Final Exam Solutions, PY541

1). An ergodic system passes through all points in phase space on the fixed energy hypersurface (or arbitrarily close to all points) during its evolution under the classical equations of motion, with almost any initial conditions. A system obeys the mixing hypothesis if an arbitrary initial density function evolves into one which uniformly fills the entire equal energy hypersurface. More accurately it evolves into one which looks uniform when viewed in a "smeared" approximation. That is it say the density integrated over an arbitrary small smooth finite region of the equal energy hypersurface is the same for all regions of the same volume. A single harmonic oscillator is ergodic but does not obey the mixing hypothesis. A pair of harmonic oscillators is not ergodic.

2)

$$N d\mu = dG = -SdT + VdP. \tag{1}$$

(Here I used $G = \mu N$ and the fact the N is constant.) This equation implies the two identies that you were asked to prove.

3)

$$n = 2 \int \frac{d^3p}{(2\pi\hbar)^3} \theta(\epsilon_F - |\vec{p}|c) \tag{2}$$

Here θ is the step function and the factor of 2 is for spin. Thus

$$\begin{aligned} n &= 2(4\pi/3)p_F^3/(2\pi\hbar)^3 = (\epsilon_F/\hbar c)^3/(3\pi^2) \\ \epsilon_F &= \hbar c(3\pi^2 n)^{1/3}. \end{aligned} \tag{3}$$

$$E_0 = 2V \int \frac{d^3p}{(2\pi\hbar)^3} \theta(\epsilon_F - |\vec{p}|c)|\vec{p}|c = (3/4)V\epsilon_F n = (3/4)V n^{4/3}\hbar c(3\pi^2)^{1/3}. \tag{4}$$

$$P = -\frac{dE}{dV} = (1/4)n^{4/3}\hbar c(3\pi^2)^{1/3}. \tag{5}$$

4) Setting $B = 0$, $\mu = \mu_0$, we find the minimum of f is determined by:

$$\begin{aligned} df/dm^2 &= b_0(T - T_c) + dm^4/2 = 0 \\ m &= \pm[2b_0(T_c - T)/d]^{1/4} \quad (T < T_c) \\ &= 0 \quad (T > T_c). \end{aligned} \tag{6}$$

Thus, $\beta = 1/4$. Setting $T = T_c$ and $\mu = \mu_0$, gives:

$$\begin{aligned} df/dm &= -\mu^* B + dm^5 = 0 \\ m &= \pm[dmu^*|B|/d]^{1/5}. \end{aligned} \tag{7}$$

Thus $\delta = 5$. Evaluating f at its minimum gives:

$$\begin{aligned} f_0 = b_0(T - T_c)[2b_0(T_c - T)/d]^{1/2} + (d/6)[2b_0(T_c - T)/d]^{3/2} &= -(2/3)(T_c - T)^{3/2} b_0[2b_0/d]^{1/2} \quad (T < T_c) \\ &= 0 \quad (T > T_c) \end{aligned} \tag{8}$$

Using $C_F \propto d^2f/dT^2$, we see that, near T_c,

$$\begin{aligned} C_V &\propto (T_c - T)^{-1/2} \quad (T < T_c) \\ &= 0 \quad (T > T_c). \end{aligned} \tag{9}$$

Thus, $\alpha = 0$ and $\alpha' = 1/2$. These exponents turn out to be correct only above 6 dimensions. The corrections to these results are believed to be quite substantial in 3 dimensions.