Problem Set 3: Solutions

1)

$$P(n) = \frac{\exp(n\mu - \epsilon_n)}{\Xi}$$

where $\exp{(\mu - \epsilon_1)/kT} = 10^{-5}10^4 = 0.1$, $\exp{(2\mu - \epsilon_2)/kT} = 10^{-10}\exp{-1.9\epsilon_1/kT} = 10^{-2.4} = 0.04$. and $\Xi = \sum_n \exp(n\mu - \epsilon_n)$.

Then,

$$P(0) = 1/1.104 = 90.6\% \quad (1)$$
$$P(1) = 0.1/1.104 = 9.06\% \quad (2)$$
$$P(2) = 0.04/1.104 = 0.36\% \quad (3)$$
$$(4)$$

Furthermore, $\overline{n} = \frac{0+0.1+2\ 0.04}{1.104} = 0.0978$ and $< n^2 >= \frac{0+0.1+4\ 0.04}{1.104} = 0.105$, then

$$\sqrt{< n^2 > - \overline{n}^2}/\overline{n} = \sqrt{0.105 - 0.0978^2)}/0.0978 = 3.16$$

2) a)

$$Z_{MB} = \frac{1}{2!}(Z_1)^2 = \frac{1}{2}\left(\sum_{i=0}^{9} e^{-i\Delta/kT}\right)^2 = \quad (5)$$
$$= \frac{1}{2}\left[\frac{1 - e^{-10\Delta/kT}}{1 - e^{-\Delta/kT}}\right]^2 \quad (6)$$

b) Particles are indistinguishable ($(i,j) = (j,i)$) and they can not occupy the same state ($i \neq j$).

$$Z_{FD} = \sum_{i<j} e^{-(i+j)\Delta/kT} = \quad (7)$$
$$= \frac{1}{2}\left[\sum_{i,j=0}^{9} e^{-(i+j)\Delta/kT} - \sum_{i=0}^{9}(e^{-i\Delta/kT})^2\right] = \quad (8)$$
$$= Z_{MB} - \frac{1}{2}\sum_{i=0}^{9} e^{-i2\Delta/kT} = \quad (9)$$
$$= \frac{1}{2}\left[\frac{1 - e^{-10\Delta/kT}}{1 - e^{-\Delta/kT}} - \frac{1 - e^{-20\Delta/kT}}{1 - e^{-2\Delta/kT}}\right] \quad (10)$$

c) Particles are indistinguishable ($(i,j) = (j,i)$) and they can occupy the same state.

$$Z_{BE} = \sum_{i \leq j} e^{-(i+j)\Delta/kT} = \quad (11)$$
$$= Z_{FD} + \sum_{i=0}^{9}(e^{-i\Delta/kT})^2 = \quad (12)$$
$$= Z_{FD} + \frac{1 - e^{-20\Delta/kT}}{1 - e^{-2\Delta/kT}} = \quad (13)$$
$$= \frac{1}{2}\left[\frac{1 - e^{-10\Delta/kT}}{1 - e^{-\Delta/kT}} + \frac{1 - e^{-20\Delta/kT}}{1 - e^{-2\Delta/kT}}\right] \quad (14)$$

d)

$$Z_{BE} + Z_{FD} = \left[\frac{1 - e^{-10\Delta/kT}}{1 - e^{-\Delta/kT}}\right] = 2\ Z_{MB}$$

e) If $kT \gg \Delta$, then

$$\frac{1 - e^{-20\Delta/kT}}{1 - e^{-2\Delta/kT}} \approx 10$$

Therefore, $Z_{MB} = 1/2(10)^2 = 50$, $Z_{FD} = 50 - 5 = 45$ and $Z_{BE} = 50 + 5 = 55$.
f) $Z_{MB} = 1/2$, $Z_{FD} = 0$ and $Z_{BE} = 1$.

3) a) $\exp(-\beta H) = 1 - \beta H + (\beta H)^2/2 - ....$
To lowest non-vanishing order:

$$Z = Tr(e^{-\beta H}) \approx Tr(1) = 2^N \tag{15}$$

and

$$Tr\left((e^{-\beta H} S^a_{i'} S^b_{j'}\right) \approx \tag{16}$$

$$\approx Tr\left((1 - \beta H) S^a_{i'} S^b_{j'}\right) = \tag{17}$$

$$= Tr(S^a_{i'} S^b_{j'}) - \beta J\ Tr\left(\sum_{<i,j>,c} S^c_i S^c_j S^a_{i'} S^b_{j'}\right) \tag{18}$$

But for $i' \neq j'$, $Tr(S^a_{i'} S^b_{j'}) = Tr(S^a_{i'}) Tr(S^b_{j'}) = 0$. Furthermore, if $a \neq b$, $Tr(S^a_{i'} S^b_{i'}) = 0$. Then,

$$< S^a_{i'} S^b_{j'} >= Tr(e^{-\beta H} S^a_{i'} S^b_{j'})/Z \approx \tag{19}$$

$$-\beta J \sum_{<i,j>} Tr\left((S^a_i)^2 (S^b_j)^2\right) \delta_{ab}/2^N = \tag{20}$$

$$-\beta J\ Tr((1/4)^2) \delta_{ab}/2^N = -\delta_{ab} \beta J\ /16 \tag{21}$$

b) If $< i, j >$ are not nearest-neighbors then we can assume $i \neq i', j'$ *and* j. Therefore:

$$Tr\left((e^{-\beta H} S^a_{i'} S^b_{j'}\right) \approx \tag{22}$$

$$\approx -\beta J\ Tr\left(\sum_{<i,j>,c} S^c_i S^c_j S^a_{i'} S^b_{j'}\right) = \tag{23}$$

$$= -\beta J \sum_{<i,j>} Tr\left(S^c_j S^a_{i'} S^b_{j'}\right) Tr\left(S^c_i\right) = 0 \tag{24}$$

c)

$$\chi(T) = \frac{2\mu_B}{kT}\left[\sum_i < (S^z_i)^2 > + \sum_{i \neq j} < S^z_i S^z_j >\right] = \tag{25}$$

$$= \frac{2\mu_B}{kT}\left[1/4 + 6\ (-\beta J\ /16)\right] \tag{26}$$

4)

$$\frac{\partial(A,B)}{\partial(C,D)}\frac{\partial(E,F)}{\partial(G,H)} = \left[\frac{\partial(A,B)}{\partial(G,H)}\frac{\partial(G,H)}{\partial(C,D)}\right]\frac{\partial(E,F)}{\partial(G,H)} = \tag{27}$$

$$= \frac{\partial(A,B)}{\partial(G,H)}\left[\frac{\partial(G,H)}{\partial(C,D)}\frac{\partial(E,F)}{\partial(G,H)}\right] = \tag{28}$$

$$= \frac{\partial(A,B)}{\partial(G,H)}\frac{\partial(E,F)}{\partial(C,D)} \tag{29}$$