDM).
Baym, G., and Dashen, R. 1976, private communication.
Bludman, S. A., and Ruderman, M. A. 1975, *Ap. J.* (*Letters*), **195**, L19.
Brown, G. E., and Weise, W. 1975, *Phys. Letters*, **58B**, 300.
———. 1976, *Phys. Reports*, **27C**, 1.
Campbell, D. K., Dashen, R. F., and Manassah, J. T. 1975*a*, *Phys. Rev.*, **D12**, 979.
———. 1975*b*, *Phys. Rev.*, **D12**, 1010.
Hartle, J. B., Sawyer, R. F., and Scalapino, D. J. 1975, *Ap. J.*, **199**, 471.
Hofberg, M., Glassgold, A. E., Richardson, R. W., and Ruderman, M. 1970, *Phys. Rev. Letters*, **24**, 775.
Kogut, J., and Manassah, J. T. 1972, *Phys. Letters*, **41A**, 129.
Lamb, D. Q., and Pethick, C. J. 1976, *Ap. J.* (*Letters*), **209**, L77.
Lifshitz, E. M., and Pitaevskii, L. P. 1974, *Relativistic Quantum Theory* (New York: Pergamon Press), Pt. 2, p. 576.
Migdal, A. B. 1973, *Nucl. Phys.*, **A210**, 421.

Migdal, A. B., Markin, O. A., and Mishustin, I. N. 1974, *Soviet Phys.—JETP*, **66**, 443.
Sawyer, R. F., and Scalapino, D. J. 1972, *Phys. Rev.*, **D7**, 953.
Sawyer, R. F. and Soni, A. 1977, *Ap. J.*, **216**, 73.
Tamagaki, R. 1970, *Prog. Theor. Phys.*, **44**, 905.
Tsuruta, S., and Cameron, A. G. W. 1965, *Nature*, **207**, 364.
———. 1966, *Canadian J. Phys.*, **44**, 1863.
Wolff, R. S., Kestenbaum, H. L., Ku, W., and Novick, R. 1975, *Ap. J.* (*Letters*), **202**, L77.

G. E. Brown and O. Maxwell: Department of Physics, State University of New York, Stony Brook, NY 11794

D. K. Campbell: T-8, MS-453, Los Alamos Scientific Laboratory, Los Alamos, NM 87544

R. F. Dashen and J. T. Manassah: The Institute for Advanced Study, Princeton, NJ 08540