PY541 Problem Set 1: Due Thursday, September 12, 2002 in class

1) I showed in class that the single one dimensional classical harmonic oscillator is ergodic. Show that it does not, however, obey the mixing hypothesis. That is, show that a general initial density function $\rho(q, p, 0)$ which is a constant on some compact subset of the equal energy surface for some fixed energy, E, and is zero elsewhere *does not* evolve under the classical equations of motion, at late times, into the time-independent density function which is constant on the entire fixed energy hypersurface (and zero elsewhere). Show that instead, it remains periodic at all times, even when $t \to \infty$. That implies that a closed system consisting of a single harmonic oscillator does not reach thermal equilibrium beginning from a general non-equilibrium state. This is not terribly alarming since thermal equilibrium is generally only expected for large complicated closed systems and even for those systems it may require small perturbations from an external reservoir.

2) Show that a system of two coupled classical harmonic oscillators, with Hamiltonian,

$$H = \frac{1}{2}\left[p_1^2 + p_2^2 + x_1^2 + x_2^2 + 2x_1x_2\right], \tag{1}$$

is not ergodic. Do this by starting with arbitrary initial conditions and showing that the future time evolution of the system *does not* entail it spending equal amounts of time arbitrarily close to all points on the fixed energy hypersurface of phase space. This is made easier by making a canonical transformation to different co-ordinates which diagonalize the Hamiltonian.

3) Consider general density matrices for a single free particle in one dimension which has a fixed energy E. Introducing $k = \sqrt{2mE}/\hbar$, a general wave-function with this fixed energy is:

$$\psi(x) = a_1 e^{ikx} + a_2 e^{-ikx}. \tag{2}$$

Now suppose that there is an ensemble of M such systems. As usual, we define the density matrix as:

$$\rho_{mn} = \frac{1}{M}\sum_{k=1}^{M} a_m^{k*} a_n^k. \tag{3}$$

Express the probability of the particle being at a particular position, x, in terms of ρ_{mn}. Find the most general density matrix for which the particle is equally likely to be at any point x. Show that the density matrix has this property if the phases of a_m^k are random. Show that the microcanonical density matrix is one of the density matrices obeying this property.