Problem Set 1: Solutions

1) The Hamiltonian of the system is

$$H = \frac{p^2}{2m} + \frac{kq^2}{2}$$

and the surface of constant energy $H = E_0$ is therefore an ellipse:

$$\frac{p^2}{2mE_o} + \frac{q^2}{2E_0 m\omega^2} = 1$$

The equations of motion imply that any point on the ellipse eveolves with time as:

$$q(t) = A\cos(\omega t + \phi); p(t) = -A\omega\sin(\omega t + \phi)$$

where $E_0 =$ max. PE $= 1/2kA^2$ and $k = m\omega^2$.

Because $q(t)$ and $p(t)$ are periodic functions of t, then if we look at a segment on the ellipse $H = E_0$ we will find that after a time $t = n2\pi/\omega$ (n=1,2,...) the segment will occupy the same portion of the ellipse, i.e. the segment will not occupy the whole ellipse for infinite times.

2) The Hamiltonian for the system is

$$H(x_1, x_2, p_1, p_2) = \frac{1}{2}\left[p_1^2 + p_2^2\right] + \left[x_1 + x_2\right]^2. \tag{1}$$

Defining the new canonical coordinates

$$x = (x_1 + x_2)/\sqrt{2}; p = (p_1 + p_2)/\sqrt{2}$$

$$X = (x_1 - x_2)/\sqrt{2}; P = (p_1 - p_2)/\sqrt{2}$$

we get,

$$H(x, X, p, P) = \frac{1}{2}\left[p^2 + x^2\right] + \frac{1}{2}P^2. \tag{2}$$

The bracket in this expresion is the Hamiltonian of an harmonic oscillator(k=2, m=1) while the last term is the Hamiltonian of a particle with mass 1.

The time dependence of these coordinates is therefore,

$$P = constant; X = X_0 + Pt \tag{3}$$

$$q(t) = A\cos(\omega t + \phi); p(t) = -A\omega\sin(\omega t + \phi) \tag{4}$$

Note that he hypersurphace $H = E_0$ is in 4-dimension phase space where X can assume any value (H does not depend on X). The motion of the representative point of the system on this hypersurphace discribes a line on the $X - P$ plane and an ellipse on the $x - p$ plane.

These two "subsistems" are decoupled (the Halmiltonian des not have a term "mixing" the coordinates (x,p) with (X,P)), so the two subsystems can not exchange energy. Now, if the total energy of the whole system is E_0 then the hypersurface $H = E_0$ must contain points (x,X,p,P) where any of the 2 subsytem could have, in principle, any energy between 0 and E_0. The fact that they can not exchange energy implyes that for any initial energies of the subsystems, they will have the same energies for ever. So the system is not ergodyc.

3) A general wave-function is

$$\psi^j(x) = a_1^j e^{ikx} + a_2^j e^{-ikx}, \tag{5}$$

and the density matrix is defined as

$$\rho_{mn} = \frac{1}{M} \sum_{j=1}^{M} a_m^{j*} a_n^j, \tag{6}$$

where here m,n=1,2.

The probability to find a particle at a position x is

$$P(x) = \frac{1}{ML} \sum_{j=1} |\psi^j(x)|^2 = \tag{7}$$

$$= \frac{1}{ML} \sum_{j=1} \left[a_1^j (a_1^j)^* + a_2^j (a_2^j)^* + a_1^j (a_2^j)^* e^{2ikx} + a_2^j (a_1^j)^* e^{-2ikx} \right] \tag{8}$$

(we are assuming that the x-axis has finite length. Therefore, L is a normalization factor). Using the definition for the density matrix above, we get

$$P(x) = \frac{1}{L} \left[\rho_{11} + \rho_{22} + \rho_{12} e^{2ikx} + \rho_{21} e^{-2ikx} \right]$$

Therefore, $P(x)$ is independent of x iff $\rho_{21} = \rho_{12} = 0$.

For any non-diagonal term, we have an expression like

$$\rho_{12} = \frac{1}{M} \sum_{j=1}^{M} |a_1^j||a_2^j| e^{\theta_1^j - \theta_2^j}$$

Now, if the phases are random then for any value of $|a_1^j||a_2^j|$ we will have a uniform distribution of values of $\theta_1^j - \theta_2^j$. The sum of these complex numbers (on a circle in the complex plane) will cancel out. Therefore the non diagonal terms ρ_{mn} would be zero.