First Exam Supplementary Question Solutions, PY541, Wednesday, October 16, 4:00-5:00

a)

$$\frac{1}{\kappa_S} = -V \left(\frac{\partial P}{\partial V}\right)_{S,N}. \tag{1}$$

Using:

$$\left(\frac{\partial E}{\partial V}\right)_{S,N} = -P, \tag{2}$$

we obtain:

$$\frac{1}{\kappa_S} = V \left(\frac{\partial^2 E}{\partial V^2}\right)_{S,N}. \tag{3}$$

b) This result is only true, in general, at $T = 0$. We may use the fact that, when $T = 0$, $S = 0$, sometimes known as the third law of thermodynamics. $S = 0$ when $T = 0$ for any N or V, so we may disregard all variations of E with S in taking derivatives. That is we may regard $E(V, N)$ as a function of E and V only, at $T = 0$. Now using the fact that E, V and N are all extensive we can write:

$$E = Ne(V/N), \tag{4}$$

where e is a function of a single variable, V/N, only. I let e' and e'' refer to first and second derivatives of the function e with respect to its argument. Then:

$$\left(\frac{\partial E}{\partial V}\right)_N = e' \tag{5}$$

and:

$$\left(\frac{\partial^2 E}{\partial V^2}\right)_N = \frac{1}{N}e''. \tag{6}$$

On the other hand,

$$\left(\frac{\partial E}{\partial N}\right)_V = e - \frac{V}{N}e', \tag{7}$$

and:

$$\left(\frac{\partial^2 E}{\partial N^2}\right)_V = -\frac{V}{N^2}e' + \frac{V}{N^2}e' + \frac{V^2}{N^3}e''. \tag{8}$$

Thus:

$$\left(\frac{\partial^2 E}{\partial N^2}\right)_V = \frac{V^2}{N^2}\left(\frac{\partial^2 E}{\partial V^2}\right)_N. \tag{9}$$

Note that these powers follow from dimensional analysis plus extensivity. Thus

$$\frac{1}{\kappa_S(T=0)} = \frac{N^2}{V}\left(\frac{\partial^2 E}{\partial N^2}\right)_V (T = 0). \tag{10}$$

The only property used here, once I got rid of S, was extensivity.