Cluster algorithm for the Ising model

Define bond index corresponding to pair of interacting spins

$$\text{bond} \quad b = 1, 2, \ldots, N_b, \quad \text{interacting spins } \sigma_{i(b)}, \sigma_{j(b)}$$

Number of bonds N_b = dN for a d-dimensional cubic lattice

Write the energy of the Ising ferromagnet as

$$E = -|J| \sum_{b=1}^{N_b} [\sigma_{i(b)} \sigma_{j(b)} + 1] = -\sum_{b=1}^{N_b} E_b$$

Write the partition function as

$$Z = \sum_{\sigma} \mathrm{e}^{-E(\sigma)/T} = \sum_{\sigma} \prod_{b=1}^{N_b} \mathrm{e}^{E_b/T} = \sum_{\sigma} \prod_{b=1}^{N_b} [1 + (\mathrm{e}^{E_b/T} - 1)]$$

Define bond functions with arguments 0,1 (bond variable):

$$F_b(0) = 1$$
$$F_b(1) = \mathrm{e}^{E_b/T} - 1$$

$$Z = \sum_{\sigma} \prod_{b=1}^{N_b} [F_b(0) + F_b(1)]$$

Introduce bond variables

$$\tau_b = 0, 1, \quad \tau = \{\tau_1, \tau_2, \ldots, \tau_{N_b}\}$$

Partition function can be written as sum over spins and bonds

$$Z = \sum_{\sigma} \prod_{b=1}^{N_b} [F_b(0) + F_b(1)] = \sum_{\sigma} \sum_{\tau} \prod_{b=1}^{N_b} F_b(\tau_b)$$

The functions F_b depend on the spins:

$$F_b(0) = 1$$

$$F_b(1) = \mathrm{e}^{E_b/T} - 1 = \begin{cases} \mathrm{e}^{2|J|/T} - 1, & \text{if } \sigma_{i(b)} = \sigma_{j(b)} \\ 0, & \text{if } \sigma_{i(b)} \neq \sigma_{j(b)} \end{cases}$$

$\tau_b = 1$ allowed only between parallel spins

Probabilities: For everything else fixed, probability for a given b

$$P(\tau_b) = \frac{F(\tau_b)}{F(0) + F(1)} = \frac{F(\tau_b)}{\mathrm{e}^{2|J|/T}}$$

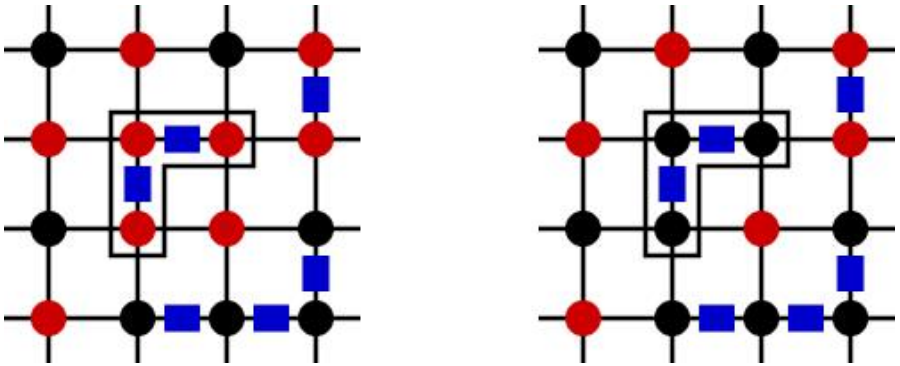
If parallel spins on bond b, probabilities for the bond variable

$$P(\tau_b = 0) = \mathrm{e}^{-2|J|/T}, \quad P(\tau_b = 1) = 1 - \mathrm{e}^{-2|J|/T}$$

If anti-parallel spins on bond b

$$P(\tau_b = 0) = 1, \quad P(\tau_b = 1) = 0$$

For a fixed bond configuration, spins forming clusters (spins connected by "filled" bonds) can be flipped and then give a configuration (term) with the same weight in Z (F_b=1 for all bonds between clusters, F_b unchanged inside cluster).



$N(\tau_b = 1) =$ No. of filled bonds

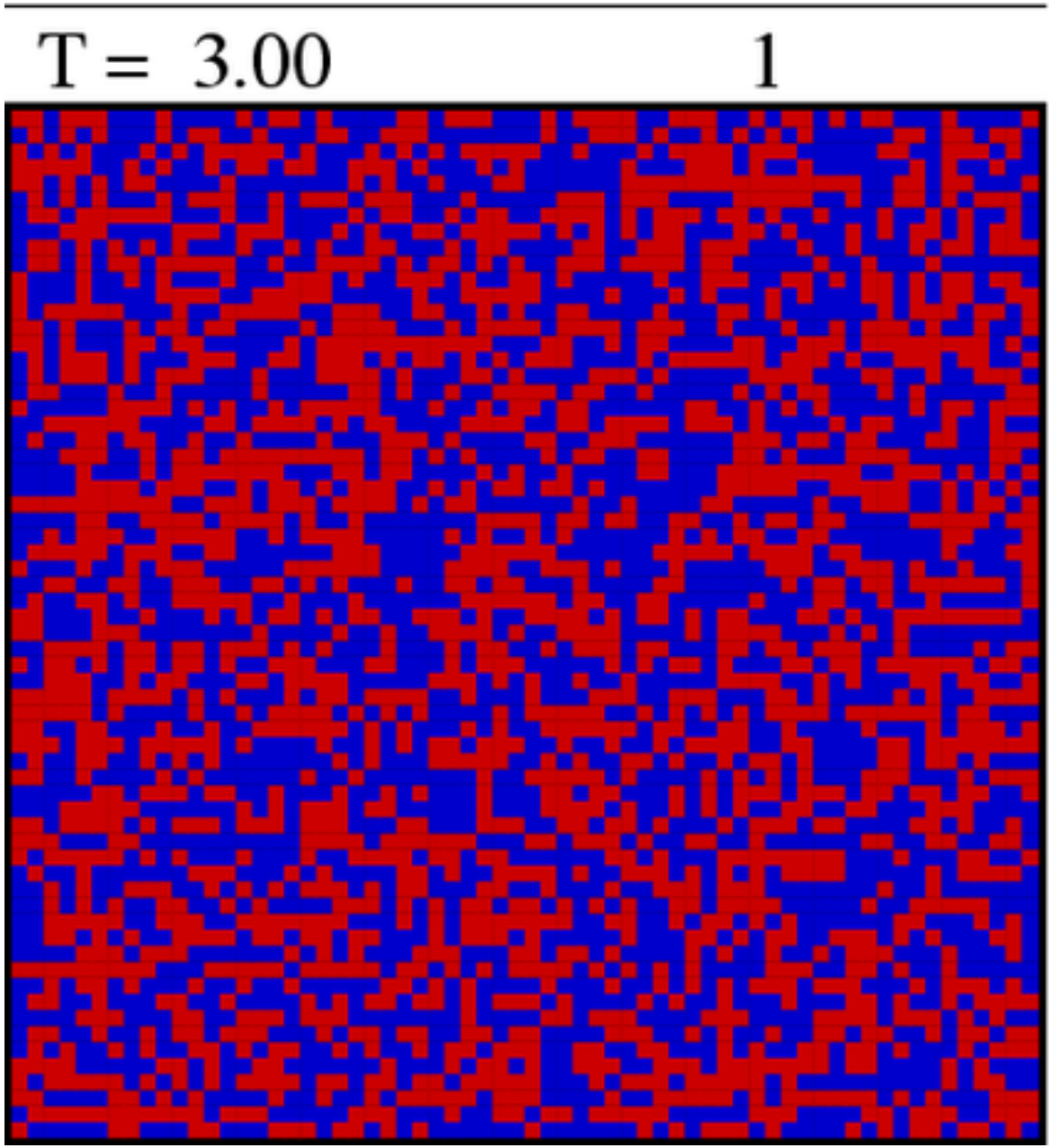
$$W = (e^{2|J|/T} - 1)^{N(\tau_b=1)}$$

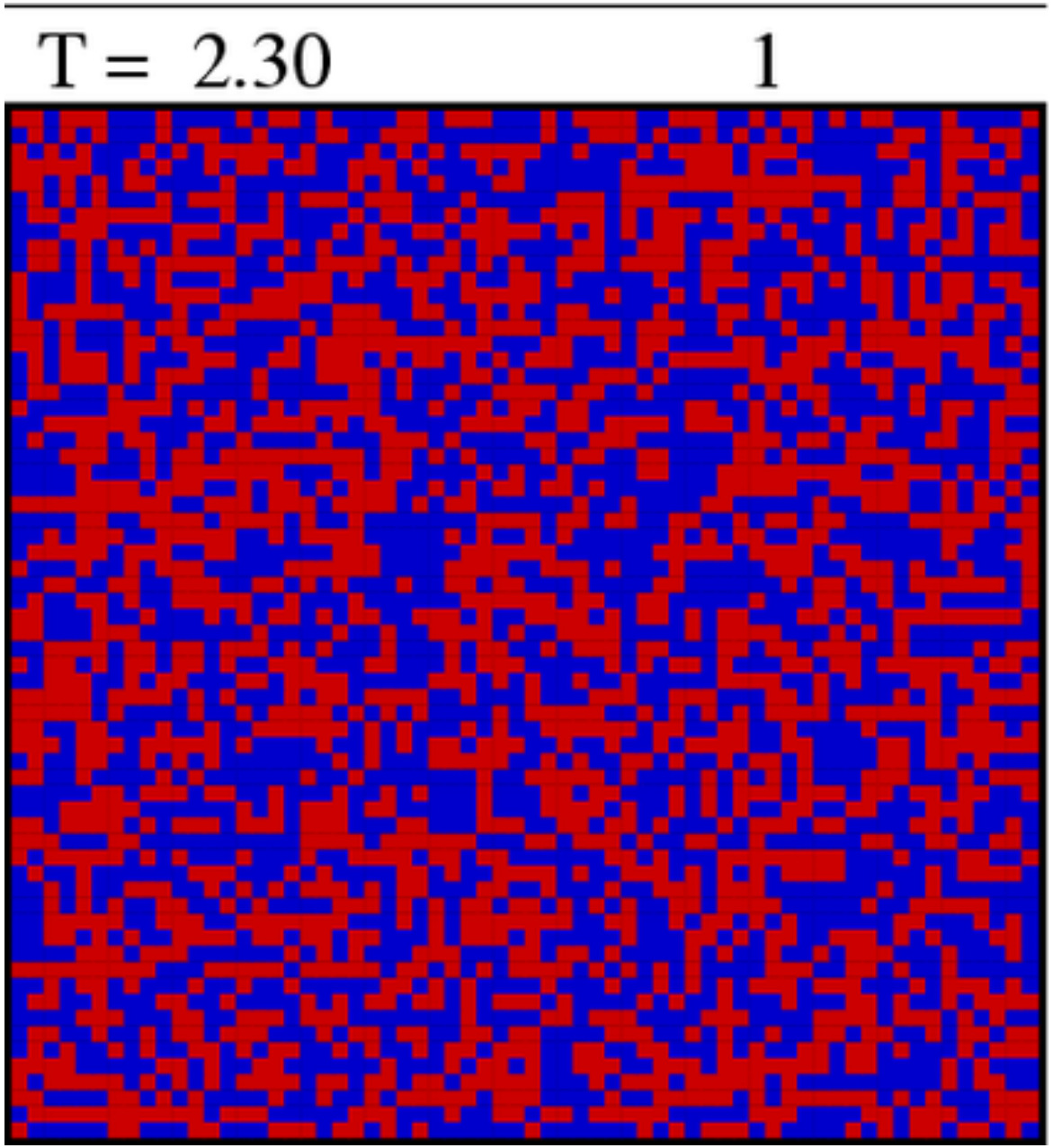
(unchanged after flip)

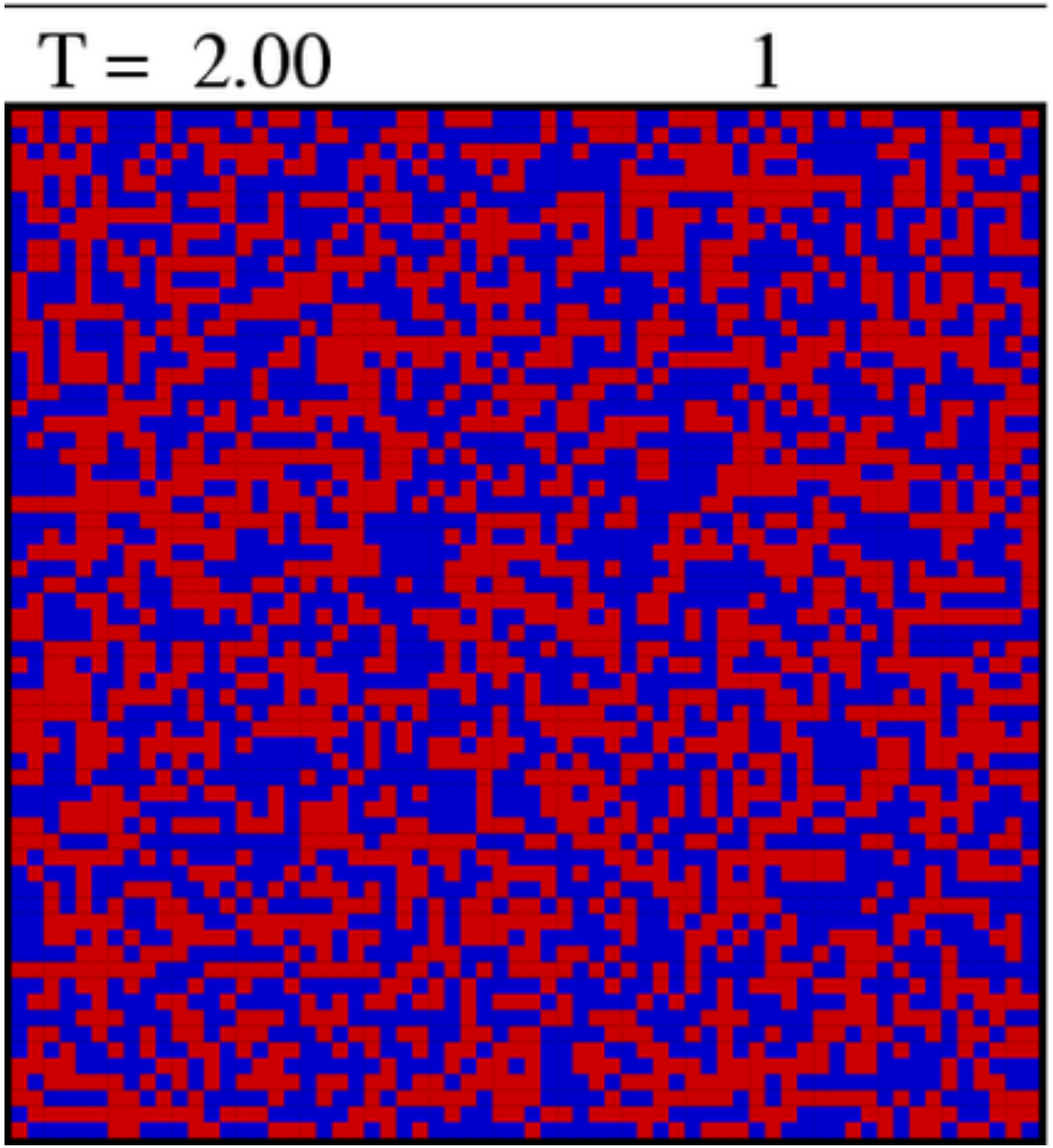
Spins not connected to any filled bonds are single-spin clusters

Swendsen-Wang algorithm

- Start from spin configuration
- Generate bond configuration
- Identify clusters of spins connected by bonds
- Flip each cluster with probability 1/2
- Generate new bonds with the current spins, etc

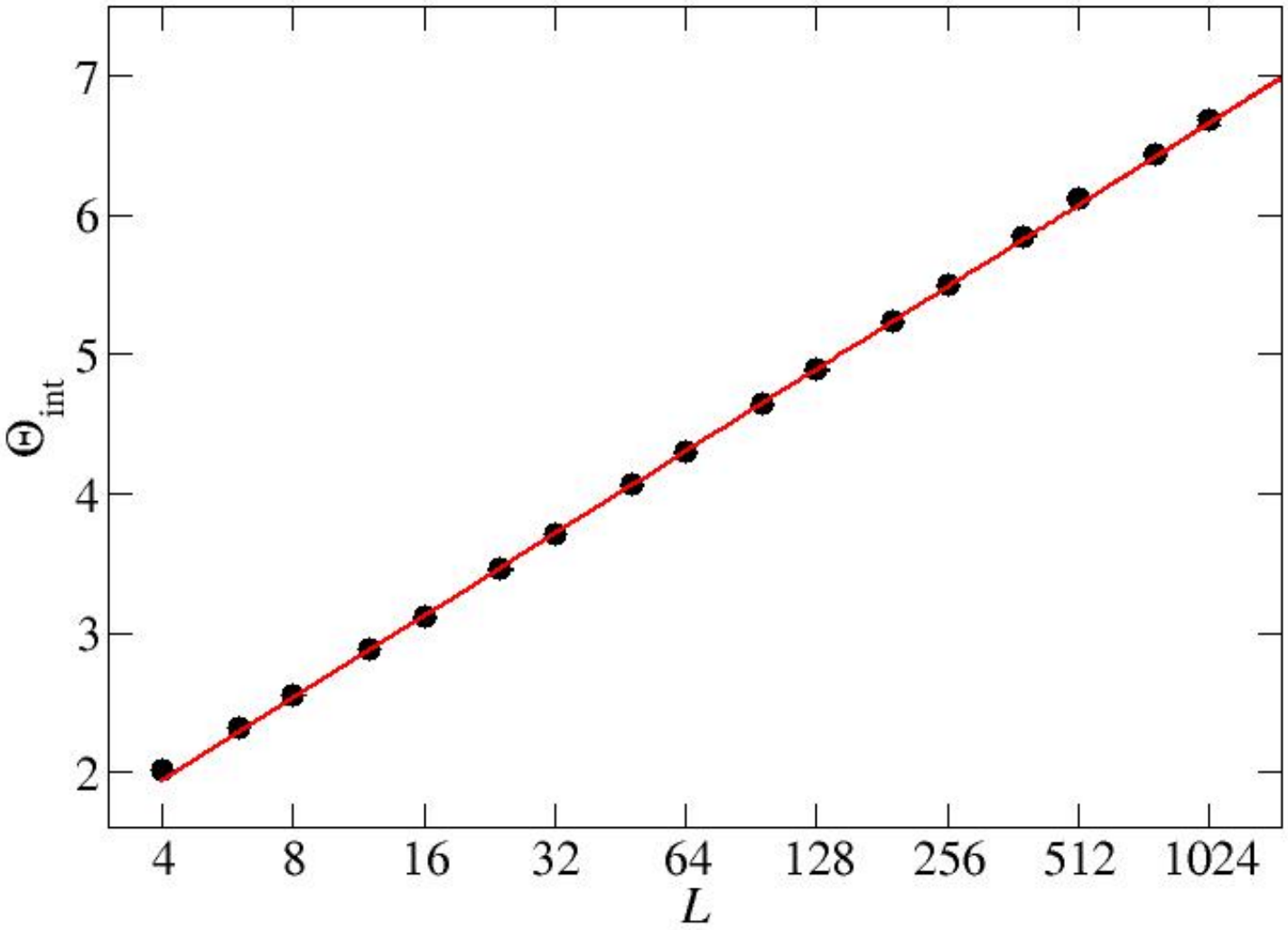

T = 3.00
1


T = 2.30
1


T = 2.00
1

Integrated autocorrelation times

- z=0 for Swendsen-Wang in two dimensions?
- log-divergence of autocorrelation time?
- More likely z ~ 0.3 (hard to distinguish L^z and log if z small)



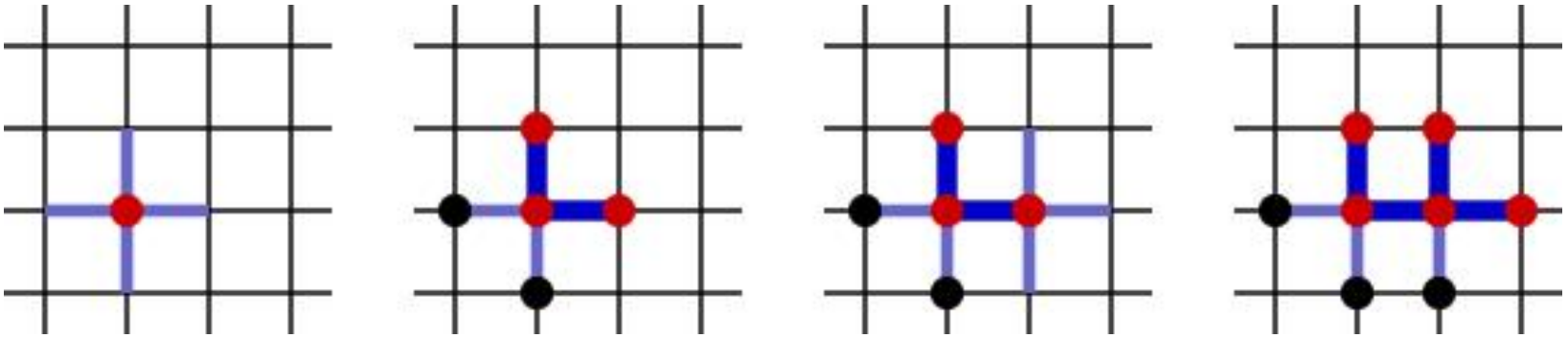
Cluster finding/flipping

Clusters can be constructed and flipped in the same process

➢ Decide whether or not to flip (50% probability) before starting

Store array with flags for spins visited

- Start with spin that has not been visited; seed of cluster
- Add connected (by filled bonds) neighbors to cluster
- Examine the non-visited neighbors of the new spins added
- Add connected neighbors to cluster
- Until no more spins in the cluster with non-visited neighbors



Use stack to store spins with neighbors to be examined