First Exam Solutions, PY541, Tuesday, October 1, 2:00-3:30

1a) The most general solution of the equations of motion is:

$$\begin{aligned} L(t) &= L_0 \quad \text{a constant} \\ \theta(t) &= (L_0/I)t + \theta_0. \end{aligned} \tag{1}$$

The complete surface in phase space with energy E is the circle $0 \leq \theta < 2\pi$ and $L = \pm L_0 = \pm\sqrt{2IE}$. The classical solution covers this circle, spending equal amounts of time at each value of θ, so the system might appear to be ergodic. However, technically it is not, since there are two disconnected parts of the equal energy surface with $L = \pm L_0$. The value of L does not change during the time evolution so both parts of the equal energy surface are not covered.

If we begin with an ensemble of systems which are uniformly spread over a range of initial angles, $0 < \theta(0) < \phi$, then, as time goes by, they stay spread over exactly the same anglular range, ϕ. At time t they are spread over $(L_0/I)t < \theta < (L_0/I)t + \phi$. They don't spread out to uniformly fill all of phase space at $t \to \infty$. Therefore, this system *does not* obey the mixing hypothesis.

b) We may diagonalize this Hamiltonian by making the canonical change of variables to:

$$\begin{aligned} q_\pm &\equiv (q_1 \pm q_2)/\sqrt{2} \\ p_\pm &\equiv (p_1 \pm p_2)/\sqrt{2}. \end{aligned} \tag{2}$$

The Hamiltonian becomes:

$$H = p_+^2/2 + p_-^2/2 + 3q_+^2/4 + q_-^2/4. \tag{3}$$

A general solution has (q_+, p_+) travelling around an ellipse and (q_-, p_-) travelling around a different ellipse. The total energy can be partitioned between the two independent oscillator modes ($+$ and $-$) in an arbitrary way. As time goes by no energy is ever exchanged between the two modes. The complete fixed energy manifold would have an arbitrary sharing of energy between the two modes. Therefore, this system is not ergodic.

Note that exactly this argument implies that *any* collection of N coupled harmonic oscillators is not ergodic. Energy never gets exchanged between the different modes under the classical time evolution. The situation is much richer and more interesting for anharmonic oscillators (with terms of higher order than quadratic in the potential energy.)

2a) . On leg $2 \to 3$, there is no work done since V is fixed. Therefore the heat adsorbed equals the increase in energy. This is:

$$Q_h = \Delta E = Nk\Delta T = (3/2)V\delta P = (3/2)V_1(P_3 - P_2). \tag{4}$$

Similarly, on leg $4 \to 1$, the heat emitted is:

$$Q_c = (3/2)V_2(P_4 - P_1). \tag{5}$$

No heat is adsorbed or emitted on the adiabatic legs.

b)

$$\eta = 1 - Q_c/Q_h = 1 - \frac{V_2}{V_1}\frac{(P_4 - P_1)}{(P_3 - P_2)}. \tag{6}$$

c) Since they are connected by adiabats we have:

$$\begin{aligned} P_3 V_1^{5/3} &= P_4 V_2^{5/3} \\ P_2 V_1^{5/3} &= P_1 V_2^{5/3}. \end{aligned} \tag{7}$$

Thus:

$$(P_3 - P_2)V_1^{5/3} = (P_4 - P_1)V_2^{5/3} \tag{8}$$

which imples:

$$\frac{(P_4 - P_1)}{(P_3 - P_2)} = \left(\frac{V_1}{V_2}\right)^{5/3}. \tag{9}$$

Substituting this into Eq. (6) gives:

$$\eta = 1 - \left(\frac{V_1}{V_2}\right)^{2/3}, \tag{10}$$

which is, remarkably, independent of the P_i's.

3) It follows from:

$$\delta S = (\partial S/\partial T)\delta T + (\partial S/\partial V)\delta V \tag{11}$$

that

$$C_P = T(\partial S/\partial T)_P = T(\partial S/\partial T)_V + T(\partial S/\partial V)_T \cdot (\partial V/\partial T)_P. \tag{12}$$

Thus:

$$C_P - C_V = T(\partial S/\partial V)_T \cdot (\partial V/\partial T)_P. \tag{13}$$

We can transform the first derivative in Eq. (13) using the Maxwell relation that follows from the Helmoltz free energy:

$$(\partial S/\partial V)_T = (\partial P/\partial T)_V. \tag{14}$$

We transform the second derivative using:

$$(\partial V/\partial T)_P = \frac{\partial(V,P)}{\partial(T,P)} = \frac{\partial(V,P)}{\partial(V,T)} \cdot \frac{\partial(V,T)}{\partial(T,P)} = -(\partial P/\partial T)_V \cdot (\partial V/\partial P)_T. \tag{15}$$

Substituting these two equations in to Eq. (13) gives the desired result:

$$C_p - C_V = -T(\partial P/\partial T)_V^2 (\partial V/\partial P)_T. \tag{16}$$

So our previous result that the isothermal compressibility is positive implies (assuming $T > 0$) that $C_P > C_V$.