Chaotic Motion

- here "motion" could refer to some general time-evolution
- **dynamical system** governed by some differential equation(s)

In the theory of dynamical systems, **chaos (deterministic chaos) involves**

- aperiodic motion
- sensitive dependence on initial condition ("butterfly effect")
    + in practice unpredictable long-time behavior
- chaotic motion is not completely random
    + there is some structure in phase space
- universality; many systems exhibit same type of chaos

Chaos theory and nonlinear dynamics is a big field

- here we just discuss basic aspects in the context of Newton's equations

To have chaos in **one-dimensional Newtonian dynamics,** we need:

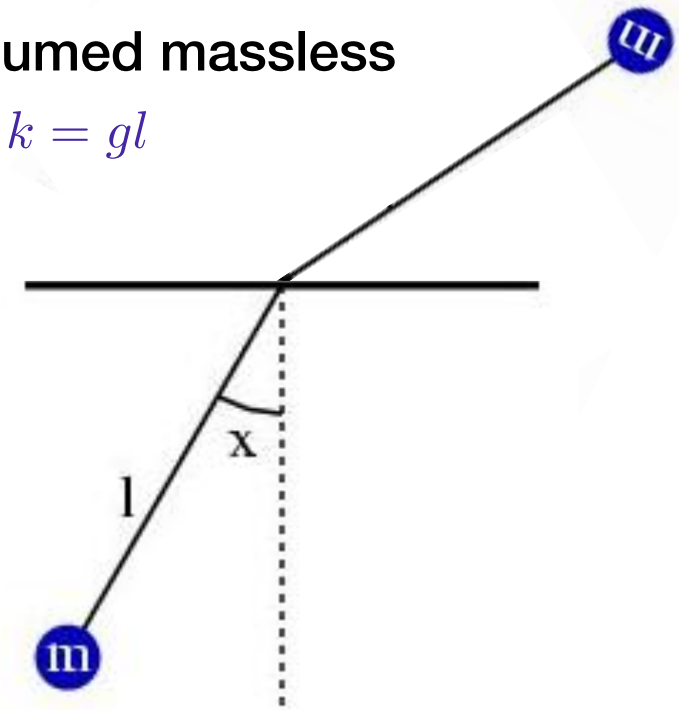
- dissipation (frictional forces)
- periodic driving (energy not conserved)

In higher dimensions (or > 1 "particle"), these external factors are not needed

<u>Example: Damped, driven pendulum</u>

Mass m in gravitational field, rod of length *l* assumed massless

$$V(x) = mgl[1 - \cos(x)] \qquad F = -km\sin(x),\ k = gl$$



Harmonic motion for small angles x

- we keep full potential; x ∊ [0,2π)

adding driving and damping

- both could be achieved by some mechanism at end of rod

The equation of motion for the angle is

$$\ddot{x}(t) = -k\sin(x) - \gamma v + Q\sin(\Omega t)$$

When we solve, x ∊ [-∞,∞]. We can bring back to x ∊ [0,2π) if needed

This is a standard example in which chaotic motion can be studied

- still not all aspects are completely understood (very rich behavior)
- same equation can be realized with other systems (incl. experimens)
- we will study some basic aspects and learn how to analyze data

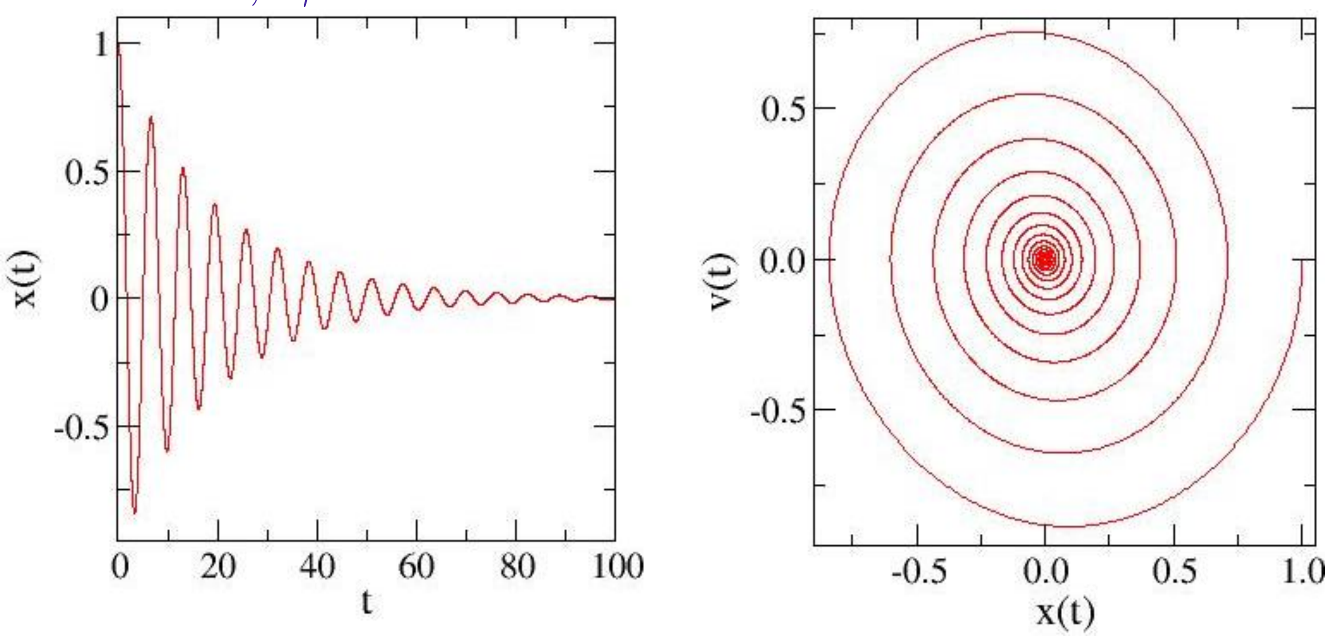
Phase space trajectories and attractors

Solving the equation numerically, short enough time step

- graphing the trajectory in phase space, [x(t),v(t)]

First, **no driving (Q=0)**, initial condition x=1,v=0

$k = 1,\ \gamma = 0.1$



Damped oscillations; eventually x→0, v→0

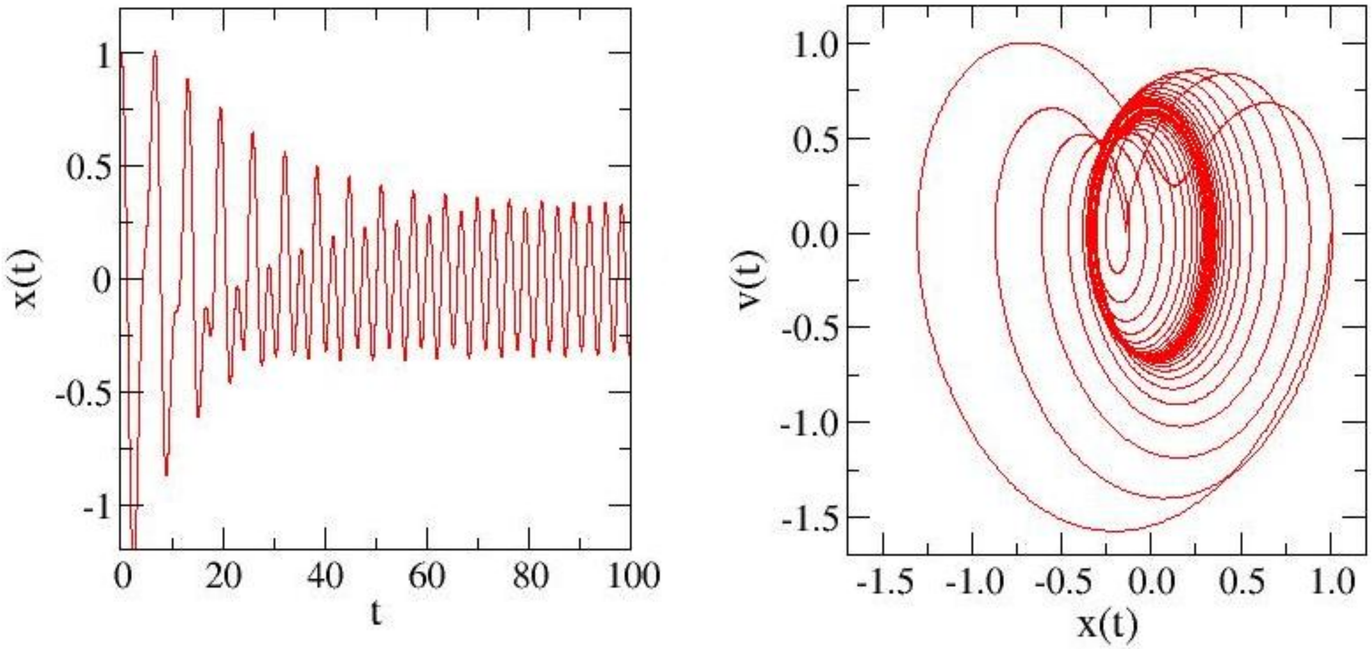
- (x=0,v=0) is the unique **attractor** of the motion in this case
- the spiral in phase space is **transient motion** (leading to the attractor)

<u>Next, adding driving (Q>0)</u>, same initial conditions as before

- now the pendulum cannot come to rest

For the chosen parameters, periodic motion eventually sets in

$k = 1, \gamma = 0.1, Q = 1, \Omega = 2$



- the attractor (limit cycle) here is a loop in phase spece
- after transient motion; the attractor is approached asymptotically for $t \to \infty$

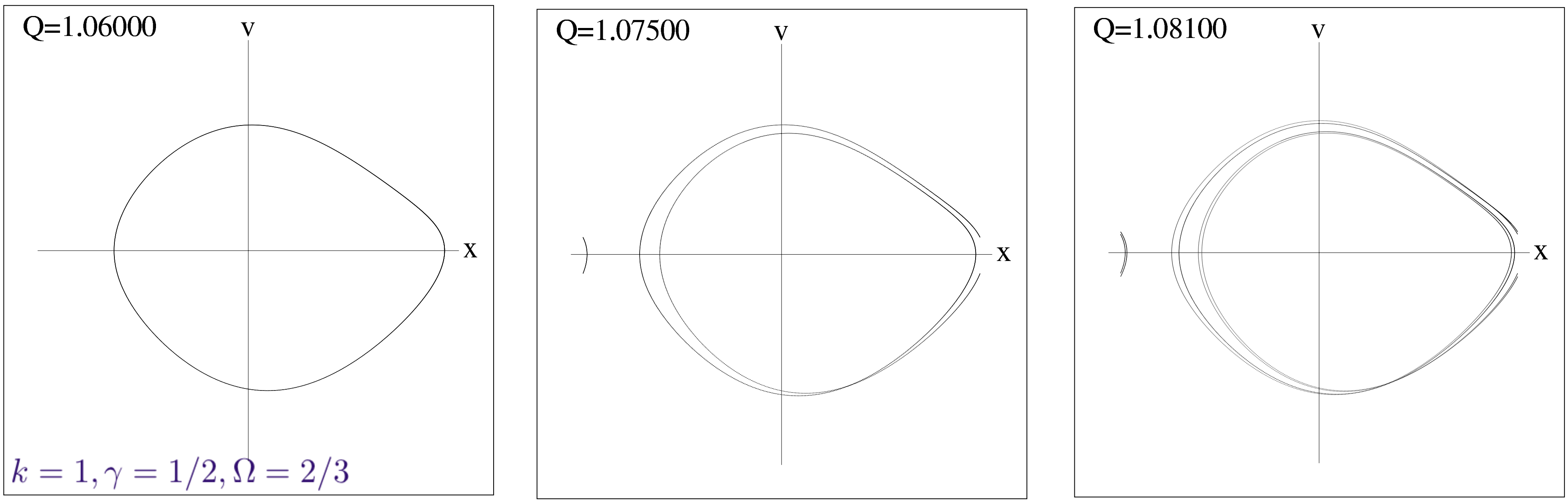
In general there can be different attractors depending on initial conditions

- basins of attraction (one for each attractor) in phase space

Period doubling (bifurcation)

The period of the motion can be any integer multiple of the driving period

Sequence of period doublings can occur versus Q; $T_{\mathrm{P}} = 2\pi/\Omega, 4\pi/\Omega, 8\pi/\Omega, \ldots$



Same initial conditions, evolved a long time before saving path

- transient has decayed away

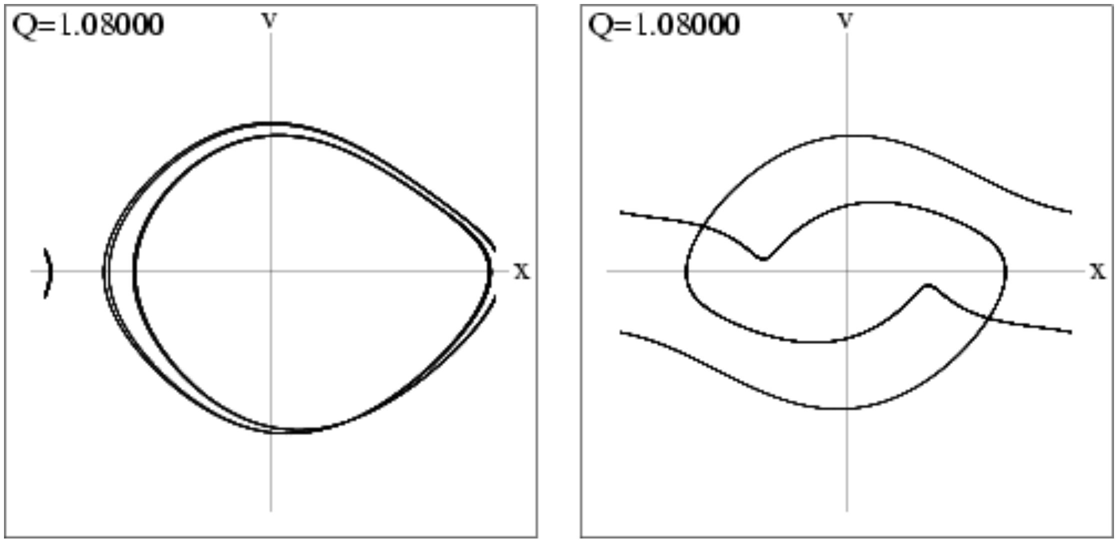
Infinite sequence of bifurcations for some Q values

- depending on other parameter values
- also depends on initial conditions (basins of attraction)

Example; dependence on boundary conditions

- different starting (x_0,v_0) combinations

$k = 1, \gamma = 1/2, \Omega = 2/3$



<u>left case:</u> attractor breaks parity symmetry

- there is another attractor obtained by x→-x, v→-v

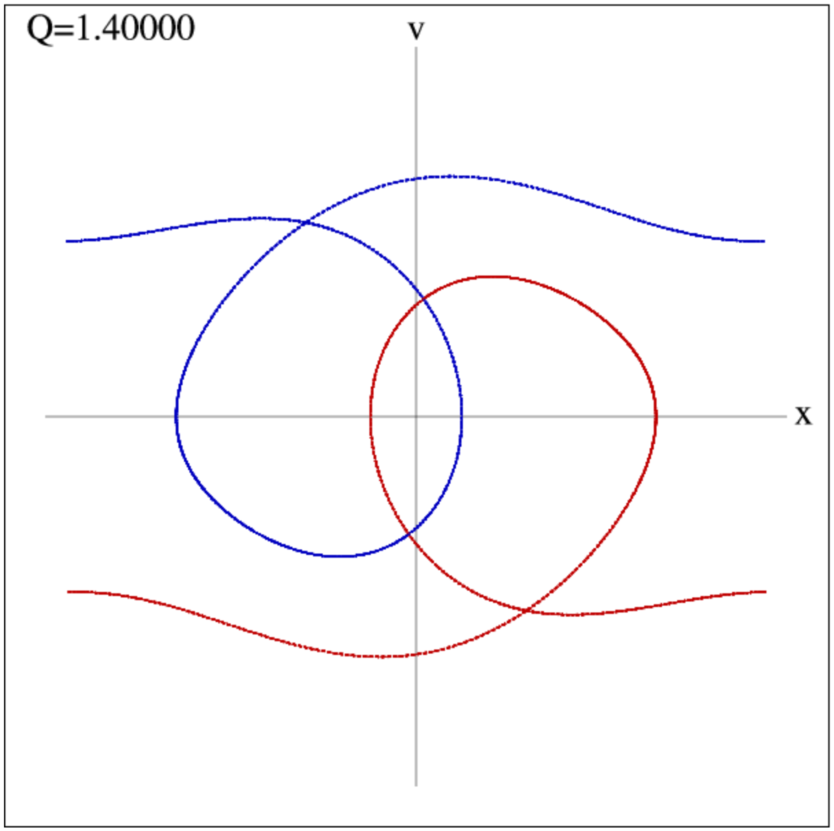
<u>right case:</u> attractor is invariant under parity transformation

3 attractors in total for these parameters and energy

- 3 basins of attraction in (x_0,v_0) space

Example: same initial conditions, different Q values

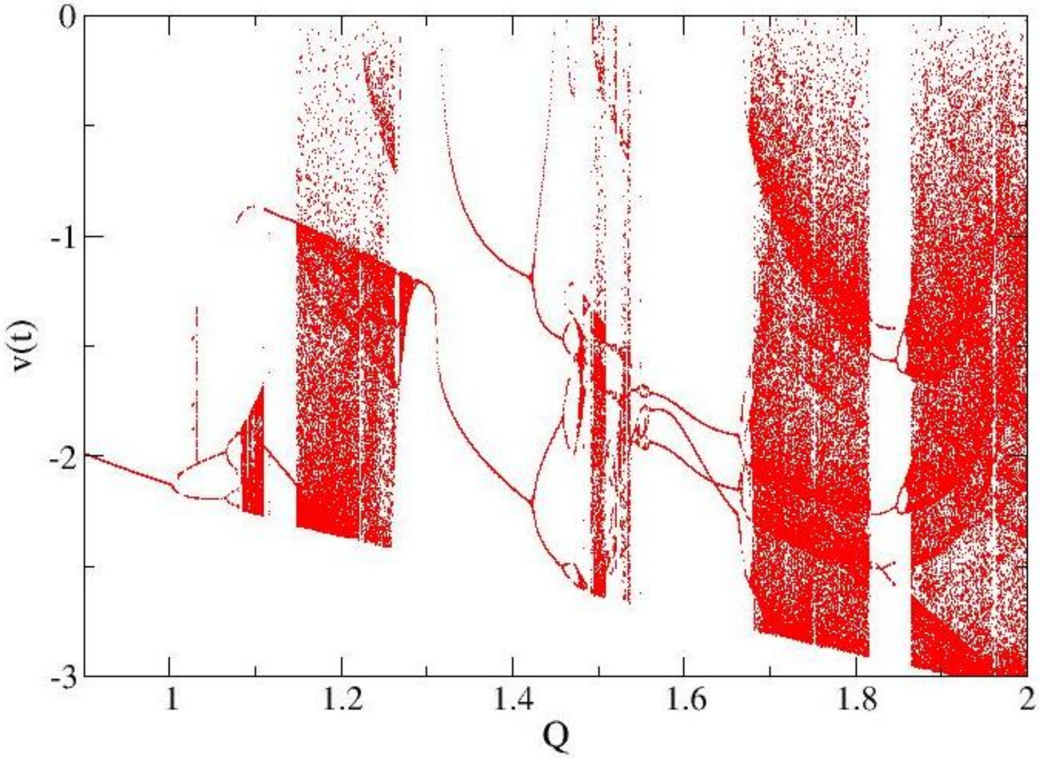
- other parameters fixed, same values as before



Visualizing basins of attraction; Poincare sections

Reducing the complexity of the phase space by graphing “cuts”

- e.g., plot velocity each time the pendulum gues through x=0 from above
- do this versus some system parameter, e.g., Q



Note 1: In some cases the transient may have been long-lived, showing up as “unexpected” vertical lines

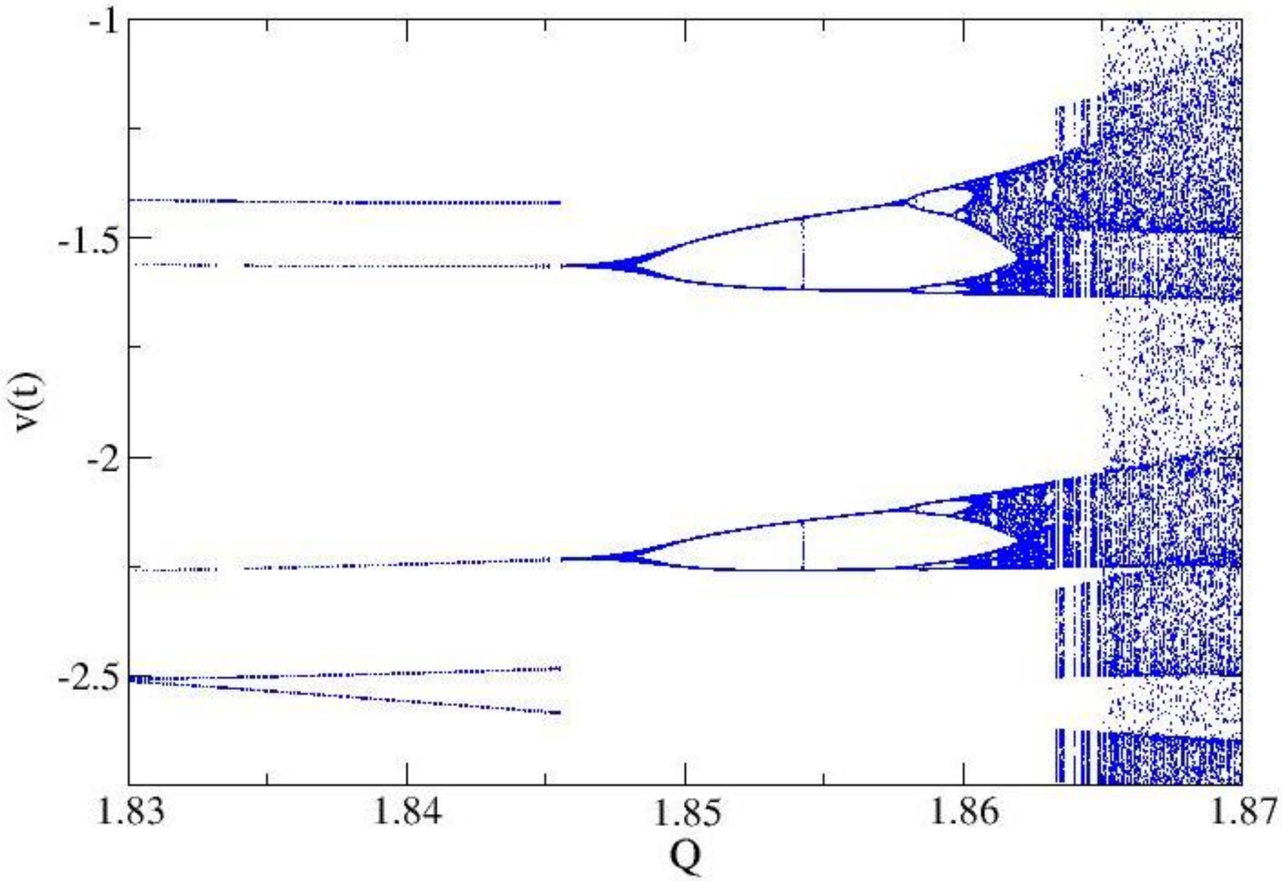
Note 2: different types of “dense regions”; how they look in a plot depends on the simulation time

Here we can see sequences of period doublings

- transitions to aperiodic motion; chaos

There are several interesting aspects, we focus on period doubling

Blow-up of Q=1.83-1.87 region



Note: Broadening of bifurcation points due to transients and integration errors

More careful work can show infinite sequence of period doublings - “route to chaos”

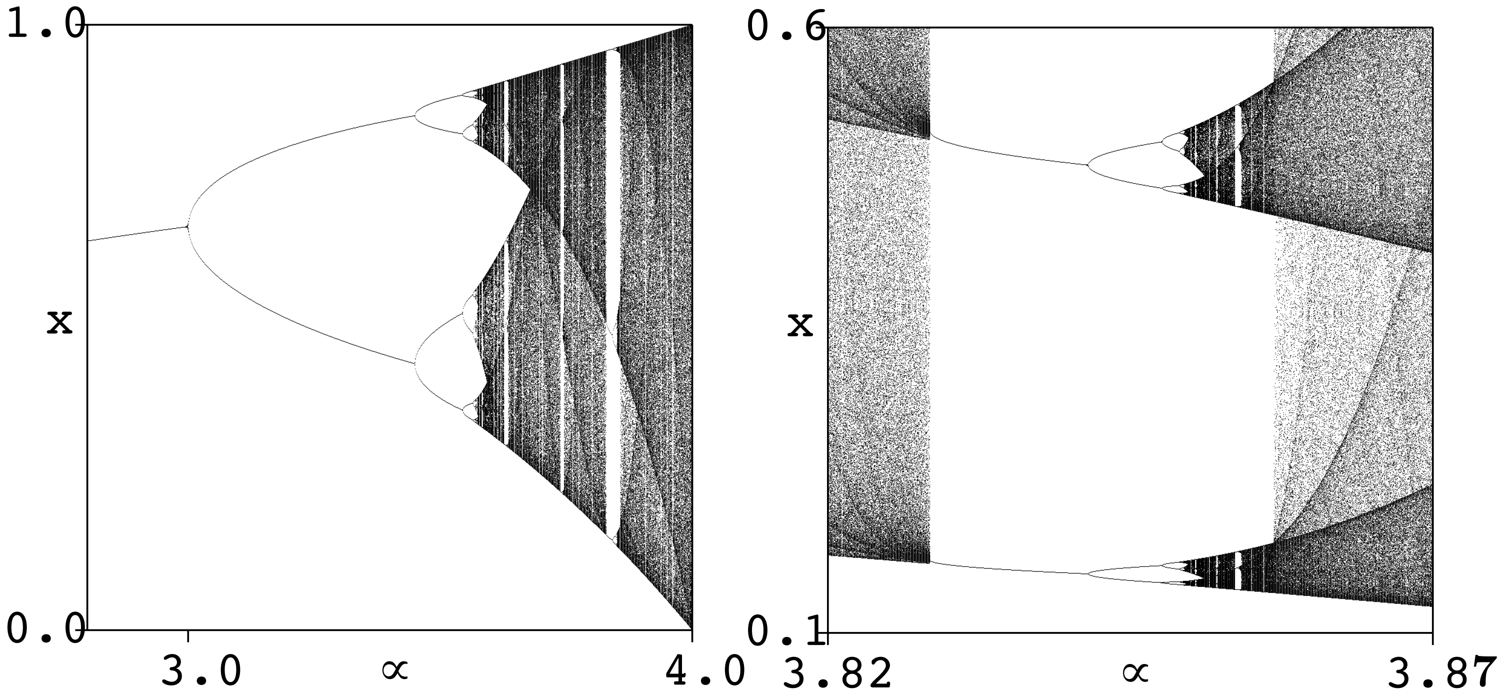
Bifurcations and chaos in a simpler setting - the Logistic map

Iterative map with a control parameter:

$$x_{n+1} = \mu x_n (1 - x_n) \quad x_0 \in (0, 1)$$

After transient, settles into periodic or aperiodic (chaotic) sequences

- bifurcations exactly as for pendulum (and many other systems)

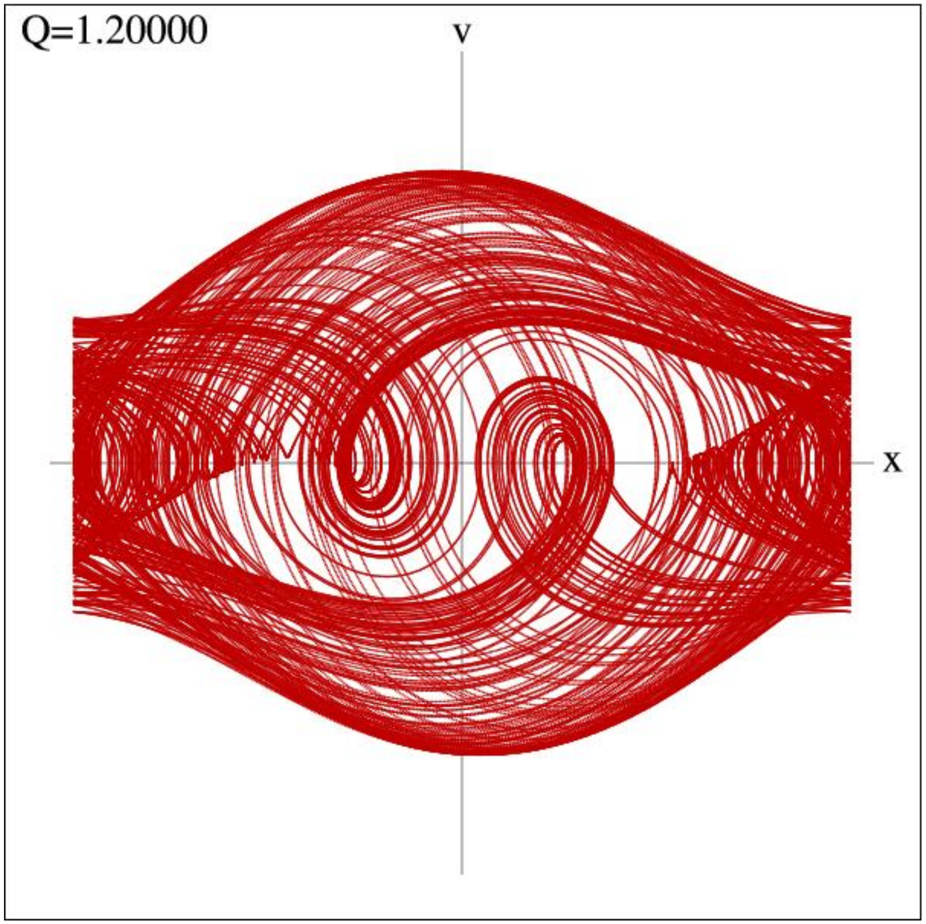


Universal behavior can be demonstrated rigorously (Feigenbaum)

- by analyzing differential equations

Attractor in the chaotic regime (often called “strange attractor”)

Running a very long time, some fractal density patterns emerge



Also, Julia animation for the Lorenz attractor (web site, examples)

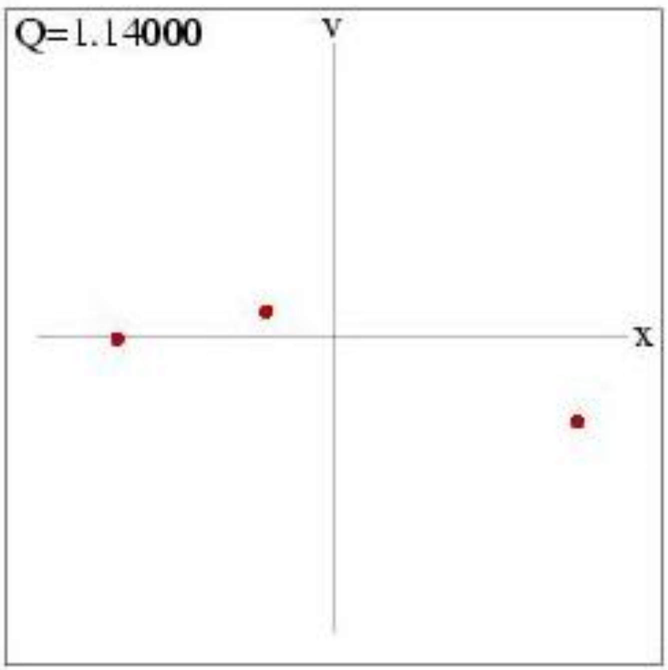
Alternative way to visualize trajectories: Stroboscopic sampling

Plot a point $[x(t_i),v(t_i)]$ for t_i being a multiple of the driving period

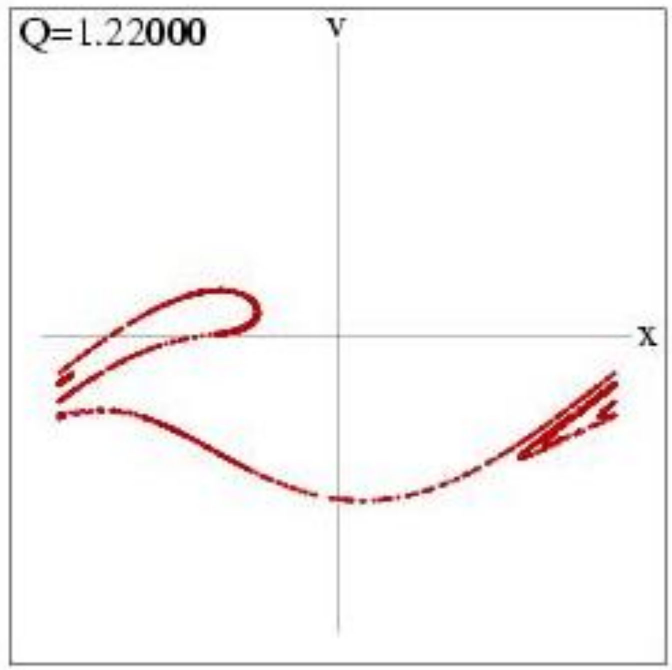
$$t_i = iT_P = i2\pi/\Omega \qquad i = 0, 1, 2, \ldots$$

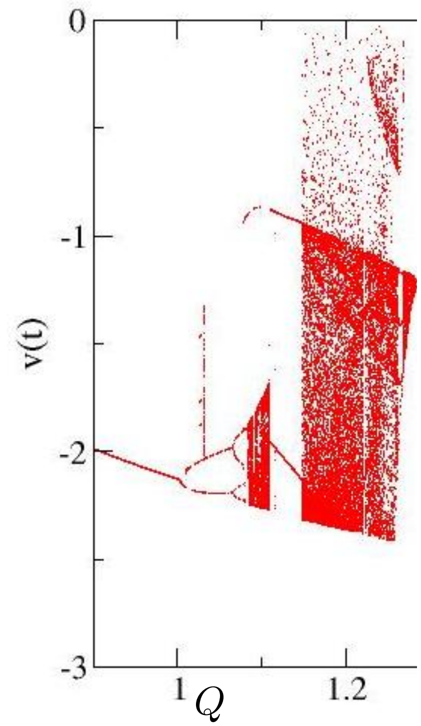
set $t_0 = 0$ after transient have decayed away

Periodic motion



chaotic motion





The strange attractor is more clearly fractal in this case

Stroboscopic sampling, the movie

- starring: red dots and blue dots (parity twins)