Second version of Second Exam, Solutions PY541

1) The figure is drawn below. Note that there are 3 independent rotation modes and 3 vibration modes. (In general the number of vibration modes is 3n-6 for a non-collinear molecule of n atoms; we substract off the 3 rotation modes and 3 center of mass translation modes from the 3n degrees of freedom.)
2a) As one student observed, this problem is not physical. Orbital angular momentum must be an integer, so L=1/2 is impossible. (I should have said L=1, S=1/2, for example.) Nonetheless, it is straightforward to answer the question. The figure is drawn on the next page. At low T the entropy per molecule is k times the logarithm of the groundstate degeneracy which is zero. At high T it is k times the total number of state, $k \ln 4$. In between the entropy must increase since its derivative givess the heat capacity which is generally positive.
b)

$$Z_1 = 1 + 3e^{-\Delta/kT}. \tag{1}$$

(I ignore ground state energy which doesn't have any effect on the entropy.) Using the Helmholtz free energy, $A = -kT \ln Z$, I obtain the internal entropy per atom from these degrees of freedom:

$$S/N = -\frac{\partial A}{\partial T} = k \ln\left(1 + 3e^{-\Delta/kT}\right) + \frac{3\Delta}{T\left(e^{\Delta/kT} + 3\right)} \tag{2}$$

Noting that the argument of the logarithm becomes 1 at low T and 4 at high T and that the second term is exponentially small at low T and drops off as Δ/T at high T we see that this has the low and high T limits of the figure drawn below.
3) The maximum density which is possible without macroscopic occupation of the groundstate, acheived when $\mu = 0$, is:

$$n_e(T) = \int \frac{d^2p}{(2\pi\hbar)^2} \frac{1}{e^{pc/kT} - 1}. \tag{3}$$

This integral is finite since it behaves as:

$$\int \frac{dpp}{p} \tag{4}$$

at small p, a finite integral. Therefore, when the actual density exceeds $n_e(T)$, Bose condensation must occur. Thus T_c is determined by the condition: $n = n_e(T_c)$ where the function $n_e(T)$ is determined by Eq. (3). We may reduce this to a dimensionless integral by a rescaling. Defining $x = pc/kT$, Eq. (3) becomes:

$$n_e(T) = (kT/2\pi\hbar c)^2 2\pi \int_0^\infty \frac{dxx}{e^x - 1}. \tag{5}$$

Letting I stand for this dimensionless integral:

$$I \equiv \int_0^\infty \frac{dxx}{e^x - 1}, \tag{6}$$

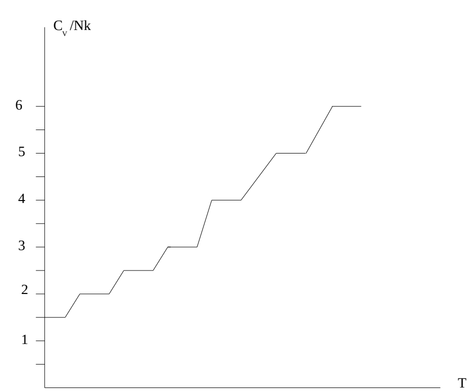

FIG. 1:

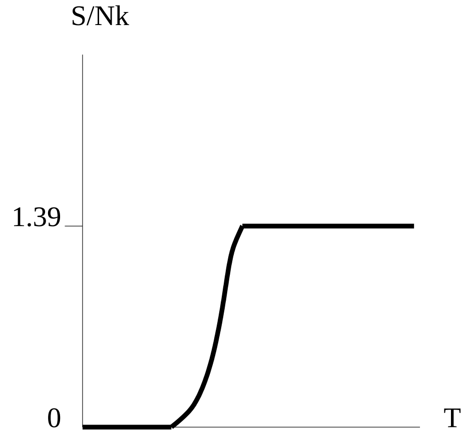

FIG. 2:

we obtain:

$$T_c = \sqrt{2\pi n/I}(\hbar c/k) \tag{7}$$