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CHAOS: CHTO DELAT?

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I provide a brief overview of the current status of the field of deterministic "chaos", stressing its interrelations and applications to other fields and suggesting a number of important open problems for future study.

1. INTRODUCTION

I should start by explaining the significance of my title and my motivations for choosing it. The question, "Chto Delat?", which is Russian for "What is to be Done?", is the title of a tract written by V.I. Lenin in 1902, outlining a course of action for his fellow revolutionaries. In a sense, the current practioners of deterministic chaos – who have on occasion been called a "chaos cabal"[1] – are also revolutionaries, for they have overthrown the "clockwork" universe of Laplace and are now plotting to bring order into the chaos they themselves discovered. One obvious issue, both for the conference and in particular for this overview, is "What are the future directions of the revolution that is deterministic chaos?": hence my primary motivation to choose as my title, "Chaos: Chto Delat?".

At a second level, my motivation for this particular title is to pay tribute to the contributions of our Soviet colleagues, who were unfortunately absent from the meeting. From the Kolmogorov-Arnol'd-Moser theorem and Arnol'd diffusion through the Melnikov method and the Chirikov overlap criterion to Sinai billiards and Lyapunov exponents, their many contributions have left a permanent impact on this subject and, more generally, on "nonlinear science". Indeed the question "Chto Delat?" was used as the title of a similar conference overview given by V. E. Zakharov at a meeting on "solitons" – in a sense, the nonlinear "flip side" of chaos, in that they reflect an astounding order and regularity in complex nonlinear systems – held in the late 1970's.

Consistent with my title, I will not attempt a detailed recapitulation of the meeting: the individual articles in the foregoing impressive proceedings speak eloquently for themselves and render such a summary unnecessary. Rather, I will seek to provide an impressionistic synthesis of the common themes that ran through the conference presentations and to focus on the open issues – the unanswered questions, both near- and long-term – that arose.

To provide a clear point of departure for the discussion let me start with a working definition of deterministic chaos: namely, the (apparently) random, "unpredictable" motion *in time* of deterministic dynamical systems having no external stochastic forces, so that the randomness derives from "sensitive dependence on initial conditions". More precisely, and for clarity in the later discussion, I will apply the term "chaos" to deterministic but unpredictable motions that occur, at least asymptotically, in low dimensional phase space. Thus, for dissipative systems, we can anticipate that there will be a (low) finite (fractal) dimensional strange attractor with at least one positive Lyapunov exponent and a broadband power spectrum. The following sections analyze how "chaos" currently interacts with a range of more traditional topics and offer some thoughts about the future interactions.

II. CHAOS AND DYNAMICAL SYSTEMS

We have come a long way in the past twenty-five years both in our understanding of deterministic chaos in low-dimensional dynamical systems and in the awareness and acceptance of this phenomenon among our scientific colleagues[1]. But as the presentations in this volume indicate, there are still many elegant, exciting, and subtle questions that remain to be resolved in this particular area.

The "strange attractors" – or more generally, the "strange sets" – that arise in deterministically chaotic dynamical systems are generally not simply "fractals" but are in fact "multi-fractals", having a range of scalings. The introduction of the "$f(\alpha)$" approach[2] and its subsequent thermodynamic interpretation[3] have greatly clarified the structure of the strange sets occurring in several universal routes to chaos, including the period-doubling and quasi-periodic cases[4]. But the full details of the dynamics[3,4] of the generation of the sets – in what order are the points generated? – remain to be explained[3]. Further, the hierarchical structure and scalings of the Henon attractor and of the two-dimensional area preserving maps lie beyond – but probably just beyond[5] – our current methods. A broader open challenge in both real and numerical experiments is to predict invariant properties of the strange sets on the basis of a (small (!)) limited amount of data; here the analysis of unstable periodic orbits that lie in the chaotic region may play a critical role[6]. On a grander scale, one can speak of the general "inverse problem": to what extent can one, given $f(\alpha)$ and related quantities, reconstruct the actual fractal set, including the dynamics? Is the long-range goal articulated by Sreenivasan[7] of characterizing by a fractal description the geomtric features in physical space of unconfined, fully-developed turbulence really achievable?

Finally, two broad challenges – one for experimentalists, one for theorists – were posed in several contributions and late-night discussions. For the experimentalists, the discovery of new physical systems exhibiting low-dimensional chaos remains an on-going challenge. And for theorists, there is the intriguing prospect of using the underlying deterministic nature for (short-term) prediction or forecasting of the behavior of these chaotic systems[8].

III. CHAOS AND EXPERIMENTAL TECHNIQUES

One of the most important developments that led to the widespread acceptance of deterministic chaos was the emergence of *high precision* experiments on real systems that gave *quantitative* confirmation of theoretical predictions; perhaps the most celebrated example is the confirmation by Libchaber and Maurer[9] of the "universality" predictions of Feigenbaum[10] for the period doubling transition to chaos.

These high precision experiments continue to be a central aspect of studies of chaotic systems. Perhaps it is just my naivete as a theorist, but I continue to be truly amazed by the ingenious ways in which our experimental colleagues extend the limits of the measurable by, in essence, "cheating" nature. At this meeting alone, the techniques discussed ranged from elegantly subtle through fiendishly clever to just plain cute. They involved such tricks as changing the Rayleigh number (Ra) by seven orders of magnitude by controlling the density in a (cryogenic) helium gas cell[11], altering the geometry of fluid cells in mid-experiment by movable boundaries[12], building "non-wall" walls to control thermal effects at the boundaries[13], constructing a ring cavity laser configuration to produce the "third equation" crucial to the existence of chaos[14], and using the measuring device itself to nucleate spatially localized instabilities[15]. In fairness, I should note that the theorists themselves have on occasion[16] used similar clever tricks; for example, by looking at binary fluid mixtures, theorists have found a tractable case in which to study co-dimension two instabilities [17]. All this bodes well for the future studies of chaotic systems.

Even given an ingenious experimental set-up — or, for that matter, a well-conceived numerical experiment — chaotic systems pose the difficulty that, at present, enormous amounts of data are necessary to interpret the motions: one is not looking at a simple fixed point or limit cycle. Fortunately, a number of developments reported at this meeting suggest promising ways for approaching this problem in the future. First, for at least some classes of systems, one can hope that a better understanding of the various "universal" mechanisms generating chaotic behavior will reduce the need for masses of data; for example, unstable periodic orbits may provide essential insight into chaotic motions[6]. Second, substantial improvements in analysis of time-series data from both experiment and numerical simulations can be obtained using "singular value decomposition" techniques[18]; the application of this approach to meterological data has yielded impressive results[18]. Third, in the spatial domain, the use of "Fourier transform truncations"[13] offers, at least in some geometries, a systematic method for spatial image enhancement.

For Hamiltonian dynamics, where the absence of attractors makes long-time studies very subtle, control over numerical methods becomes even more important; in these proceedings this issue is elegantly examined in the context of the stability of the solar system[19]. The result, incidentally, should any anxious reader be holding his breath, is that the solar system will most likely be stable at least for "billions and billions" of years. The subtlety of the numerics needed to obtain these results is definitely worth noting[19].

When a clever experimental technique is combined with sophisticated data analysis, the consequences can be truly impressive. A good example treated in these proceedings is the measurement in a semi-conductor device of the $f(\alpha)$ curve for the quasi-periodic transition to chaos[20]: the agreement between the theoretical prediction and the high precision experimental results is indeed remarkable.

Finally, as in all attempts to compare theory and experiment, it is essential to ask the right question. In his contribution, Bob May stressed that the non-homoegeneous nature – the "patchiness" – in field biological systems can mask the true mechanisms of chaos.

IV. CHAOS AND TURBULENCE

From the first beginnings of the chaos revolution, one clear goal of many, if not most, of the participants, was a deeper understanding of turbulence. Although we have made remarkable progress towards this deeper understanding, there is still much to be done. Indeed, if one had to choose just one "most important" area for future studies of chaos, it would have to be the relationship of chaos to turbulence.

In Section I I have already given a working definition of "chaos", in which a crucial aspect was the low effective phase space dimension of the "unpredictable" deterministic system. In distinction, by "turbulence" I shall mean what is often called "fully developed" – or, more colorfully, "macho"[22] – turbulence in which, in addition to temporal disorder, there is "disordered" spatial structure on (at least apparently) all scales. Thus the spatial power spectrum is, like the temporal spectrum, broadband, and the phase space dimension of any attractor appears (at least a priori) very high. In turbulence, different spatial regions of the system appear independent, and spatial correlation functions are short-ranged. Further, there is no mean spatial structure in time: any overall spatial pattern observed at time t is uncorrelated to that observed at time t + T, for large enough T. This result can be elegantly visualized using the shadowgraph technique described by Berge[12]. To summarize, "turbulence" involves strong disorder and unpredictability in both space and time and consequently high phase space dimension, whereas "chaos" refers to the unpredictablility in time of low effective phase space dimensional systems.

From these remarks it is clear that in principle there need not be a sharp boundary between chaos and turbulence, but rather a continuum in which the effective phase space dimension goes from small to large. In practice, this continuum is in fact generally observed: this is indicated in Fig. 1 in the context of an idealized Rayleigh-Benard convection experiment. As we shall see, the different regimes shown schematically in the figure are observed in a wide variety of actual experiments. In the figure the primary control parameter dictating the regime of a given experiment is the Rayleigh number (Ra), which represents the strength of the driving force. As Ra is increased, the system becomes increasingly more "turbulent".

Given the obvious complexity of turbulent motions, it is useful to begin with a very simple question: is real fluid turbulence, for example, "chaotic", in our technical sense? More precisely, do "turbulent" solutions to the Navier-Stokes equations exhibit sensitive dependence on initial conditions, positive Lyapunov exponents, and so forth? Although this

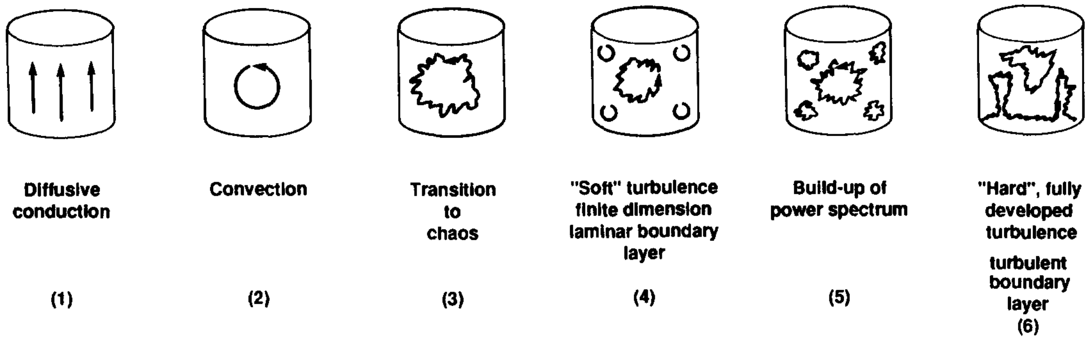

FIGURE 1

A schematic depiction of the different regimes of flow patterns observed in Rayleigh-Benard convection cells as the driving force (related to the Rayleigh number) increases from left to right. (After A. Libchaber, private communication).

question may seem naive and the answer obvious, it is important as a test for whether any of the techniques developed for chaos can actually be applied to turbulent flows: can we, for example, measure reliably the value of the Lyapunov exponents (λ) or the fractal dimension (d_f) of the strange attractor? One recent theoretical study[23] has addressed this question explicitly by studying numerically solutions to the three-dimensional Navier-Stokes equations in a (weakly) turbulent regime with Reynolds number about 13. The results show that the motion is indeed chaotic, exhibiting sensitive dependence on initial conditions with $\lambda \sim 2.7$ and an attractor dimension $d_f \sim 10$. Apart from providing explicit evidence that turbulence is indeed chaotic, this simulation also indicates that both the Lyapunov exponent and the fractal dimension increase fairly smoothly with Reynolds number. A second study[24], addressing the question in the context of real Couette-Taylor flow, found that for Reynolds numbers up to about 30 dimension of the strange attractor remained < 5. Thus we can indeed say that real fluid turbulence is chaotic and that, as Figure 1 suggests, the techniques familiar from the chaotic dynamics of low dimensional dynamical systems can be used to analyze this chaos, at least near the transition.

But what of more general flows at Reynolds numbers of 2000, let alone 10^6? Here another very recent theoretical investigation[25] provides insight into the difficulties of a direct "chaotic dynamics" approach to fully-developed turbulence. This study pushed the techniques of chaos far into the turbulent regime by simulating turbulent Poiseuille flow at a Reynolds number of 2800[25]. Based on calculations of the Lyapunov exponent spectrum, the results offer strong evidence that the "turbulent" solutions to the Navier-Stokes equations for this flow do indeed lie on a "stange attractor" and further that the dimension of the attractor is $d_f \sim 400$ (!). Obviously, although it is perhaps comforting to know that this flow is indeed "chaotic" in the technical sense, it is disconcerting to contemplate trying to analyze similar flows experimentally or to model them theoretically in terms of a dimension 400 dynamical system. So how *can* we confront the truly complex structure expected of turbulence in these high-phase-space-dimension regimes? To attack this daunting problem, it is best to try to break it up into less daunting pieces. This "divide and conquer" philosophy has proven very successful in studies of the transition to irregular motion, where the many different routes to chaos – period doubling, quasi-periodic, and intermittent – were understood *quantitatively* only when they were treated separately. In the case of turbulence, we expect that focusing on special, simplifying features of particular flows will again be the key to quantitative understanding. Let me describe how I believe this can be done, both at present and in the near future, first in experimental and then in theoretical studies.

From an experimental perspective, it is clear that one needs to extend to stronger driving and more fully developed turbulence the high precision studies that proved crucial in establishing "chaos". Clever techniques, such as those discussed in Section III above, will be essential to this effort, as will the choice of systems: binary fluid mixtures and cryogenic fluids in unusual geometries (such as annuli[12]) are good examples of what will be needed. Good control over the driving parameters, as well as the ability to vary them over a wide range, will be essential. The crucial role played by geometric constraints such as boundaries requires considerable further study: whereas in Rayleigh-Benard flow "chaos" can readily be observed in "small boxes"[9,12] with aspect ratios $\Gamma \sim 2$ in which geometry constrains the number of rolls, turbulence, with its attendant spatial disorder, arises in the same flow in large cells ($\Gamma \sim 10^{12}$). Experimental studies are also needed for "open flow" systems, in which there are essentially no boundary constaints. High quality, *spatially resolved* data – obtained via by now conventional techniques such as laser Doppler interferometry or more exotic methods such as NMR tomography[26] or low temperature scanning electron microscopy[15] – will be essential to identifying the nature of spatial structures and complexity in the turbulent motion.

Fortunately, as these proceedings testify, the experimentalists seem well equipped to meet these challenges. In fluids, experiments covering systematically a good fraction of the diagram in Fig. 1 were reported in helium[11], silicon oil[12], and binary mixtures[13,27]. In solid state, in both semi-conductors[15,20] and magnetic systems[28] the region beyond the transition to chaos has been thoroughly examined and initial steps toward studying spatial structures are underway[15]. Experiments involving lasers[14]) and chemical reactions[29] are also revealing further details of the regions beyond the transition to chaos.

Theoretically, studies focusing on specific aspects of turbulence in particular flows – again, our "divide and conquer" philosophy – have also shown progress. Over the past several years, two crucial general concepts have emerged as valuable guiding principles for these studies: "mode reduction" and "hierarchies of equations". Let me examine in detail how these concepts can be applied, both now and in future studies.

The central idea of "mode reduction" can be most easily visualized in fluid flow problems. In any given flow, not all of the (infinitely !) many possible modes are "active" – think of laminar flow, with the fluid moving en bloc as the extreme example – as that the *effective* phase space dimension is much smaller than the full (infinite) dimension of the equations. The examples of Couette-Taylor flow experiments ($d_f \sim 5$,[24]) and the numerical simulations of low-Reynolds number ($R \sim 13, d_f \sim 10$,[23]) and intermediate-Reynolds number ($R \sim 2800, d_f \sim 400$[25]) flows provide concrete realizations of this reduced effective phase space. "Mode reduction" involves (somehow!) reducing the number of degrees of freedom being modeled explicitly to the minimum necessary to capture the essential features – spatial patterns and energy, mass, and concentration transport, for example – of the motion. Intuitively, one hopes that the "modes" themselves can be related to any persistent spatial structures observed in the flow; I will discuss this point in detail in the next section. But even if this is not possible, mode reduction can provide crucial insights into, and a simplified description of, the flow.

Ideally, the process of mode reduction should be *deductive*, *controlled*, and *constructive*: that is, one would hope to be able to derive the equations governing the reduced modes, to bound the error made in the reduction, and to "construct" the actual modes themselves. In general, this remains an elusive goal, although substantial recent progress in this area has occurred. Historically, the process has more typically been either deductive and constructive but not controlled, *or* non-deductive and non-constructive (but sometimes remarkably accurate !). Examples of the former case include the celebrated Lorenz equations[30], which are a three-Fourier-mode truncation of the (Oberbeck-Boussinesq) equations applicable to a particular two- dimensional flow. In deriving the Lorenz equations, coupling terms generate Fourier modes not included in the truncation and these are simply ignored: hence the a priori uncontrolled nature of the reduction. Indeed, as further studies of related problems have established[31], as more Fourier modes are included, the detailed description – for example, the bifurcation sequence observed as a function of the driving parameters – depends crucially on the number of modes, so that here the uncontrolled nature of the approximation is a potentially serious drawback. An example of the latter case is the quantitative description of actual period doubling observed in experimental fluid flow[9] by a one-dimensional map[10]. Although (as yet) non-deductive, this link is made powerful by the universality of the period doubling phenomenon. More generally, the "geometry from a time series" (see, for example,[24]) reconstructions based on embedding theorems[32] give information about the dimension of the reduced system but not about the nature of the modes themselves.

Clearly, then, the challenge is to construct methods that are deductive, controlled, and constructive. In view of some substantial very recent progress, the future prospects of this approach are quite bright. Let me focus on three specific areas of great near-term potential. First, as exemplified in these proceedings[33], results are now available establishing rigorously that the long-time behavior of certain partial differential equations is controlled by a finite number of modes (for systems in finite geometry !). At present, the successes of the method for the Kuramoto-Sivashinsky equation for unstable chemical fronts[33] and for the (one-space dimensional) complex Ginzburg-Landau equation[34] describing certain situations near the onset of turbulence have depended on the existence of (increasing) gaps in the (discrete) spectrum of the (linear) dispersion relation to control the nonlinear effects and hence to produce the rigorous bounds. The important next step is to deal with models – involving, for example, hyperbolic, "wave-like" equations – in which these increasing gaps are not present; significantly, it appears that this step will be possible in the near future[35]. Second, the intuition that if one observes a particular flow, it should be possible to determine a "best" set of modes – spatial "basis vectors" to be used in a truncated Galerkin approximation – to describe that flow has recently been concretely realized in a study of coherent structures in the wall region of a turbulent boundary layer[36]. Interestingly, this approach involves a "singular value decomposition" similar to that successfully applied in these proceedings[18] to the analysis of a chaotic time series. Third, the vital question of the relation of mode reduction to the actual structures observed in space is rapidly becoming approachable. I will illustrate this in more detail in Section V.

The second useful guiding principle – the "hierarchy of equations" – is also most conveniently illustrated in the context of fluid flows, where the technique in various forms has long been established[37]. Starting from the Navier-Stokes equations, one derives for the specific type of flow under study an approximate (typically partial) differential equation (PDE). Among the desired features of this approximate equation are that it is simpler to solve than Navier-Stokes, and/or that it contains obvious coherent spatial structures, and/or that it has controllable temporal asymptotics. This approximate PDE may be further reduced – via rigorous "inertial manifold" techniques[33,34] or more heuristic methods – to a (finite) set of coupled ordinary differential equations (ODE's). Finally, in some cases this finite set of ODE's may fall into a "universality class" that permits further reduction to the simplest element of the class – for example, a period doubling transition to chaos being model by the logistic map.

Although this "hierarchy" approach is conceptually clear, implementing even pieces of it successfully remains more an art than a science. Future progress in this area will depend upon establishing systematic answers to two very broad questions:

(1) How does one derive the hierarchy of equations?

and (2) How can one "match" solutions between the various levels of the hierarchy?

Let me discuss briefly various aspects of these questions.

The first step in establishing the hierarchy of equations – deriving an approximate PDE for specific types of flows – has by now become fairly systematic[17,37]. Although the details naturally vary from case to case, two broad classes of approximate equations have emerged. The first, known as "amplitude equations" – of which the complex Ginzburg-Landau equation is an example[34] – are generally valid near the onset of a particular instability; in a sense, the amplitude equations at the non-equilibrium analogs[38] of the phenomenological equations describing phase transitions[39]. The second, known as "phase equations", apply in the case of slow spatial modulation of a system in which there is a dominant wave vector – call it k_0 – so that there exists some spatial periodicity. Such "phase equations" can be viewed as the non-equilibrium analogs[38] of hydrodynamics, which applies to the low-frequency, long-wavelength motions near equilibrium.

Of course, "phase" and "amplitude" equations can, in some cases, arise in combined forms. An example discussed in the present proceedings is a set of coupled equations describing the evolution of coherent structures in shear flows and having the form[40]

$$\frac{\partial A}{\partial t} = \mu A - |A|^2 A + \left(\frac{\partial^2}{\partial x^2} + \gamma \frac{\partial^2}{\partial z^2}\right) A - i \frac{\partial \psi}{\partial z} A \tag{1.a}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\psi = \sigma \frac{\partial^2}{\partial x \partial z}(|A|^2) \tag{1.b}$$

For a full explanation of the interpretation, see Ref. (40). Our purpose in writing these equations explicitly is in part to emphasize that the approximate PDE's need not *appear* simpler than the original Navier-Stokes equations to represent in fact a simplification.

The second step in establishing a hierarchy of equations – deriving a finite dimensional, coupled ODE description of the approximate PDE's – is (partially) illustrated in these proceedings in the case of the Kuramoto-Sivashinsky equation[33] and has also been (partially) carried out in the case of the complex Ginzburg-Landau equation[34]. I stress "partially" carried out because although the existence of a finite dimensional inertial manifold has been proven, the explicit equations governing the motion on it have not been established.

The final possible step in the hierarchy of equations – reduction of the coupled set of ODE's, via universality, to the simplest example of a given universality class – typically can be expected to hold only for certain limited parameter ranges and thus remains very problem specific. Actually, *deriving* such reductions – *predicting*, for example, for which parameter ranges the damped, driven pendulum will exhibit a period-doubling transition to chaos[41] – remains an open problem.

In general, the issue of "matching" solutions between the various levels in the hierarchy represents *terra incognita* and as such is clearly an important area for future work. An example[42] will suffice to indicate both the difficulty and the importance of this "matching" problem. Suppose, for example, one level of approximate equations supports spatial structures which become singular in finite time: a familiar case is the two-space dimensional (cubic) nonlinear Schroedinger equation (NLSE), for which the localized, "self-focusing" solutions – the analogs of the solitons in the one dimensional NLSE – do blow up in finite time. As these solutions collapse, they transport their energy from large spatial scales (small k) to very small spatial scales (large k). Clearly understanding a single, identifiable – and possibly "isolatable" – nonlinear mode that cuts across a range of spatial scales is of considerable importance for general turbulence modeling. However, as the solution becomes singular, one expects that physical effects not incorporated into the approximate model will enter to prevent this singularity. To follow the flow beyond this point, one must "match" onto the (higher level) equations incorporating these additional physical effects. Thus the "matching" problem across the hierarchy can be absolutely critical to the whole approach.

A concrete, albeit qualitative, example of an attempt to understand turbulence via this hierarchy of equations and matching approach is the "turbulence engine" suggested by Yakhot and Orszag in these proceedings[43]. A three-dimensional, turbulent, nearly Beltrami ($\vec{\omega} \times \vec{v} \sim 0$) in which energy is transported from small k to large k becomes unstable via a hyperscale instability and leads (with appropriate matching assumed) to a quasi-two-dimensional flow. Here a rapid energy build-up from large k to small k occurs, and a secondary instability leads back (via another matching problem) to the three-dimensional nearly Beltrami flow, and the cycle repeats. The central role that detailed matching would play in the quantification of this scheme is obvious.

Finally, in searching for more insight into the relation of chaos and turbulence, one must always be attuned to novel approximations or calculational techniques. Recent examples, many of which were discussed at the meeting, include renormalization group methods[43], "cellular automata"/"lattice gas" models for hydrodynamics[44], the use of (real space) lattices of point vortices[45], and "vortex tube" dynamics[46].

V. CHAOS AND COHERENT STRUCTURES: PATTERN SELECTION

As I have already suggested briefly in the introduction, "solitons", or more generally "coherent structures" – that is, spatially localized, long-lived nonlinear excitations – are a striking but remarkably common feature of spatially extended nonlinear systems. These coherent structures can dominant the long-time behavior and are thus, in a sense, the natural "modes" of the system. It is therefore equally natural to ask how these coherent structures can interact with the chaos that we now know is also a common feature of nonlinearity[47]. This question looms as even more important when one recognizes the possible connection between coherent structures, viewed as spatial nonlinear "modes", and the process of "mode reduction", described in Section IV as a key ingredient in the potential relation between chaos and turbulence. Explicitly, to what extent can one identify the coherent structures with the spatial modes described by the reduced equations ?

Considerable progress has recently been made in answering this question, but in view of its broad scope it nonetheless promises to remain one of the most important and exciting challenges in studies of chaos for years to come. One clear line of attack is to start with a system that has exact solitons – the sine-Gordon or nonlinear Schroedinger equations, or the Toda lattice, for example – , perturb the system by damping and driving, and then look for evidence of chaos. In the case of the sine-Gordon equation, this approach has been investigated extensively[48] and a very rich phenomenology has been developed. This general approach, of course, is intended to work in a regime of "strong" nonlinearity in the sense that it requires the "soliton" modes to be the dominant coherent structures.

An alternative approach, which in a sense starts from the weakly nonlinear limit, is aimed at systems where some mechanism causes very strong growth in some linear – typically Fourier – modes while damping others. A clear example discussed in these proceedings is the effective "negative viscosity" in systems like the Kuramoto-Sivashinsky equation[33]. As noted above, here one can establish that even for the *partial* differential equation (in a finite box !), only a *finite* number of modes are necessary to capture the long-time dynamics. Further, in many regions of the parameters, interesting bifurcation sequences leading to chaotic motions arise[33]. An essential subtlety, however, is that the modes that diagonalize the asymptotic "inertial manifold"[33] are not quite the Fourier modes themselves, so the precise nature of the coherent structures is not immediately known; this is in clear constrast to the "soliton" case discussed above. Nonetheless, particularly in view of the recent success in extending this approach to other systems – for example, the complex Ginzburg-Landau equation[34] – this method promises to be very important in future studies of the interaction of chaos and coherent structures.

Between these two limits there are numerous examples of coherent structures in nonlinear systems, many of which are treated in these proceedings. The convection rolls in fluid experiments[12,13,27,44,50] come immediately to mind, as do the vortices[45] and the irregular structures in turbulent boundary layers[36]. Even singular solutions can play a role as (transient) coherent structures; examples include the collapsing "cavitons" of the two-dimensional nonlinear Schroedinger equation discussed in Section IV and the (near singular) vortex filaments that emerge in the numerical simulations of the Red Spot of Jupiter[45]. A less visually apparent possible coherent structure is the "helicon" – an object obtained by averaging over bands in Fourier space with $\vec{v}$ parallel to $\vec{\omega}$ (the vorticity) – discussed by Orszag[43]. Incidentally, as this last example suggests, if you are lucky enough to discover a novel coherent structure, history shows that you should pay special attention to naming it properly. The celebrated "solitons", discovered by Martin Kruskal and Norman Zabusky[51] in their asymptotic and numerical studies of the Korteweg-de Vries equation, had actually been "seen" –in the form of localized, nonlinear, kink-like waves that preserved their form exactly under collisions – in earlier *analytic* studies of the sine-Gordon equation[52]! But they were not properly named!

In all these examples, the central issue is to identify a dominant coherent structure and then to understand its role in the chaotic or turbulent motion, first qualitatively and then quantitatively. In general, this last step is a wide-open problem and represents one area of great importance for future research. Among the specific issues that come to mind are

- What is the nature of the interactions among the coherent structures? Can one write effective equations that capture the essence of these interactions? Clearly, these questions are a restatement of the problem of "hierarchies of equations" discussed above; the reappearance of these questions here correctly reflects the crucial importance of coherent structures for understanding the relation between chaos and turbulence.

- To what extent can one expect "superposition" of these coherent structures to apply, and in what "space"? The expansions of the velocity field in a fluid into a sequence of Beltrami components[43] and of a solution to the damped, driven sine-Gordon equation into "inverse spectral transform" modes[48] – "breathers", "kinks", and "radiation" – illustrate that superposition may be valid, even if not naively in real space.

- Can one establish the "long-time" dominance of coherent structures and, if so, on what time scales? Here one must recognize the distinction between rigorous asymptotic results – as for the Kuramoto-Sivashinsky equation[33] – and the (often physically more relevant) dominance over long, but not infinite, times.

In a large aspect ratio finite system, or more generally in an open flow system, the number of coherent structures can be very large. Further, these structures, particularly when driven by chaotic dynamics, can arrange themselves in a bewildering array of patterns. A very clear visual example of this is provided by Fig. 2 of reference[12], which shows just a few of the many patterns observed in experiments on Rayleigh-Benard convection in a high Prandtl number fluid. The resulting "pattern selection" problem is one that exists in virtually all spatially extended, chaotic nonlinear systems. Although exciting preliminary results are beginning to appear[40,53], this problem promises to occupy both theoretical and experimental "chaoticians" for years to come, for it is one of the central problems of nonlinear, nonequilibrium systems. Here, some obvious but difficult specific questions include

- To what extent can one view the static patterns as multiple (perhaps nearly degenerate) minima in a (vague) "accessibility" space? Is there an analog, at least for some of these nonequilibrium systems, of the "free energy" functional for equilibrium systems ?
- What is the dynamics of the competition among patterns[40,53]? Can one define basins or attraction for the "minima" corresponding to the patterns ? Do these basins have the complicated (fractal) boundaries observed in simpler finite dimensional dynamical systems[54]? Can one in some cases find a Lyapunov functional that allows determination of the "winning" pattern?.
- In systems with constrained geometry (see, for example,[12], can one understand quantitatively the observed selection of more symmetric patterns over less symmetric ones? Here the analogy to "pinning" phenomena in commensurate/incommensurate solid state transitions should be useful[55].

Although I have thus far focused my remarks on fluid problems, it is important to note the broad impact of the pattern selection question. In these proceedings the problem of dendritic growth[56,57,58] was shown to depend critically on a pattern selection mechanism that is exquisitely sensitive to anisotropy and to extrinsic noise[59]. Further, in solid state physics applications, an important issue not yet fully resolved is that of the nature of the *communication* between patterns. Since here one does not have the Navier-Stokes equations to rely on as a macroscopic model, one cannot simply naively assume diffusive coupling but must look closely at various competing mechanisms[15,60].

VI. CHAOS AND STATISTICAL MECHANICS

For over a century, successive generations of mathematicians and physicists have sought a rigorous, or at least fully compelling, answer to the fundamental question of statistical mechanics: how does one go from a reversible, Hamiltonian microscopic dynamics to the irreversible, "ergodic", "thermal equilibrium" motion assumed in statistical mechanics[61,62]? The chaos revolution, with its sharp quantification of deterministic chaos and its predictions of "sensitive dependence on initial conditions" and long time behavior "as random as a coin toss", clearly offers a renewed prospect of answering this question. However, despite this strong intuitive connection, a full understanding of the relation between chaos and statistical mechanics is not yet at hand. In this section I will discuss briefly several issues related to this problem and try to indicate why it remains one of the most subtle and profound challenges in studies of chaos.

Let me start with a somewhat more focused version of the general question being addressed: "What is the minimum necessary set of conditions/circumstances/features for a large (but not necessarily infinite) degree of freedom system described by a Hamiltonian involving *realistic* interactions among the sub-systems to exhibit (at least effectively) the features we associate with statistical mechanics, for example, time averages = ensemble averages ?"

To confront this question, we must first shift our focus from dissipative dynamical systems – to which the bulk of this overview has been devoted – to conservative (indeed Hamiltonian) ones. This has a number of important (and generally complicating) consequences. First, and most importantly, for most (finite dimensional) Hamiltonian systems, the KAM theorem[63] shows that, in addition to "chaotic" regions of phase space, there are non-chaotic regions of finite measure. These invariant tori imply that ergodicity does not hold for most finite dimensional Hamiltonian sytems. Second, the (few) Hamiltonian systems for which the KAM theorem does not apply, and for which one can prove[64] ergodicity and the approach to thermal equilibrium, involve "hard spheres" and consequently contain interactions which are not "realistic" from a physicist's perspective. Third, because of the absence of "attractors", the numerical simulations needed to stimulate intuition are often more subtle[19] than in the dissipative case. Further, when one does perform such experiments – as in the celebrated Fermi-Pasta-Ulam problem[65] – the dominant feature, despite non-integrability and the presence of chaos, can be "solitons" and surpising recurrences of the initial state[51]! Finally, in finite dimensional systems it is in general possible for the paths in phase space to wander among the chaotic regions – and hence to explore "most" of the energy surface – via "Arnold diffusion"[63,66]. But calculating the resulting "Arnold web" is a very difficult and subtle task even in simple systems.

For many years, most researchers have believed that these subtleties become irrelevant in the "thermodynamic limit" in which the number of degrees of freedom (N) and the energy (E) go to infinity such that $e = E/N$ remains (a non-zero) constant. For instance, the KAM regions of invariant tori may approach zero measure in this limit. However, recent evidence suggests[67] that there may exist (non-trivial) counterexamples to this result. Given the increasing sophistication of our analytic understanding of chaotic dynamics and the growing ability to simulate numerically systems with large N, the time seems ripe for quantitative investigations that can establish (or disprove !) this belief. Among the specific issue which should be addressed in a variety of physically "realistic" models are

- How does the measure of phase space occupied by KAM tori depend on N? Is there a class of models with realistic interactions for which this measure goes to 0? Are there (non-integrable) models for which a finite measure is retained by the KAM regions? If so, what are the characteristics that cause this behavior?

- How does the rate of Arnold diffusion depend on N in a broad class of models? What is the structure of important features – such as the Arnold web – in the phase space as N approaches infinity?

- If there is an approach to equilibrium, how does the time-scale for this approach depend on N? Is it less than the age of the universe?

Currently there is considerable work in this area, and many results are beginning to appear[67,68,69]. Limitations of space and the detail required to make precise the various assumptions preclude fuller discussion here. But clearly this is an exciting, and profound, area for future research in chaos.

To render these fairly abstract considerations somewhat more concrete, let me present a specific example that nicely illustrates the subtlety of the integrability/chaos/KAM issues in models of physical systems with many degrees of freedom. This example involves "effects beyond all orders" in perturbation theory, and apart from its relevance to the relation between chaos and statistical mechanics, is particularly appealing because of its strong conceptual connection to several other topics discussed in these proceedings. The general problem concerns a comparison of the type of "coherent structures" possible in integrable versus non-integrable "classical field theories", which in mathematical terms are just nonlinear PDE's and hence model dynamical systems with an infinite number of degrees of freedom. The specific question posed (and answered (!)) in an important recent study[70] is, "Does the (non-integrable) ϕ^4 theory possess a "breather" solution analogous to that found in the (integrable) sine-Gordon theory?" Recall that the sine-Gordon "breather" is a spatially (exponentially) localized, time-periodic nonlinear coherent structure having the explicit analytic form

$$u_B(x,t) = 4tan^{-1}\left(\frac{\varepsilon sin(t/\sqrt{1+\varepsilon^2})}{cosh\varepsilon x/\sqrt{1+\varepsilon^2}}\right) \tag{2.a}$$

$$\simeq 4\varepsilon sech\varepsilon x sin(t(1-\varepsilon^2/2)) + \cdots \tag{2.b}$$

where the approximate equality holds for small ε. By direct differentiation, one can verify that eqn. (2.a) represents an exact solution to the (unperturbed) sine-Gordon equation,

$$u_{tt} - u_{xx} + sinu = 0 \tag{3}$$

As we indicated earlier, the role of the breathers in the temporal chaos and spatial complexity of the damped, driven sine-Gordon equation has been extensively studied recently[48].

At issue in[70] is whether the ϕ^4 equation[71]

$$\phi_{tt} - \phi_{xx} - \phi + \phi^3 = 0 \tag{4}$$

also can support a breather solution. Since ϕ^4 is known to be non-integrable[72], one does not expect to find a closed-form analytic expression for such breathers. However, using multiple scale techniques[73,74] similar to those used to derive the "amplitude equations" discussed in Section IV[17,37], one can search for breathers in terms of an (asymptotic) perturbation expansion in the amplitude of the breather. Explicitly, one defines $\phi_B = 1 + z_B$ and expands z_B as

$$z_B \simeq \varepsilon[A(X,T)e^{i\sqrt{2}t} + c.c.] + 0(\varepsilon^2) \tag{5}$$

Here X and T represent scaled variables X=ϵ x and T = $\varepsilon^2 t$. The resulting "amplitude equation" for A(X,T) is the celebrated nonlinear Schroedinger equation (NLSE)

$$i2\sqrt{2}\frac{\partial A}{\partial T} - \frac{\partial^2 A}{\partial X^2} - 12|A|^2 A = 0 \tag{6}$$

which admits (exponentially) localized solutions – the solitons – for A. Specifically, one finds

$$A(X,T) = \sqrt{\frac{1}{6}} sechX e^{-\frac{iT}{2\sqrt{2}}} \tag{7}$$

so that (to this order in ε) z_B (x,t) does represent a breather solution in the sense defined above. Importantly, if one applies the same multiple scale technique to the sine-Gordon equation, one again obtains the NLSE (with trivially different coefficients) as the leading order amplitude equation; one immediate indication of this is the form of the small ε expansion of u_B in eqn. (2.b). Further, carrying out the expansion to all orders in ε, one finds that both ϕ_B and u_B remain localized in space and periodic in time; at no order in ε does one see a distinction between the "breather" solutions to the integrable versus the non-integrable case.

And yet there is a difference. Using asymptotic matching techniques, one can show[70] that in the ϕ^4 theory there are "effects beyond all orders" in ε, so that the true solution to which z_B is asymptotic has the form

$$z_{True} = z_B + c_1 e^{-c_2/\epsilon} cos(k_c x - \omega_c t) + \cdots \tag{8}$$

where $\omega_c = 2\sqrt{2}(1-\varepsilon^2/2)$, $k_c = \sqrt{\omega_c^2 - 2}$, and c_1 and c_2 are two (non-zero) constants. Hence z_{true} is not localized in space and in the ϕ^4 theory there is in fact no true "breather" solution. The "miracle" of the integrable sine-Gordon case – for which Eq. (2.a) represents an exact, closed-form solution – is that when similar "beyond all order" techniques are applied to it, the constant analogous to C_1 vanishes identically (as do all higher order non-localized terms).

Why is this illustration relevant to our discussion of chaos and statistical mechanics and, more generally, of chaos itself? There are several reasons. First, it illustrates the subtlety of the integrable/non-integrable(and hence potentially chaotic) distinction in the many-degree-of-freeom systems relevant to statistical mechanics (and the thermodynamic limit). Second, very recently a conceptually related "beyond all orders" analysis[75] has shown rigorously that for a certain driven one-degree of freedom Hamiltonian system, the splitting of separatrices is also of order $e^{-\frac{1}{\epsilon}}$. Among the implications of this result are that similar "beyond all orders" splittings should occur in separatrices in KAM theory and in the unfolding of degenerate singularities in certain cases involving the interaction of bifurcations. Third, in Section V, I indicated that the problem of dendritic growth[56,57,58] represents a spatial analogoue of the "sensitive dependence on initial conditions" characteristic of deterministic chaos. In fact, as shown in these proceedings[57] and elsewhere[76], "beyond all orders" effects occur in these systems and can play an essential role in the growth of dendrites. Fourth, in view of the similarity of the derivation of Eq. (6) to the standard amplitude equations discussed in Section IV, one must always be aware of the possibility that similar beyond all order effects could be present. Here, however, the distinction between conservative and dissipative systems may be crucial. Finally, studies of the role of "coherent structures" in chaotic nonlinear systems must take account of the (very long but finite) life times induced by these $O(e^{-\frac{1}{\epsilon}})$ effects. In sum, the phenomena of effects "beyond all orders in perturbation theory" and the associated techniques are subjects about which students of chaos should be knowledgeable, for they promise to play an essential role in many future problems related to chaos.

VII. CHAOS AND QUANTUM MECHANICS

Many would argue that the less said about the often controversial subject of "quantum chaos" the better. I do not agree, and I believe that anyone who studies the articles describing a quantum version[77] of the classical stadium problem, suggesting a novel "quantum chaotic" underpinning for resistivity in solids[78], and reporting on the chaotic ionization of hydrogen atoms[79] will discover how subtle yet exciting this subject can be. It is essential, however, to proceed from a clear starting point. To this end, I offer for consideration an unambiguous, albeit obvious, definition of "quantum chaos": namely, "quantum chaos" is the study of the quantum mechanics of those systems whose classical Hamiltonians allow chaotic motion. A good example in this spirit of tying quantum and classical chaos directly together is the discussion of the role of unstable periodic (classical) trajectories both in "scarring" the quantum wave functions (or probability densities)[77] and in analyzing the structure of the strange sets in purely classical problems[6].

VIII. CONCLUSIONS: CHAOS: CHTO DELAT?

From the extensive lists of open problems discussed in the previous sections, it is clear that the brief answer to my question "Chaos: Chto Delat?" is simply "Much !". In a broad array of subjects, ranging from fluid turbulence to semiconductors and from stability of the solar system to multi-fractal strange sets, an exciting set of challenges awaits future researchers. Further, since the organizers of CHAOS '87 correctly chose to focus primarily on topics from mathematics and the natural sciences – areas in which the impact of chaos has most strongly been felt – a number of other subjects in which chaos may play a role could be mentioned only in passing, if at all. In particular, there were no detailed discussions of the implications of chaos in, for example, engineering or medicine[80], let alone economics or the social sciences. As the field of deterministic chaos matures, recognition of its implications will continue to spread to other disciplines, opening up still further questions that we can at present hardly anticipate. In this sense, the chaos revolution is only beginning, and we can all look forward with enthusiasm to a continuing role as revolutionaries.

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