

Dynamics of entanglement in a dissipative Bose-Hubbard dimerTadeusz Pudlik,¹ Holger Hennig,² D. Witthaut,³ and David K. Campbell^{1,*}¹*Department of Physics, Boston University, Boston, Massachusetts 02215, USA*²*Department of Physics, Harvard University, Cambridge, Massachusetts 02138, USA*³*Network Dynamics, Max Planck Institute for Dynamics and Self-Organization (MPIDS), 37077 Göttingen, Germany*

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We study the connection between the semiclassical phase space of the Bose-Hubbard dimer and inherently quantum phenomena in this model, such as entanglement and dissipation-induced coherence. Near the semiclassical self-trapping fixed points, the dynamics of Einstein-Podolski-Rosen (EPR) entanglement and condensate fraction consists of beats among just three eigenstates. Since persistent EPR entangled states arise only in the neighborhood of these fixed points, our analysis explains essentially all of the entanglement dynamics in the system. We derive accurate analytical approximations by expanding about the strong-coupling limit; surprisingly, their realm of validity is nearly the entire parameter space for which the self-trapping fixed points exist. Finally, we show significant enhancement of entanglement can be produced by applying localized dissipation.

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I. INTRODUCTION

Ultracold gases in optical traps bring unprecedented control and resolution to the study of quantum systems [1–4]. Their prospective applications range from simulation of solid-state phenomena [5–7] to quantum metrology [8] to quantum information processing [9], but realizing their potential requires a thorough understanding of quantum coherence. An opportune system for the study of coherence is a Bose-Einstein condensate loaded into a double-well optical trap, known as a “BEC dimer” or “bosonic Josephson junction”: it is both amenable to theoretical analysis and realizable in current experiments. Highlights of past work on the BEC dimer include the demonstrations of matter-wave interferometry [10], number squeezing [11–15], and measurements transcending the standard quantum limit [16,17], as well as applications such as gravity detectors [18] and noise thermometers [19].

The macroscopic dynamics of the BEC dimer is well described by a semiclassical mean-field model. The mean-field dynamics, including the emergence of self-trapping fixed points in bifurcations, has been studied both theoretically [20–23] and experimentally [24,25]. These works imply a connection between the structure of the classical phase space and truly quantum phenomena such as entanglement, a connection studied in greater detail recently [26].

Decoherence and dissipation due to interactions with the environment are major obstacles to long-time control of the coherence of quantum systems [27]. However, it was recently demonstrated that dissipation can be a versatile tool for the manipulation of quantum systems, if it can be carefully controlled [28–30]. In a BEC dimer, interactions with a thermal bath or the escape of atoms from the trap may strengthen rather than destroy coherence [31,32]. Work has now begun on analyzing how the effects of noise and phase-space structure interact [33].

In this paper, we use the global phase-space picture [26] to offer insight into the dynamics of entanglement and

dissipation-induced coherence in the double-well optical trap. We show that the wells are entangled in the Einstein-Podolski-Rosen (EPR) sense only in the neighborhood of the classical fixed points. The time dependence of the entanglement can be entirely explained in terms of beats among three eigenstates of the system. These eigenstates can only be found numerically, but we derive analytical results in the limit of weak coupling between the wells. A perturbative expansion in this limit produces excellent agreement with numerically exact calculations. The dynamics of other observables, such as well population imbalance or condensate fraction, can be understood in the same framework.

These simple patterns in the dimer’s behavior not only are interesting in their own right but also suggest a new approach to understanding the less numerically tractable behavior of BECs in multiwell optical lattices. The self-trapping fixed points of the dimer are analogous to discrete breathers in larger systems. We hope to pursue this connection in future work.

II. ENTANGLEMENT AND COHERENCE IN THE BOSE-HUBBARD DIMER

Consider a collection of N bosonic atoms in a double-well optical trap sufficiently deep that only the lowest state in each well is populated. In this so-called two-mode approximation, the atoms’ dynamics is described by the Bose-Hubbard Hamiltonian [34],

$$\hat{H} = -J(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1) + \frac{U}{2}[\hat{n}_1(\hat{n}_1 - 1) + \hat{n}_2(\hat{n}_2 - 1)], \quad (1)$$

where \hat{a}_i is the annihilation operator for an atom in well i and $\hat{n}_i = \hat{a}_i^\dagger \hat{a}_i$ is the number operator. The same Hamiltonian can be realized in related systems, such as two spin states of atoms in a single optical well [25]. It is also mathematically equivalent to the Lipkin-Meshkov-Glick model [35–38].

The macroscopic dynamics of the BEC dimer is well described by a mean-field approximation. This approximation implicitly assumes that the atoms remain at all times in a

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product state,

$$|z, \phi\rangle = \frac{1}{\sqrt{N}} [\sqrt{(1+z)/2} \hat{a}_1^\dagger + \sqrt{(1-z)/2} e^{i\phi} \hat{a}_2^\dagger]^N |0\rangle, \quad (2)$$

where z is the population imbalance and ϕ is the relative phase of the two modes. For general quantum states, these observables are defined by $z = (\langle \hat{n}_1 \rangle - \langle \hat{n}_2 \rangle) / (\langle \hat{n}_1 \rangle + \langle \hat{n}_2 \rangle)$ and $\langle \hat{a}_1^\dagger \hat{a}_2 \rangle = \|\langle \hat{a}_1^\dagger \hat{a}_2 \rangle\| e^{i\phi}$. In the mean-field approximation, the system is described solely in terms of z and ϕ , and the evolution of these variables is determined by the classical Hamiltonian,

$$H_{\text{MF}} = \frac{\Lambda z^2}{2} - \sqrt{1-z^2} \cos(\phi), \quad (3)$$

where the parameter $\Lambda = U(N-1)/2J$ captures the strength of the repulsive interaction between the bosons [39]. The trajectories $(z(t), \phi(t))$ are given by the contours of constant “energy” $H_{\text{MF}} = \text{const}$, shown in Fig. 1(a). For $\Lambda < 1$, the model has two stable fixed points, at $(z, \phi) = (0, 0)$ and $(0, \pi)$. As Λ is increased above 1, a supercritical pitchfork bifurcation takes place, and the stable fixed point at $\phi = \pi$ is replaced with

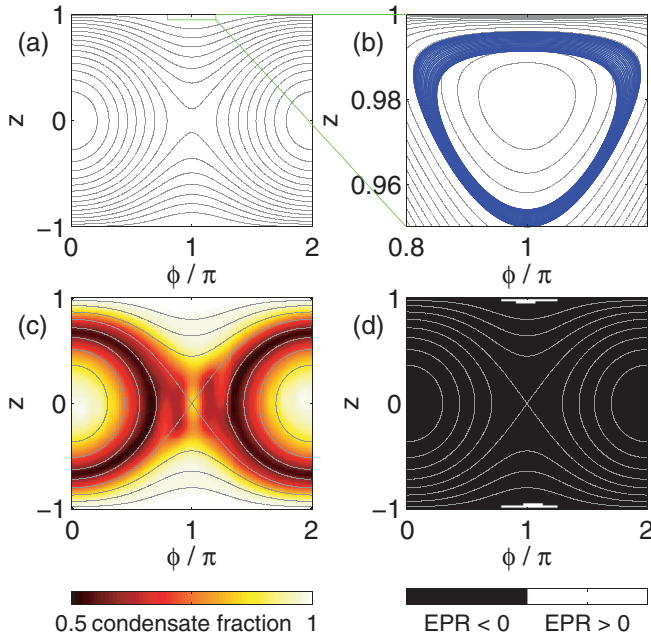


FIG. 1. (Color online) The global phase-space picture of the BEC dimer. (a) Mean-field trajectories. (b) The expectation values of z and ϕ over time (phase-space trajectory) for an initially coherent state close to the stable fixed point. The phase trajectory is drawn in blue (dark gray); nearby mean-field trajectories are in light gray. Note that the actual trajectory has a “thickness” associated with it, a phenomenon beyond the mean-field description [40]. (c) The condensate fraction after 1 s of evolution for initially coherent states uniformly sampled in z and ϕ : condensate fraction remains high for initial conditions in the neighborhood of the stable fixed points. (d) EPR entanglement for initially coherent states after 1 s of evolution: the only states still EPR entangled are those initially very near the self-trapping fixed points. All plots are for $N = 40$, $J = 10\hbar/s$, and $U = 100/39 \approx 2.6\hbar/s$, so $\Lambda = 5$.

a pair of stable fixed points at $z \neq 0$, $\phi = \pi$ and an unstable fixed point at $(z, \phi) = (0, \pi)$.

In a global phase-space picture we analyze the quantum dynamics of an initially pure BEC $|z, \phi\rangle$ as a function of the starting position (z, ϕ) . The condensate fraction or purity, defined as the largest eigenvalue of the single-particle density matrix,

$$\rho = \begin{pmatrix} \langle \hat{a}_1^\dagger \hat{a}_1 \rangle & \langle \hat{a}_1^\dagger \hat{a}_2 \rangle \\ \langle \hat{a}_2^\dagger \hat{a}_1 \rangle & \langle \hat{a}_2^\dagger \hat{a}_2 \rangle \end{pmatrix},$$

measures how close the many-body state is to a pure BEC [32, 41]. The EPR entanglement, an important resource in quantum metrology, is quantified by the observable [42,43],

$$E = \langle \hat{a}_1^\dagger \hat{a}_2 \rangle \langle \hat{a}_2^\dagger \hat{a}_1 \rangle - \langle \hat{a}_1^\dagger \hat{a}_1 \hat{a}_2^\dagger \hat{a}_2 \rangle. \quad (4)$$

The wells are said to be EPR-entangled whenever $E > 0$. The global phase-space picture [26] shows that the condensate fraction remains large near all of the stable fixed points, while EPR entanglement is found only near the $z \neq 0$ fixed points (see Fig. 1).

The global phase-space picture suggests a new method for generating EPR entanglement in the Bose-Hubbard dimer: driving the system closer to the mean-field fixed points using controlled atom loss. Since the mean-field dynamics is particularly simple near the fixed points, one might hope the full quantum dynamics is simple as well, allowing for a clear yet quantitative understanding. To develop such an understanding, in Sec. III we describe the full quantum dynamics of the Bose-Hubbard dimer near the mean-field fixed points, and in Sec. IV we consider the effects of controlled atom loss on this dynamics.

III. DYNAMICS NEAR THE SELF-TRAPPED FIXED POINTS

Let us consider the behavior of the system near the so-called self-trapping fixed points, located at $z = \pm\sqrt{1-1/\Lambda^2}$, $\phi = \pi$. In their neighborhood the observables defined in the previous section exhibit peculiar dynamics, the most striking feature of which is the presence of two distinct frequencies (see Fig. 2).

The higher frequency is expected on the basis of the mean-field model [20]. Linearizing the equations of motion obtained from the Hamiltonian of Eq. (1) about the fixed point yields

$$f_{\text{MF}} = \frac{\sqrt{\Lambda^2 - 1} J}{\pi \hbar}. \quad (5)$$

The mean-field prediction works for a broad range of Λ (see Fig. 3). The lower frequency, however, cannot be explained within the mean-field approximation. To see this, consider the trajectory of the system in z, ϕ space [see Fig. 1(b)]. In the mean-field picture, this trajectory is expected to coincide with the energy contours of H_{MF} . However, simulation of the full quantum dynamics reveals a “thick” orbit, the size of which oscillates with the low frequency [44]. (Similar low-frequency phenomena were noted before [34] but were not discussed quantitatively.)

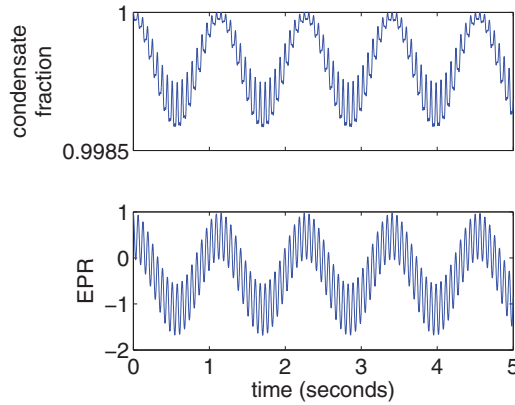


FIG. 2. (Color online) The condensate fraction and EPR entanglement over time for an initially coherent ($z = 0.95$, $\phi = \pi$) state of the Bose-Hubbard dimer with $N = 40$, $J = 10\hbar/s$, and $U = 100/39 \approx 2.6\hbar/s$, so $\Lambda = 5$. The stable fixed point is at $z = 2\sqrt{6}/5 \approx 0.98$, $\phi = \pi$. Results are obtained by numerical integration of the Schrödinger equation.

A. Eigenstate decomposition

To explain the low-frequency oscillations, let us decompose the evolving quantum state into the energy eigenstates:

$$|\psi(t)\rangle = \sum_{n=0}^N a_n e^{-iE_n t/\hbar} |E_n\rangle. \quad (6)$$

At first glance, this decomposition does not offer much insight, as the Hamiltonian has a large number of eigenstates and their energies can only be found numerically. However, in the neighborhood of the system's fixed points only a few states contribute appreciably to the wave function (see Fig. 4). This is not entirely surprising: the stable fixed points are the extrema of the mean-field energy, so in the neighborhood of these points only the eigenstates with most nearly extremal energy values should contribute to the coherent state. Indeed, for the $z = 0.95, \phi = \pi$ coherent state of Fig. 2, we find the contributions of the three highest-energy eigenstates

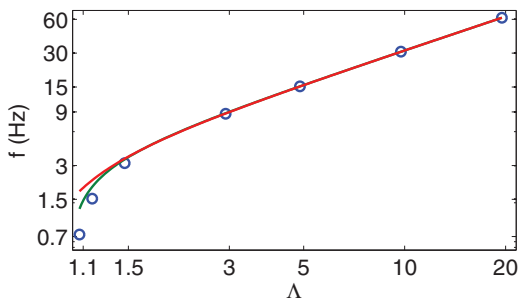


FIG. 3. (Color online) The high frequency observed near the fixed point for different values of Λ (blue circles) on a log-log scale. The mean-field prediction [green (light gray) line] is consistent with the numerically exact results. The $O(\Lambda^{-2})$ perturbative result [shown in red (medium gray)] agrees with the mean-field down to $\Lambda = 1.5$, below which it overestimates the frequency; see Sec. III B for a discussion.

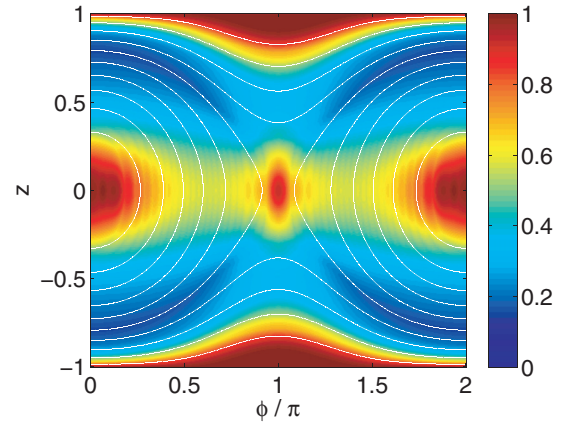


FIG. 4. (Color online) A projection onto just three eigenstates is effective near the mean-field fixed points. Consider a coherent state $|z, \phi\rangle = \sum_{n=0}^N a_n |E_n\rangle$ and its projection $|\psi'\rangle = \sum_{n=0}^2 a_n |E_n\rangle$ onto the three energy eigenstates with the largest coefficients a_n in the energy eigenstate expansion. The plot above shows the norm $|\langle\psi'|\psi'\rangle|^2$ of this projection as a function of z and ϕ . Note that the norm of the projection is nearly 1 (perfect) near all of the fixed points, including the unstable one, suggesting the three-eigenstate description will be informative for those initial conditions. The system parameters are the same as those in Fig. 2, namely, $J = 10\hbar/s$, $\Lambda = 5$, and $N = 40$. The mean-field stable fixed points are at $z = 0, \phi = 0$ and at $z = \pm 0.98, \phi = \pi$, while the unstable fixed point is at $z = 0, \phi = \pi$. Contours of constant mean-field energy are shown in white.

to be

$$\begin{aligned} a_0 &= 0.9353, & E_0 &= 2040\hbar/s, \\ a_1 &= 0.3474, & E_1 &= 1942\hbar/s, \\ a_2 &= 0.0653, & E_2 &= 1850\hbar/s. \end{aligned}$$

These three eigenstates together account for

$$|a_0|^2 + |a_1|^2 + |a_2|^2 = 0.9997$$

of the probability weight of the coherent state. We might therefore expect the frequencies observed in the data to be beats between the eigenstates,

$$\begin{aligned} (E_0 - E_1)/2\pi &= 15.56 \text{ Hz} \equiv f_{\text{fast}}, \\ (E_1 - E_2)/2\pi &= 14.64 \text{ Hz}, \\ (E_0 - E_2)/2\pi &= 30.23 \text{ Hz}, \end{aligned}$$

or perhaps higher-order beats, such as

$$\frac{E_0 - E_1}{2\pi} - \frac{E_1 - E_2}{2\pi} = 0.8805 \text{ Hz} \equiv f_{\text{slow}}.$$

These expectations are borne out: 15.6 Hz is the mean-field frequency given by Eq. (5), while 0.85(5) Hz is the measured frequency of the large-amplitude oscillation in Fig. 2. The two other beats are also seen in the power spectrum of the condensate fraction [at 14.65(5) and 30.27(5) Hz], although not in that of EPR [45]. The projection onto the three most important eigenstates recovers not only the frequencies but essentially all of the observables' dynamics (see Fig. 5). A projection onto just two states is sufficient to recover the mean-field motion but not the low-frequency oscillations.

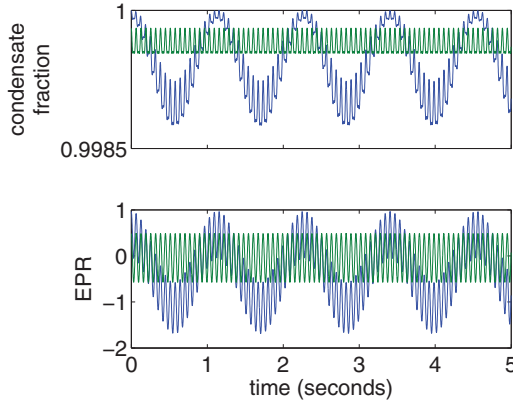


FIG. 5. (Color online) Validity of the two-frequency approximation. The condensate fraction and EPR entanglement calculated by approximating the initial coherent state with only three eigenstates [blue (dark gray)] is virtually indistinguishable from the numerically exact results shown in Fig. 2. An approximation with only two eigenstates [green (light gray)] reproduces the fast but not the slow oscillations.

B. $J \rightarrow 0$ limit

To gain more insight into the two frequencies, consider the limit $J \rightarrow 0$ in which the Hamiltonian can be diagonalized exactly [46]. The eigenstates are the Fock states $|N_1, N - N_1\rangle \equiv |N_1\rangle$, and the associated energies are

$$\epsilon_{N_1} = \frac{N_1(N_1 - 1)}{2}U + \frac{(N - N_1)(N - N_1 - 1)}{2}U, \quad (7)$$

which we denote with ϵ rather than E to distinguish the $J \rightarrow 0$ limit from the general case. Note the $N_1 \rightarrow N - N_1$ twofold degeneracy of the spectrum, reflecting the symmetry of the system with respect to a relabeling of the wells. The frequencies analogous to f_{fast} and f_{slow} computed numerically in Sec. III A are

$$\begin{aligned} \frac{\epsilon_0 - \epsilon_1}{2\pi\hbar} &= \frac{U(N-1)}{2\pi\hbar} \approx 15.9 \text{ Hz}, \\ \frac{\epsilon_0 - \epsilon_1}{2\pi\hbar} - \frac{\epsilon_1 - \epsilon_2}{2\pi\hbar} &= \frac{U}{\pi\hbar} \approx 0.82 \text{ Hz}. \end{aligned}$$

The $J \rightarrow 0$ estimate of f_{fast} coincides with the limit of the mean-field expression:

$$\begin{aligned} f_{\text{MF}} &= \frac{\sqrt{\Lambda^2 - 1}}{\pi} \frac{J}{\hbar} = \frac{\sqrt{U^2(N-1)^2 - 4J^2}}{2\pi\hbar} \\ &= \frac{U(N-1)}{2\pi\hbar} \sqrt{1 - \left(\frac{2J}{U(N-1)}\right)^2} \\ &= \frac{U(N-1)}{2\pi\hbar} \left[1 - \frac{1}{2}\Lambda^{-2} + \mathcal{O}(\Lambda^{-4})\right]. \end{aligned} \quad (8)$$

More interesting is $f_{\text{slow}} = U/\pi\hbar$. The slow oscillations are a purely quantum phenomenon, as $U/\pi\hbar$ goes to zero in the classical limit of $N \rightarrow \infty$ with $\Lambda = U(N-1)/2J$ fixed. A first hypothesis might identify them with the quantum revivals, in which *all* of the components of the coherent state rephase [47]. This is almost correct. Consider an initial state $|\psi(0)\rangle$ decomposed into energy eigenstates [Eq. (6)]. Evolving the

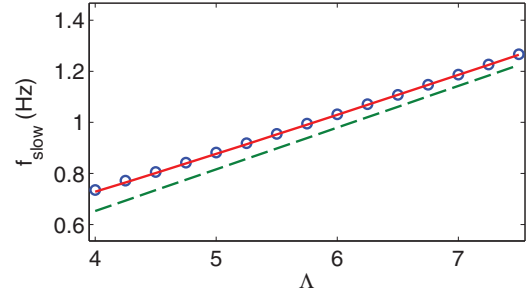


FIG. 6. (Color online) The slow frequency near the fixed point as a function of Λ . The numerically exact values (blue circles) are well described by the second-order perturbative results (solid red line). Zeroth-order perturbation theory (dashed green line) slightly underestimates the frequency.

state over one period $\tau = 1/f_{\text{slow}}$ of the slow oscillation yields

$$|\psi(\tau)\rangle = \begin{cases} \sum_{n=0}^N a_n |n\rangle & \text{for } N \text{ odd,} \\ \sum_{n=0}^N (-1)^n a_n |n\rangle & \text{for } N \text{ even,} \end{cases} \quad (9)$$

up to an overall phase (see Appendix A for a proof). For odd N , we observe a full revival, as expected. For N even, the relative phases of the eigenstates are altered, and a revival occurs only after a translation by 2τ . However, the additional phases present after a τ translation cancel when the condensate fraction and EPR are computed (see Appendix B for a proof). In the limit $J \rightarrow 0$ one therefore expects revivals in these observables with a frequency $1/\tau = \frac{U}{\pi\hbar}$ for all values of N .

Surprisingly, the $J \rightarrow 0$ result is close to the observed frequencies even when $J \gg U$ (see Fig. 6). To shed light on this, one may compute the shifts in the frequencies due to $J \gtrsim 0$ using degenerate perturbation theory (see Appendix C for a derivation following [48]). The resulting corrections to the $J = 0$ result are proportional to $\Lambda^{-2} \approx (J/NU)^2$:

$$\begin{aligned} \frac{\epsilon_0 - \epsilon_1}{2\pi\hbar} &= \frac{U(N-1)}{2\pi\hbar} \\ &\times \left[1 - \frac{1}{2} \frac{N+1}{N-3} \Lambda^{-2} + \mathcal{O}(\Lambda^{-4})\right], \\ \frac{\epsilon_0 - \epsilon_1}{2\pi\hbar} - \frac{\epsilon_1 - \epsilon_2}{2\pi\hbar} &= \frac{U}{\pi\hbar} \left[1 + \frac{3}{2} \frac{(N-1)(N+1)}{(N-5)(N-3)} \Lambda^{-2} \right. \\ &\left. + \mathcal{O}(\Lambda^{-4})\right]. \end{aligned} \quad (10)$$

The perturbative high-frequency estimate agrees with the mean-field result [Eq. (8)] in the limit of large N , as one would expect. Close to the bifurcation the mean-field expression performs better than the perturbative one (see Fig. 3), presumably because we dropped terms of order Λ^{-4} and higher. But above $\Lambda \approx 2$, the agreement of the perturbative expressions with the observed frequencies of both the mean-field motion (Fig. 3) and the quantum revival (Fig. 6) is excellent.

C. Region of validity

How far from the fixed point can we expect the dynamics to be dominated by the two-frequency pattern described above? To avoid introducing additional frequencies at the outset, the

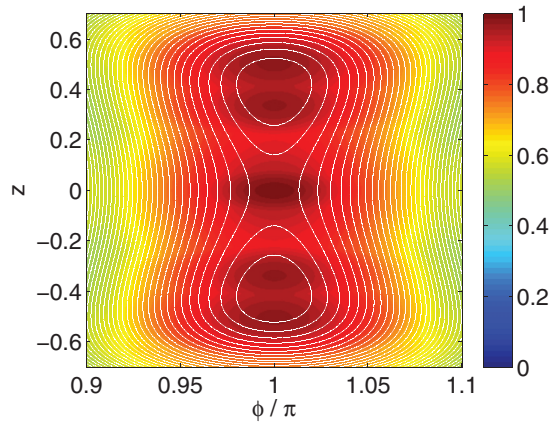


FIG. 7. (Color online) Breakdown of the three-eigenstate approximation near the bifurcation. The squared norm $\|\langle\psi'|\psi'\rangle\|^2$ of the projection of coherent states onto the three energy eigenstates with the largest coefficients in the energy eigenstate expansion is plotted for $\Lambda = 1.1$. The mean-field trajectories are overlaid in white. Note that the projection norm is not conserved along the mean-field trajectories, indicating the breakdown of the mean-field approximation and the two-frequency pattern.

initial coherent state $|z, \phi\rangle$ must have an appreciable projection onto just three eigenstates: that is, the projection $|\psi'\rangle = \sum_{n=0}^2 a_n |E_n\rangle$ must satisfy $\|\langle z, \phi|\psi'\rangle\|^2 = \|\langle\psi'|\psi'\rangle\|^2 \approx 1$. But in addition, coherent states at every point of the mean-field trajectory must be well approximated by the three eigenstates: if $\|\langle z, \phi|\psi'\rangle\|^2$ deviates significantly from 1 anywhere along an orbit, a breakdown of the two-frequency pattern is expected. An instructive example of such a breakdown is observed as the system approaches the bifurcation ($\Lambda \rightarrow 1_+$). Although in the neighborhood of the stable fixed points the norm of the three-eigenstate projection remains high, the mean-field orbits venture out of this neighborhood (see Fig. 7). The true quantum dynamics involves tunneling from one stable fixed point to the other [49], which is classically forbidden and does not conform to the two-frequency paradigm described in this section.

Thus, the two-frequency description is valid only for initial conditions in some neighborhood of the stable fixed points. However, this is generally sufficient to understand the generation of EPR entanglement: On long time scales EPR entanglement is present only for initial conditions close to the fixed points, as shown in Fig. 1.

What is more, the slow oscillations set the time scale for which EPR entanglement is present in the system. To obtain a global picture of entanglement generation, we simulated the dynamics of 10 000 initially coherent states uniformly sampled from the Bloch sphere. In Fig. 8 we plot, as a function of time, the fraction of these in which the two wells are entangled. Pronounced revivals occur with the frequency f_{slow} analyzed above. The implication, supported by an examination of individual phase-space trajectories [50], is that entanglement is only observed in those regions of phase space where its dynamics is dominated by the two-frequency behavior. In this sense, the two-frequency model explains the dynamics of the dimer's entanglement quite generically.

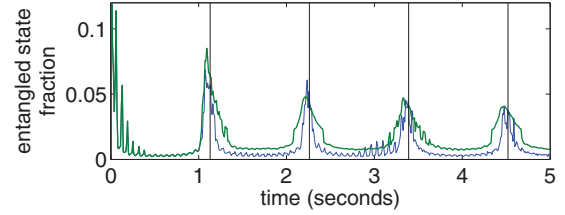


FIG. 8. (Color online) EPR entanglement is predominantly found at the quantum revival times and can be promoted by applying localized dissipation. We plot the fraction of coherent-state initial conditions for which the two wells are EPR entangled (EPR > 0) as a function of time for $\Lambda = 5$. The initial conditions are uniformly sampled on the Bloch sphere. The quantum revival times near the fixed point (multiples of $\tau = 1.13$ s) are marked with black vertical lines. The thin blue line is obtained in the absence of dissipation; the thick green line is the result seen when atom loss at the second site is induced between seconds 1 and 1.25 of the simulation. Applying dissipation increases the fraction of initial conditions for which the wells are persistently entangled.

IV. DISSIPATION-INDUCED COHERENCE

Atoms can be removed from a double-well optical trap with single-site resolution using strong resonant laser blasts or a focused electron beam [51,52]. This process can be described by the quantum master equation in Lindblad form for the density matrix $\hat{\rho}$ [53],

$$\frac{d}{dt}\hat{\rho} = -i[\hat{H}, \hat{\rho}] - \frac{1}{2} \sum_{j=1}^2 \gamma_j (\hat{a}_j^\dagger \hat{a}_j \hat{\rho} + \hat{\rho} \hat{a}_j^\dagger \hat{a}_j - 2\hat{a}_j \hat{\rho} \hat{a}_j^\dagger), \quad (11)$$

where γ_j is the loss rate at site j . Instead of solving the master equation directly, we use the quantum jump method [54–58], discussed further in Appendix D. Previous studies carried out along these lines show that controlled atom loss may lead to improved coherence (as measured by, among other indicators, the condensate fraction) in the Bose-Hubbard dimer [31,32] and in multiwell systems (i.e., lattices) [33,59–62].

An example of dissipation-induced coherence, simulated using the quantum jump method, is shown in Fig. 9. The initial condition is a coherent state near the self-trapping fixed point. After a second of free evolution, atoms are removed from the less populated site for half a second. The result is a long-term increase in condensate fraction and a transition from intermittent entanglement to a persistently entangled state. This process can be understood within the phase-space picture: the system's trajectory is driven towards the stable fixed point, which is a region of high entanglement.

How representative is the picture presented above? Consider again the evolution of 10 000 coherent states uniformly spaced in z and ϕ , shown in Fig. 8. Between the quantum revivals, the fraction of entangled states is substantially increased by the application of dissipation. This implies the mechanism shown in Fig. 9 operates for an appreciable range of initial conditions.

V. SUMMARY AND OUTLOOK

We have used the global phase-space picture of the BEC dimer to illuminate the dynamics of entanglement in this

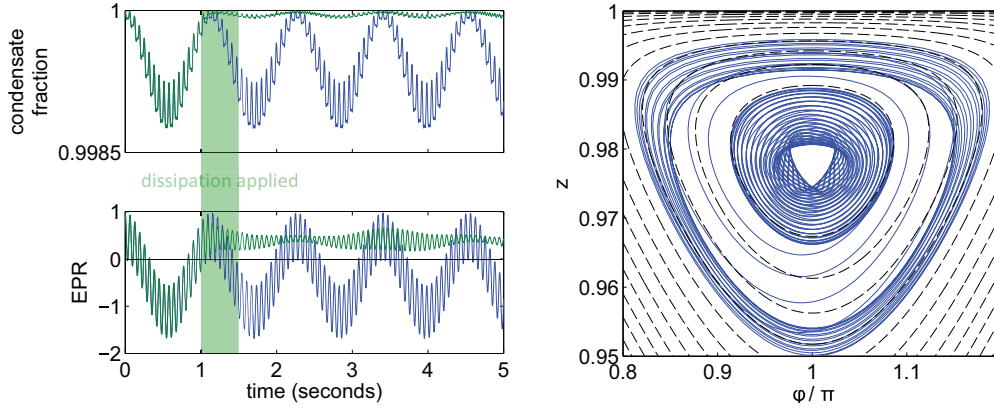


FIG. 9. (Color online) Dissipation-induced coherence, with signatures in condensate fraction, EPR entanglement, and phase-space trajectory (expectation values of z and ϕ). The initial condition is the same as in Fig. 2, but atom loss ($\gamma_2 = 5$ J/h) is induced at site 2 between seconds 1 and 1.5 of the simulation. The plots of condensate fraction and EPR compare the results without [blue (dark gray)] and with [green (light gray)] dissipation.

system and provide a perspective on dissipation-induced coherence. We showed that for initial conditions close to the mean-field self-trapping points the dimer's dynamics is completely captured by a projection onto just three eigenstates. Where the projection is successful, two frequencies appear prominently in the observables: f_{fast} , due to the mean-field motion, and f_{slow} , associated with a quantum revival. These frequencies are accurately analytically approximated by a second-order expansion about the strong-coupling limit. The frequency f_{slow} sets the dominant time scale for the dynamics of EPR entanglement in the BEC dimer. This is because the regions of phase space in which our description is valid coincide with the regions where EPR entanglement persists. It is also within these regions that dissipation-induced entanglement can be induced.

The significance of this work is twofold. First, the patterns we describe, two-frequency motion near the fixed point, the driving of the system into the fixed point by dissipation, and the resulting enhanced coherence, should be observable in ongoing experiments. Second, and more broadly, analogous patterns may be present in larger, multiwell systems of cold atoms in optical lattices. These systems are potential platforms for quantum information processing but by virtue of their size cannot be analyzed via exact techniques. Consequently, relationships between approximate but tractable semiclassical dynamics and inherently quantum behavior such as are described here offer an attractive path to large-scale quantum engineering.

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APPENDIX A: REVIVALS OF THE WAVE FUNCTION

Consider a Bose-Hubbard dimer with $J = 0$ and N atoms. The energy eigenstates of this system are the Fock states $|n\rangle \equiv |N_1, N - N_1\rangle$, with energies

$$\epsilon_n = \frac{n(n-1)}{2}U + \frac{(N-n)(N-n-1)}{2}U. \quad (\text{A1})$$

It will prove convenient to define a dedimensionalized energy,

$$h_n \equiv \frac{\epsilon_n}{U}. \quad (\text{A2})$$

The coherent states of the dimer are of the form

$$|\psi(t)\rangle = \sum_{n=0}^N a_n e^{-i\epsilon_n t/\hbar} |n\rangle, \quad (\text{A3})$$

with the expansion coefficients a_n all nonzero. Consider translating the coherent state in time by

$$\tau \equiv \frac{\pi\hbar}{U}. \quad (\text{A4})$$

As was asserted in the main text, the results of this translation depend on the value of N :

$$|\psi(\tau)\rangle = \begin{cases} |\psi(0)\rangle & \text{for } N = 1 + 4p, \\ -\sum_{n=0}^N (-1)^n a_n |E_n\rangle & \text{for } N = 2 + 4p, \\ -|\psi(0)\rangle & \text{for } N = 3 + 4p, \\ \sum_{n=0}^N (-1)^n a_n |E_n\rangle & \text{for } N = 4 + 4p, \end{cases} \quad (\text{A5})$$

with $p \in \mathbb{Z}_{\geq 0}$. Let us prove this assertion case by case.

Case 1. If $N = 1 + 4p$ for $p \in \mathbb{Z}_{\geq 0}$, then $|\psi(\tau)\rangle = |\psi(0)\rangle$.

Proof. Note that

$$\exp(-i\epsilon_n \tau/\hbar) = \exp(-i\pi h_n). \quad (\text{A6})$$

The dedimensionalized energy h_n satisfies

$$\begin{aligned} 2h_n &= n(n-1) + (N-n)(N-n-1) \\ &= n(n-1) + (1+4p-n)(1+4p-n-1) \\ &= n^2 - n + 4p - n + 16p^2 - 4np - 4np + n^2 \\ &= 2(n^2 - n) + 4p + 16p^2 - 8pn, \\ h_n &= n(n-1) + 2(p + 4p^2 - 2pn). \end{aligned}$$

The right-hand side is even for any $n \in \mathbb{Z}_{\geq 0}$ and any $p \in \mathbb{Z}_{\geq 0}$. Therefore, h_n is even and $\exp(-i\pi h_n) = 1$. It follows that $|\psi(\tau)\rangle = |\psi(0)\rangle$. ■

Case 2. If $N = 3 + 4p$ for $p \in \mathbb{Z}_{\geq 0}$, then $|\psi(\tau)\rangle = -|\psi(0)\rangle$.

Proof. We proceed as in the previous case.

$$\begin{aligned} 2h_n &= n(n-1) + (N-n)(N-n-1) \\ &= n(n-1) + (3+4p-n)(3+4p-n-1) \\ &= n^2 - n + 6 + 12p - 3n + 8p + 16p^2 \\ &\quad - 4np - 2n - 4np + n^2 \\ &= 2n^2 - 6n + 20p + 16p^2 - 8np + 6, \\ h_n &= [b(b-1) + 2(-b+5p+4p^2-2np)] + 3. \end{aligned}$$

Note that the number in square brackets is always even; thus, h_n is odd. Consequently, for any n ,

$$\exp(-i\pi h_n) = -1.$$

The claim then follows from the definition of $|\psi(t)\rangle$. ■

$$\exp\left(-\frac{i\epsilon_n\tau}{\hbar}\right) = \exp(-i\pi h_n) = \begin{cases} 1 & \text{if } N = 4p + 4 \text{ and } n \text{ is even or } N = 4p + 2 \text{ and } n \text{ is odd,} \\ -1 & \text{if } N = 4p + 4 \text{ and } n \text{ is odd or } N = 4p + 2 \text{ and } n \text{ is even.} \end{cases}$$

The claim follows immediately from the definition of $|\psi(t)\rangle$. ■

APPENDIX B: REVIVALS OF THE CONDENSATE FRACTION AND EPR

Consider the two observables discussed in the main text. The condensate fraction is given by the normalized largest eigenvalue of the single-particle density matrix,

$$c = \frac{1}{2N}(\rho_{11} + \rho_{22} + \sqrt{(\rho_{11} - \rho_{22})^2 + 4\rho_{12}\rho_{21}}), \quad (\text{B1})$$

where $\rho_{ij} = \langle \hat{a}_i^\dagger \hat{a}_j \rangle$. The measure of EPR entanglement is

$$E = \langle \hat{a}_1^\dagger \hat{a}_2 \rangle \langle \hat{a}_2^\dagger \hat{a}_1 \rangle - \langle \hat{a}_1^\dagger \hat{a}_1 \hat{a}_2^\dagger \hat{a}_2 \rangle. \quad (\text{B2})$$

In this section, we show that these observables take the same values at time $t = \tau = \pi\hbar/U$ as at $t = 0$, regardless of the value of N . For odd N , this follows immediately from the results of Appendix A, so assume N is even.

Consider first the condensate fraction. We have

$$\rho_{ij}(t=0) = \sum_n \sum_m a_n^* a_m \langle E_n | \hat{a}_i^\dagger \hat{a}_j | E_m \rangle$$

and, by the result proven in Appendix A,

$$\rho_{ij}(t=\tau) = \sum_n \sum_m (-1)^{n+m} a_n^* a_m \langle E_n | \hat{a}_i^\dagger \hat{a}_j | E_m \rangle.$$

At $J = 0$ the energy eigenstates are the Fock states, so the components of ρ can be found immediately:

$$\begin{aligned} \rho_{11}(t=\tau) &= \sum_n \sum_m (-1)^{n+m} a_n^* a_m n \delta_{n,m} \\ &= \sum_n (-1)^{2n} |a_n|^2 n = \sum_n |a_n|^2 n = \rho_{11}(t=0), \end{aligned}$$

Case 3. Let N be even. Then,

$$\begin{aligned} |\psi(\tau)\rangle &= \sum_{n=0}^N a_n e^{-i\epsilon_n\tau/\hbar} |E_n\rangle \\ &= \begin{cases} \sum_{n=0}^N (-1)^n a_n |E_n\rangle & \text{for } N = 4p + 4, \\ -\sum_{n=0}^N (-1)^n a_n |E_n\rangle & \text{for } N = 4p + 2, \end{cases} \end{aligned}$$

with $p \in \mathbb{Z}_{\geq 0}$.

Proof. Let $q = N/2$. We have

$$\begin{aligned} 2h_n &= n(n-1) + (2q-n)(2q-n-1) \\ &= 2n^2 + 4q^2 - 4nq - 2q, \\ h_n &= (n^2 - q) + 2(q^2 - nq) = (n^2 - q) + \text{even factor.} \end{aligned}$$

The factor $n^2 - q$ (and, by extension, h_n) is even if and only if n and q are of the same parity. Recall that q is even when $N = 4p + 4$ and odd when $N = 4p + 2$. Therefore,

$$\begin{aligned} \rho_{22}(t=\tau) &= \rho_{22}(t=0) \quad (\text{analogously}), \\ \rho_{12}(t=\tau) &= \sum_{n,m} (-1)^{n+m} a_n^* a_m \sqrt{(m+1)(N-m)} \delta_{n,m+1} \\ &= \sum_n (-1)^{2n-1} a_n^* a_{n-1} \sqrt{n(N-n+1)} \\ &= -\sum_n a_n^* a_{n-1} \sqrt{n(N-n+1)} \\ &= -\rho_{12}(t=0), \\ \rho_{21}(t=\tau) &= -\rho_{21}(t=0) \quad (\text{Hermiticity}). \end{aligned}$$

Note that $\rho(t = \tau)$ differs from $\rho(t = 0)$ only in the sign of the off-diagonal elements. But these elements enter the condensate fraction only through their product [cf. Eq. (B1)]. Therefore, $c(t = \tau) = c(t = 0)$.

The argument for EPR is similar. The first term in Eq. (B2) is equal to $\rho_{12}\rho_{21}$, and so it is the same at $t = \tau$ as at $t = 0$; the second term is equal to $N_1 N_2$, the product of the wells' populations, and so is independent of time. Therefore, $E(t = \tau) = E(t = 0)$.

APPENDIX C: PERTURBATION ABOUT THE $J \rightarrow 0$ LIMIT

In this appendix, we derive the perturbative corrections to the $J = 0$ mean-field and quantum revival frequencies [Eq. (10)].

It is convenient to rescale the problem by dividing all energies by NU . Then, the unperturbed Hamiltonian is given by

$$H_0 = \frac{1}{2N} (\hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_2), \quad (\text{C1})$$

Let $m = i + p$.

$$B_{i+p,i+n} = B_{m,m+n-p}.$$

Since p is odd and n is odd, $n - p$ is even, so $B_{i+p,i+n} = 0$ because B is an odd matrix. Thus,

$$(AB)_{i,i+n} = 0$$

for odd n , and AB is an even matrix. ■

Lemma 2. Let A be an odd $N \times N$ matrix and C be an even $N \times N$ matrix. Then, AC and CA are odd matrices.

Proof. Let n be an even integer. Proceeding as before,

$$\begin{aligned} (AC)_{i,i+n} &= \sum_{k=1}^N A_{i,k} C_{k,i+n} = \sum_{p=1-i}^{N-i} A_{i,i+p} C_{i+p,i+n} \\ &= \sum_{p \text{ odd}} A_{i,i+p} C_{i+p,i+n}. \end{aligned}$$

Let $m = i + p$.

$$C_{i+p,i+n} = C_{m,m-p+n}.$$

Since p is odd and n is even, $n - p$ is odd. Since C is an even matrix, $C_{i+p,i+n} = 0$ and

$$(AC)_{i,i+n} = 0$$

for even n , proving that AC is an odd matrix. The proof for CA is analogous. ■

The matrix V is odd, while the matrix L^{-1} is even. By the lemmas above, $W_3 = V(L^{-1}V)(L^{-1}V)$ must be odd. The diagonal entries of an odd matrix are all zero. This implies there are no third-order corrections to the energies, except for the few lowest-energy levels. Incidentally, the same argument can be used to show that the fifth-order corrections are zero.

APPENDIX D: THE QUANTUM JUMP METHOD

This appendix outlines the quantum jump method, the technique used in our simulations of the BEC dimer in the presence of dissipation.

We start with a simulation time interval $[0, T]$ and an initial state described by a wave function $|\psi(t=0)\rangle$. The time interval is divided into time steps δt . At each time step we act on the state with the time evolution operator,

$$U = \exp\left(-\frac{i}{\hbar}\hat{H}'\delta t\right),$$

where the pseudo-Hamiltonian \hat{H}' consists of the standard Bose-Hubbard Hamiltonian \hat{H} of Eq. (1) and an extra term:

$$\hat{H}' = \hat{H} - \frac{i\hbar}{2} \sum_{j=1,2} \gamma_j \hat{a}_j^\dagger \hat{a}_j.$$

In addition to the time evolution, each time step atoms are removed from wells $j = 1, 2$ with probability

$$\delta p_j = \delta t \gamma_j \langle \psi(t) | \hat{a}_j^\dagger \hat{a}_j | \psi(t) \rangle,$$

where γ_j is the atom loss rate from well j . The time δt is taken to be sufficiently small that the possibility of multiple atoms being removed during one time step can be ignored. If an atom is removed, we update the wave function,

$$|\psi\rangle \rightarrow \hat{a}_j |\psi\rangle,$$

and reduce the dimension of the Hilbert space. (The Hilbert space of the dimer has dimension $N + 1$, where N is the number of particles, so the problem shrinks as atoms are ejected.) Finally, regardless of whether an atom was removed or not, we renormalize the wave function.

A surprising feature of this algorithm is that the time evolution differs from the dissipation-free case even when no atoms are removed. This is because the absence of a removal event reveals information about the system, changing the probability distribution over the well occupation numbers and so altering the wave functions. Intuitively, if strong dissipation is applied to a well and yet no atoms are ejected from it, the well is likely to be empty. Alternately, one may interpret the suppression of tunneling into the well as a manifestation of the quantum Zeno effect [63]. See Sec. 3 of [56] for a more detailed discussion and references devoted specifically to this paradoxical “null measurement” effect.

APPENDIX E: A NOTE ON PROJECTIONS

The phase space of the semiclassical model of the BEC dimer is a sphere; representing it in a plane, as we have done in Figs. 1, 4, 7, and 9, requires a choice of projection. Throughout this work, we have used the cylindrical equal-area projection, also known as the Lambert projection. In the Lambert projection, the x and y coordinates on the map are proportional to the longitude ϕ (polar angle) and the cosine of the colatitude $z = \cos \theta$ (the usual z coordinate of spherical coordinates). In other words, the Lambert is an axial projection of the sphere onto a cylinder tangent at the equator.

As its name suggests, the cylindrical equal-area projection preserves areas: any two regions on the sphere with equal areas $S_1 = S_2$ are mapped to two regions on the map with equal areas $S'_1 = S'_2$. However, this projection deforms angles (it is not conformal), as a circle on the sphere is mapped to an ellipse, the eccentricity of which increases towards the poles.

The Lambert projection should not be confused with the Mercator projection, defined by

$$x = \phi, \quad y = \ln \tan\left(\frac{\pi - \theta}{2}\right).$$

The Mercator projection is conformal, mapping a circle anywhere on the sphere to a circle in the x, y plane, but is not area preserving: famously, Mercator maps of the Earth show South America to be smaller than Greenland, although the former land mass is in fact some eight times larger. In addition, since $y \rightarrow \infty$ as $\theta \rightarrow 0$ or π , a Mercator map must employ a θ cutoff and cannot include the poles. Some authors [41,64] have described their maps of the Bloch sphere as being drawn in the Mercator projection but labeled the y scale from $z = -1$

to $z = 1$, suggesting they might have actually used the less well-known Lambert (in which $y \propto z$ and the poles are in the range of the map).

A thorough and freely available reference on projections of the sphere is the review publication of the US Geological Survey [65].

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