Solutions Problem Set 5

1)

$$\frac{N}{V}=\frac{geB}{hc}\int\frac{dp_z}{2\pi\hbar}\sum_{n=0}^{\infty}\frac{1}{\exp\left(\beta[p_z^2/2m+(n+1/2)\hbar\omega-\mu]\right)+1}$$

where $\omega=\frac{eB}{mc}$ and $g=2$ due to spin.

From Euler-McLaurin formula:

$$\sum_{n=0}^{\infty}\frac{1}{\exp[\,]+1}=\int_{n=0}^{\infty}\frac{1}{\exp[\,]+1}+\frac{1}{24}\frac{d}{dn}\left(\frac{1}{\exp[\,]+1}\right)_{n=0}$$

The first term gives usual result for $B=0$:

$$\frac{N}{V}=\frac{1}{3\pi^2}\left(\frac{2m\epsilon_F}{\hbar^2}\right)^{3/2}$$

The correction term is:

$$\frac{d}{dn}\left(\frac{1}{\exp[\,]+1}\right)_{n=0}=-\hbar\omega\frac{d}{d\mu}\left(\frac{1}{\exp[\,]+1}\right)$$

at $T\to 0$ it gives

$$-\hbar\omega\frac{d}{d\epsilon_F}\theta\left[\epsilon_F-pz^2/2m\right]$$

where $\theta(x)$ is the step function. Therefore,

$$\frac{N}{V}\approx\frac{1}{3\pi^2}\left(\frac{2m\epsilon_F}{\hbar^2}\right)^{3/2}-\hbar\frac{\omega}{24}\frac{2eB}{hc}\int_{-\infty}^{\infty}\frac{dp_z}{2\pi\hbar}\delta\left(\epsilon_F-pz^2/2m\right)$$

where $\delta(x)$ is the Dirac δ-function. Then,

$$\frac{N}{V}\approx\frac{1}{3\pi^2}\left(\frac{2m\epsilon_F}{\hbar^2}\right)^{3/2}-\left(\frac{eB}{c}\right)^2\frac{1}{12\pi h\sqrt{em\epsilon_F}}$$

Note that here I have used $\int dx\delta[f(x)]=\sum 1/f'(x_i)$ where x_i are the zeros of $f(x)$. In this case there are 2 zeros at $p_z=\pm\sqrt{2m\epsilon_F}$. Using $\epsilon_F\approx\frac{(3\pi^2n)^{2/3}\hbar^2}{2m}$, we get

$$\epsilon_F\approx\frac{(3\pi^2n)^{2/3}\hbar^2}{2m}\left[1+\left(\frac{eB}{c}\right)^2\frac{1}{9h^2(3\pi^2n)^{1/3}n}\right]$$

2) The next correction to Ω from Euler-McLaurin expansion is

$$\delta\Omega\approx-\frac{7}{5760}\frac{2eB}{hc}(-kTV)\int\frac{dp_z}{2\pi\hbar}\frac{d^3}{dn^3}\ln\left[1+\exp\left(-\beta\left(\frac{p_z^2}{2m}+n\hbar\omega-\mu\right)\right)\right]_{n=0}= \tag{1}$$

$$=-\frac{14}{5760}\frac{eBkTV}{hc}(\hbar\omega)^3\frac{d^3}{d\mu^3}\int\frac{dp_z}{2\pi\hbar}\ln\left[1+\exp\left(-\beta\left(\frac{p_z^2}{2m}-\mu\right)\right)\right] \tag{2}$$

$$=-\frac{14}{5760}V(\hbar\omega)^3\frac{eB}{hc}\frac{d^2}{d\mu^2}\int\frac{dp_z}{2\pi\hbar}\frac{1}{\exp\left(\beta\left(\frac{p_z^2}{2m}-\mu\right)\right)+1} \tag{3}$$

and for $T \to 0$ it gives,

$$\delta\Omega \approx -\frac{14}{5760} V(\hbar\omega)^3 \frac{eB}{hc} \frac{d^2}{d\epsilon_F^2} \left(2\frac{\sqrt{2m\epsilon_F}}{2\pi\hbar} \right) = \tag{4}$$

$$= \frac{14}{5760} V \frac{3}{4} (\hbar\omega)^3 \frac{eB}{hc} \frac{\sqrt{2m}}{\pi\hbar\epsilon_F^{3/2}} = \tag{5}$$

$$= \frac{14}{5760} \frac{3}{4} \frac{V}{m^3} \left(\frac{eB}{c} \right)^4 \frac{\hbar\sqrt{2m}}{2\pi^2\epsilon_F^{3/2}} \tag{6}$$

$M = -\frac{\partial\Omega}{\partial B}$, then,

$$\delta M = -3V \frac{14 \times 3}{5760} \left(\frac{e}{c} \right)^4 B^3 \frac{\hbar\sqrt{2m}}{2\pi^2\epsilon_F^{3/2} m^3} = -V \frac{7}{1440} \left(\frac{e}{c} \right)^4 \frac{B^3}{\pi^4 m\hbar^2 n}$$

There is another correction to M from $\delta\epsilon_F$, calculated in the previous problem. Using the lowest order result for M:

$$M = -VB \left(\frac{e\hbar}{2mc} \right)^2 \frac{1}{6} \frac{(2m)^{3/2}\epsilon_F^{1/2}}{2\pi^2\hbar^3} = \tag{7}$$

$$= -VB \left(\frac{e\hbar}{2mc} \right)^2 \frac{1}{6} \frac{(2m)^{3/2}}{2\pi^2\hbar^3} \frac{(3\pi^2 n)^{1/3}\hbar}{\sqrt{2m}} \left[1 + \left(\frac{eB}{c} \right)^2 \frac{1}{18h^2(3\pi^2 n)^{1/3} n} \right] \tag{8}$$

Then,

$$\delta M = -VB^3 \left(\frac{e}{c} \right)^4 \frac{1}{m\pi^2 n h^2} \frac{1}{432}$$

Finally, combining these two terms:

$$\delta M_{TOTAL} = -VB^3 \left(\frac{e}{c} \right)^4 \frac{1}{m\pi^2 n h^2} \frac{11}{2160}$$

3) Suppose $2\frac{eB}{hc}Ap < N < 2\frac{eB}{hc}A(p+1)$ where A is the area of the system and p is a non-negative integer. Lowest p Landau levels are filled $[n = 0, 1, 2, \ldots (p-1)]$, with $(2eB/hc)A$ electrons each and the (p+1)$^{\mathrm{st}}$ level (n=p) is partially filled.

$$E_0 = \hbar\omega \left[2A\frac{eB}{hc} \sum_{n=0}^{p-1} (n + 1/2) + \left(N - A2\frac{eB}{hc}p \right) (p + 1/2) \right]$$

Now, $\sum_{n=0}^{p-1}(n + 1/2) = p^2/2$, then

$$E_0 = \hbar\omega \left[N(p + 1/2) - A\frac{eB}{hc}p(p+1) \right]$$

Therefore,

$$M = -\frac{dE_0}{dB} = -\frac{e\hbar}{mc} N(p + 1/2) + \frac{2Ae^2B}{2\pi mc^2} p(p+1)$$

This result is valid for: $\frac{(N/A)hc}{2e(p+1)} < B < \frac{(N/A)hc}{2ep}$.

At $B = \frac{(N/A)hc}{2e(p+1)}$,

$$M = \frac{e\hbar}{mc} N[-(p + 1/2) + p] = -1/2\frac{e\hbar N}{mc}$$

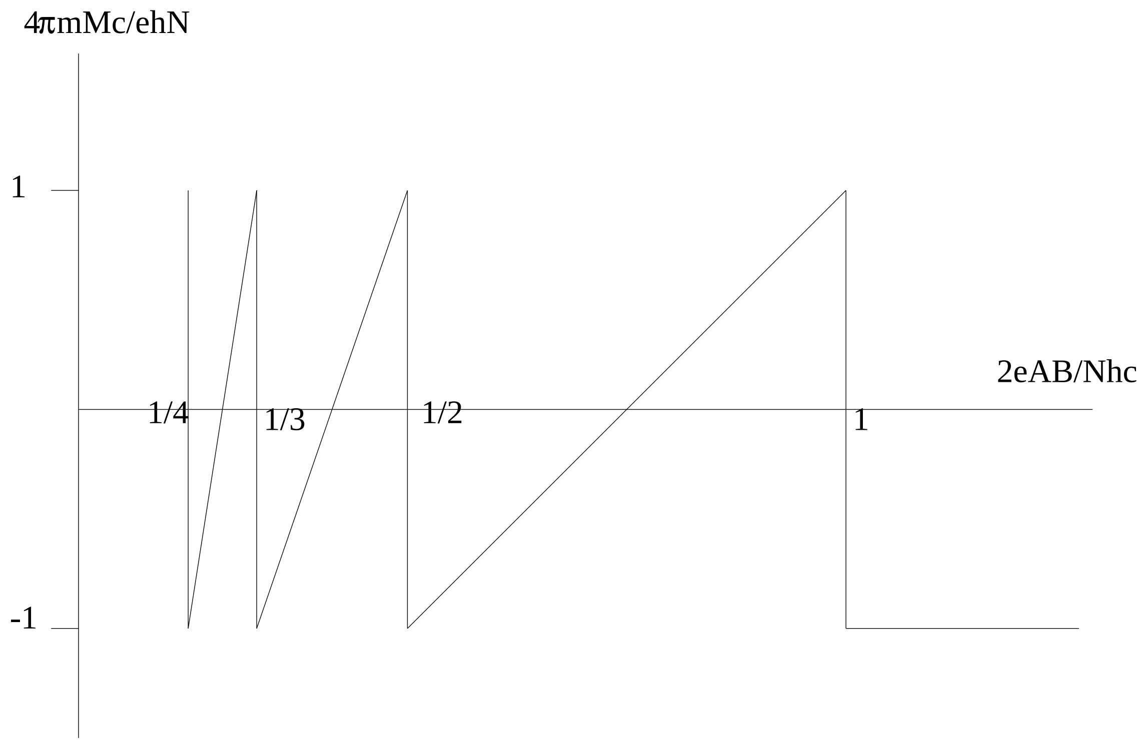

FIG. 1:

At $B = \frac{(N/A)hc}{2ep}$,

$$M = \frac{e\hbar}{mc} N[-(p+1/2)+(p+1)] = 1/2 \frac{e\hbar N}{mc}$$

So M(B) looks like the graph shown above. Note that the sawteeth get closer and closer together as B is reduced. A jump occurs each time a Landau level is completely filled. Such oscillations also occur in 3 dimensions and are called de Haas-van Alphen oscillations.
4) In class we showed that

$$\frac{C_v}{N} = \frac{\pi^2}{2}\frac{k^2 T}{\epsilon_F} = \left(\frac{\partial U}{\partial T}\right)_{V,N}$$

then,

$$\frac{U}{N} = \frac{\pi^2}{4} V \frac{(kT)^2}{\epsilon_F} + U_0$$

In general, $PV = \frac{2}{3}U$, so

$$P = P_0 + \frac{\pi^2}{6} n \frac{(kT)^2}{\epsilon_F} = P_0 + \frac{\pi^2}{6}\frac{2m}{\hbar^2}\frac{(kT)^2 n^{1/3}}{(3\pi^2)^{2/3}} \tag{9}$$

Then,

$$\left(\frac{\partial P}{\partial V}\right)_{T,N} = \left(\frac{\partial P}{\partial V}\right)_0 - \frac{\pi^2 m}{9V\hbar^2}\frac{(kT)^2 n^{1/3}}{(3\pi^2)^{2/3}} = \left(\frac{\partial P}{\partial V}\right)_0 - \frac{1}{3V}\frac{(kT)^2}{\epsilon_F}\frac{n\pi^2}{6}$$

$P_0 = \frac{2}{5} n\epsilon_F \propto n^{5/3}$, then

$$\frac{\partial P_0}{\partial V} = -\frac{2}{3V} n\epsilon_F$$

and

$$\left(\frac{\partial P}{\partial V}\right)_{T,N} = -\frac{2}{3V}n\epsilon_F - \frac{(kT)^2}{\epsilon_F}\frac{n\pi^2}{18V} = -\frac{2}{3V}n\epsilon_F\left[1+\left(\frac{kT}{\epsilon_F}\right)^2\frac{\pi^2}{12}\right]$$

The compresibility is,

$$K_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T = -\frac{1}{V}\left(\frac{\partial P}{\partial V}\right)_T^{-1} = \frac{3}{2n\epsilon_F}\left[1-\left(\frac{kT}{\epsilon_F}\right)^2\frac{\pi^2}{12}\right]$$

To get K_S we need $\left(\frac{\partial P}{\partial V}\right)_S$

$$\left(\frac{\partial P}{\partial V}\right)_S = \left(\frac{\partial P}{\partial V}\right)_T + \left(\frac{\partial P}{\partial T}\right)_V\left(\frac{\partial T}{\partial V}\right)_S$$

From Eq. (9),

$$\left(\frac{\partial P}{\partial T}\right)_V = \frac{\pi^2}{3}n\frac{k^2T}{\epsilon_F}.$$

Since c_V and hence S are linear in T at low T they are equal:

$$S = C_v = \frac{k^2T}{\epsilon_F}\frac{\pi^2}{2}$$

then,

$$T = \frac{2\epsilon_F S}{\pi^2k^2}$$

Now, $\epsilon_F \propto V^{-2/3}$, then

$$\left(\frac{\partial T}{\partial V}\right)_S = -\frac{2}{3}\frac{T}{V} = -\frac{4}{3}\frac{\epsilon_F}{\pi^2k^2}\frac{S}{V}$$

Therefore,

$$\left(\frac{\partial P}{\partial V}\right)_S = \left(\frac{\partial P}{\partial V}\right)_T - \frac{2\pi^2}{9}\frac{n(kT)^2}{V\epsilon_F} = -\frac{2n\epsilon_F}{3V}\left[1+\frac{\pi^2}{12}\left(\frac{kT}{\epsilon_F}\right)^2+\frac{\pi^2}{3}\left(\frac{kT}{\epsilon_F}\right)^2\right]$$

$$\left(\frac{\partial P}{\partial V}\right)_S = -\frac{2n\epsilon_F}{3V}\left[1+\frac{5\pi^2}{12}\left(\frac{kT}{\epsilon_F}\right)^2\right]$$

Finally,

$$K_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_S = \frac{3}{2n\epsilon_F}\left[1-\frac{5\pi^2}{12}\left(\frac{kT}{\epsilon_F}\right)^2\right]$$