PY541: Solutions to Practice Problems

1) To study the isothermal compressibility we can study the partial derivative, $(\partial P/\partial V)_{T,N}$. We then recall that the way we studied the singularity of the specific heat at T_c was by studying the singular behavior of the chemical potential and inserting this into the formula for the energy, expressed as an integral over single particle states which depended on μ. We must do the same thing here, beginning with the formula for the pressure as an integral over energies. It is convenient to use the form obtained after an integration by parts:

$$P = (2/3)(E/V) = (2/3)\frac{(2m)^{3/2}}{4\pi^2\hbar^3}\int_0^\infty d\epsilon \frac{\epsilon^{3/2}}{e^{\beta(\epsilon-\mu)}-1}. \tag{1}$$

From this formula we can see immediately that $\partial P/\partial V = 0$ below T_c. This follows from the fact that $\mu = 0$ below T_c and therefore P is independent of V, for a given T. From this formula, all the V dependence of P comes from μ. As we approach T_c from above, μ approaches a constant (zero) so its derivative vanishes, implying that $\partial P/\partial V \to 0$. We already calculated the behavior of μ near T_c in class when we studied the singularity in C_V so we can just use this result. A good starting point is obtained by differentiating Eq. (1) with respect to V:

$$\frac{\partial P}{\partial V}_{T,N} = (2/3)\frac{(2m)^{3/2}}{4\pi^2\hbar^3}\beta\int_0^\infty d\epsilon \frac{\epsilon^{3/2}e^{\beta(\epsilon-\mu)}}{\left(e^{\beta(\epsilon-\mu)}-1\right)^2}\cdot\left(\frac{\partial\mu}{\partial V}\right)_{T,N}. \tag{2}$$

Since we want this derivative near T_c we can set $\mu = 0$ and $T = T_c$ inside this integral. We can use the approximate expression for μ near T_c:

$$\mu \approx \frac{(4\pi\hbar)^3}{(2m)^3(kT)^2}[N/V - n^*(T)]^2, \tag{3}$$

where:

$$n^*(T) \equiv 2.315\frac{(2mkT)^{3/2}}{4\pi^2\hbar^3}. \tag{4}$$

Thus, close to T_c we see that:

$$\left(\frac{\partial\mu}{\partial V}\right)_{T,N} = \frac{(4\pi\hbar)^3}{(2m)^3(kT_c)^2}\left(\frac{-2N}{V^2}\right)[N/V - n^*(T)]. \tag{5}$$

The last factor in this equation tells us how this partial derivative vanishes as we approach T_c:

$$[N/V - n^*(T)] = n^*(T_c) - n^*(T) \approx 2.315(3/2)\frac{(2mk)^{3/2}T_c^{1/2}}{4\pi^2\hbar^3}(T - T_c). \tag{6}$$

Thus we see that $\partial P/\partial V$ vanishes linearly in $T - T_c$ and therfore its inverse, which gives the isothermal compressibility, diverges with an exponent $p = 1$. To calculate the coefficient, A we must assemble all the various factors that we have encountered here. In particular, we need to do the integral in Eq. (2). Having set $\mu = 0$ inside this integral we can rescale it giving:

$$\int_0^\infty d\epsilon \frac{\epsilon^{3/2}e^{\beta(\epsilon)}}{\left(e^{\beta(\epsilon)}-1\right)^2} = \beta^{-5/2}\int_0^\infty dx \frac{x^{3/2}e^x}{(e^x-1)^2}. \tag{7}$$

The value of this integral, a dimensionless constant, can be found in Pathria, for example. I don't bother to go through all the algebra to determine A. I note, however, that we can express it entirely in terms of n and constants $\hbar$ and m. The only quantity with the correct dimension, those of $1/P$, is $\hbar^2/(mn^{1/3})$. Thus the factor A must be a numerical constant times this quantity.

2) The exact formula for the $T = 0$ paramagnetic magnetization (i.e. the spin part) for an ideal Fermi gas is:

$$M = V\frac{\mu^*}{6\pi^2\hbar^3}(2m)^{3/2}[(\epsilon_F + \mu^* B)^{3/2} - (\epsilon_F - \mu^* B)^{3/2}]. \tag{8}$$

Naively, all we need to do is to expand this out to order B^3:

$$M \approx V \frac{\mu^*}{6\pi^2\hbar^3}(2m)^{3/2}[3\epsilon_F^{1/2}\mu^* B - (1/8)(\mu^* B)^3/\epsilon_F^{3/2}]. \tag{9}$$

(Here I use the binomial expansion.)

However, there is a trick here because ϵ_F depends weakly on B and this must also be taken into account. ϵ_F is determined, as usual, by the thermodynamic equation for the number of particles:

$$N = V \frac{1}{6\pi^2\hbar^3}(2m)^{3/2}[(\epsilon_F + \mu^* B)^{3/2} + (\epsilon_F - \mu^* B)^{3/2}]. \tag{10}$$

Expanding this equation to order B^2 gives:

$$n \approx \frac{1}{6\pi^2\hbar^3}(2m)^{3/2}[2\epsilon_F^{3/2} + (3/4)(\mu^* B)^2/\epsilon_F^{1/2}]. \tag{11}$$

Solving approximately for ϵ_F gives:

$$\epsilon_F \approx (3\pi^2 n)^{2/3}\frac{\hbar^2}{2m} - \frac{1}{4}\frac{(\mu^* B)^2}{(3\pi^2 n)^{2/3}}\frac{2m}{\hbar^2}. \tag{12}$$

Including this correction to ϵ_F in the first term in Eq. (9) gives the cubic term in M:

$$M^{(3)} = -\frac{V 2\mu^*(\mu^* B)^3 m^3}{3\pi^2 n\hbar^6}. \tag{13}$$

3a) Bose condensation arises if the density remains finite at $\mu \to 0$ without macroscopic occupation of the groundstate. For this dispersion relation, the density is:

$$n = \int \frac{d^3p}{(2\pi\hbar)^3}\frac{1}{e^{\beta(Ap^4-\mu)} - 1}. \tag{14}$$

Suppose we let $\mu \to 0$ in this integral. Then it diverges as:

$$\int \frac{d^3p}{p^4} \tag{15}$$

at $p \to 0$. Thus the density diverges as $\mu \to 0$ for this dispersion relation, unlike the normal situation where $\epsilon \propto p^2$. Thus Bose condensation would not occur in this case.

b) The groundstate energy is now given by:

$$E_0 = V\int \frac{d^3p}{(2\pi\hbar)^3}Ap^4, \tag{16}$$

where the integral is resticted, as usual, to the Fermi sphere, $p < p_F$. Thus:

$$E_0 = \frac{AVp_F^7}{14\pi^2\hbar^3}. \tag{17}$$

The usual relationship between p_F and the density applies:

$$p_F = (3\pi^2 n)^{1/3}\hbar, \tag{18}$$

so:

$$E_0 = \frac{AV(3n)^{7/3}\pi^{8/3}\hbar^4}{14}. \tag{19}$$

The pressure at $T = 0$ is:

$$P = -\left(\frac{\partial E_0}{\partial V}\right)_N = \frac{2A(3n)^{7/3}\pi^{8/3}\hbar^4}{21}. \tag{20}$$

Note the unusual power of n which is determined by the power of p in the energy-momentum relation.

Here I have ignored the spin of the fermions. If they have a spin degeneracy g=(2S+1), then Eqs. (16) and (17) pick up a factor of g on the right hand side and n gets divided by g in Eq. (18). This finally results in a factor of $1/g^{4/3}$ in the pressure.