The Decibel Scale

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1. Review of Logarithms

The logarithm of a number is the power to which 10 must be raised to give that number. (In this discussion we always assume base 10 for the logarighms.) We can write that as

$$10^{\log\ x} = x \tag{1}$$

Logs are particularly useful if we are dealing with a quantity which varies over many powers of 10 (or orders of magnitude as scientists would say). While the primary example we have in mind here is the range of sound pressure levels over which the human ear can function, the level of light over which the eye can function, and the scale for earthquakes (Richter Scale) are two other examples of the use of a logarithmic scale.

Let's look at a few simple examples:

Example 1)

$$1000 = 10^3 \qquad \text{so} \qquad \log(1000) = 3$$

Example 2)

$$1,000,000,000 = 10^9 \qquad \text{so} \qquad \log(1,000,000,000) = 9$$

If we use the exponent of 10 rather than the number itself we can say 3 or 9 instead of the big numbers.

Logs have the property that, if x and y are two numbers, the log of the product is the sum of the individual logs, i.e.

$$\log(x \times y) = \log(x) + \log(y) \tag{2}$$

Aside: We can show this by using the definition of the logarithm. The definition of the logarithm is

$$10^{\log(y)} = y$$

then

$$xy = 10^{\log(xy)}$$

Now xy can also be written as $10^{\log(x)}10^{\log(y)}$ so we have the equality

$$10^{\log(xy)} = 10^{\log(x)}10^{\log(y)}$$

The right hand side of the above equation is just $= 10^{\log(x)+\log(y)}$ since the exponents add when we multiply exponentials together. Equating the exponents on the two sides of the equation gives

$$\log(xy) = \log(x) + \log(y)$$

This proves the rule that if we <u>multiply</u> two numbers together we <u>add</u> the logarithms.

We also have the properties that

$$\log\left(\frac{x}{y}\right) = \log(x) - \log(y) \tag{3}$$

and

$$\log\left(\frac{1}{x}\right) = -\log(x) \tag{4}$$

We could also show

$$\log(x^y) = y\log(x) \tag{5}$$

2. The Decibel Scale

The decibel scale is useful in defining ratios of numbers, which may vary over many orders of magnitude (powers of 10). The dB scale was invented by telephone engineers, and a deci Bell is 1/10 th of a Bell, which we define below. The scale is defined by

$$\beta = 10\log\left(\frac{I}{I_0}\right) \tag{6}$$

where β is in dB. An alternative way of writing this is

$$10^{\beta/10} = \frac{I}{I_0} \tag{7a}$$

or equivalently

$$I = I_0 10^{\beta/10} \tag{7b}$$

In all these cases I_0 is some reference intensity. In discussing sound intensity levels, the reference is *the threshold of hearing at 1000 Hz*, which corresponds to $I_0 = 10^{-12} W/m^2$. We emphasize that I_0 is really arbitrary and the dB scale is used in many places, especially in sound recording and electrical engineering. Let's look at some examples.

Example 3) What intensity ratio is implied by 64 dB ?

From equation (7a) we have

$$\frac{I}{I_0} = 10^{\beta/10} = 10^{6.4} = \left(10^{0.4}\right)\left(10^{6}\right)$$

The only catch is: "What is $10^{0.4}$?" We need a table.

x (or $\log x$)	10^x (or x)
0.0	1.0
0.1	1.3
0.2	1.6
0.3	2.0
0.4	2.5
0.5	3.2
0.6	4.0
0.7	5.0
0.8	6.3
0.9	7.9
1.0	10.0

We could have labeled the columns as $\log x$ and x, or as level difference in dB and intensity ratio, these are equivalent to each other. We return to the example. From the table we see that $10^{0.4} = 2.5$, so

$$\frac{I}{I_0} = 2.5 \times 10^6 = 2,500,000$$

Example 4) Suppose the ratio was

$$\frac{I}{I_0} = 63,000 = 6.3 \times 10^4$$

How many dB does this represent?

From the table we find $6.3 = 10^{0.8}$, so we have

$$\frac{I}{I_0} = \left(10^{0.8}\right)\left(10^{4}\right) = 10^{4.8}$$

which from eq. (7a) means 48 dB.

Sometimes it is convenient to relate ratios of amplitudes rather than intensities. In electronics we measure amplitudes of wave forms coming out of an amplifier, not the intensities. We convert to amplitudes by realizing that the intensity is proportional to the amplitude squared. We can write that as

$$I = CA^2$$

where C is a constant factor. Thus

$$\frac{I}{I_0} = \frac{CA^2}{CA_0^2} = \frac{A^2}{A_0^2},$$

where A_0 is the amplitude of the reference (corresponding to I_0. What about the conversion to dB? Well,

$$\beta = 10\log\left(\frac{I}{I_0}\right) = 10\log\left(\frac{A^2}{A_0^2}\right) = 10\log\left(\frac{A}{A_0}\right)^2$$

but from eq (5) we get

$$\beta = (10)\,(2)\log\left(\frac{A}{A_0}\right)$$

which is just

$$\beta = 20\log\left(\frac{A}{A_0}\right) \tag{8}$$

One very common factor we encounter is 6 dB. Let's ask what this means in terms of amplitude ratios.

Now 6 dB means

$$\frac{I}{I_0} = 10^{0.6} = 4$$

so

$$\frac{A}{A_0} = \sqrt{\frac{I}{I_0}} = 2$$

6 dB represents an <u>amplitude</u> change of 2 (and intensity change of 4).

Another common factor is 3 dB.

$$\frac{I}{I_0} = 10^{0.3} = 2$$

so 3 dB is an <u>intensity</u> change of 2 (and an amplitude change of $\sqrt{2} = 1.414$).

In measuring sound levels it is common to speak of the sound pressure level, SPL. The sound intensity level, SIL, can be related to the sound pressure level by realizing that the intensity, I, is proportional to the max pressure squared P_m^2 (where there is no reflected sound),

$$I = (const)\, P_m^{2.}$$

In terms of ratios

$$\frac{I}{I_0} = \frac{(const)\, P_m^2}{(const)\, P_{m_0}^2} = \frac{P_m^2}{P_{m_0}^2}$$

We drop the subscript m (remembering that we always mean the maximum when we write P_0 and substitute into eq (6) to obtain

$$\beta = 10 \log \left(\frac{P^2}{P_0^2} \right) = 10 \log \left(\frac{P}{P_0} \right)^2 = 20 \log \left(\frac{P}{P_0} \right) \tag{8b}$$

or

$$\frac{P}{P_0} = 10^{\beta/20} \tag{9}$$

If the sound is reverberating in a closed room, then the SPL and SIL may differ from each other by several dB.

3. Addition of two sounds

In acoustics, we often are asked the question: If two sounds are present with intensities I_1 and I_2, what is the resulting sound level? There are two distinct cases: when the two sounds are coherent, and when they are incoherent. Coherent sources have exactly the same frequency and a definite phase relationship between the two waves, whereas incoherent sources do not. An example of coherent sources is two speakers with the amp set to mono. Another is direct and reflected sound from the same source. An example of two incoherent sources is two trumpet players trying to play in unison.

Coherent sources have the possibility of interfering – either destructively or constructively. Incoherent sources do not. If two sources are coherent, we must add the amplitudes of the individual waves and then square them to get the resulting intensity. If they are incoherent, we just add intensities.

In the following section, we derive formulas for the resulting sound level for both cases. First we consider incoherent addition, then coherent.

3.1. Incoherent Addition

If the sound sources are incoherent, (i.e., there is no definite (constant) phase difference and the frequencies are not equal) then one adds intensities.

Example 5) Consider the two sound intensity levels:

$$SIL_1 = 63\ dB \qquad \text{and} \qquad SIL_2 = 70\ dB$$

There are two ways to obtain the answer: one can convert from dB to W/m^2 and add the intensities directly and then convert back to dB, or one can use ratios, and skip the conversion to and from W/m^2.

¡¡¡ NEVER ADD THE SOUND INTENSITY LEVELS DIRECTLY!!!

3.1.1. Direct Addition of Intensities

We convert to W/m^2, add and convert back to dB. From the definition of I_0, ($I_0 = 10^{-12}\ W/m^2$) we get,

$$\frac{I_1}{I_0} = 10^{6.3}$$

so

$$I_1 = \left(10^6\right)\left(10^{0.3}\right) I_0 = 2 \times 10^6 \times 10^{-12}\ W/m^2$$

and

$$I_1 = 2 \times 10^{-6}\ \mathrm{W/m^2}$$

We can calculate I_2 in a similar fashion:

$$\frac{I_2}{I_0} = 10^{7.0}$$

so

$$I_2 = 10^7 \times 10^{-12}\ W/m^2$$

$$= 1 \times 10^{-5}\ W/m^2$$

or equivalently

$$I_2 = 10 \times 10^{-6}\ W/m^2$$

To get the combined intensity, we just add these two numbers,

$$I_1 + I_2 = (2 + 10) \times 10^{-6}\ \mathrm{W/m^2}$$

$$= 12 \times 10^{-6}\ W/m^2 = 1.2 \times 10^{-5}\ W/m^2$$

The combined level $I_c = I_1 + I_2$ is converted to dB as:

$$\frac{I_c}{I_0} = \frac{I_1 + I_2}{I_0} = \frac{1.2 \times 10^{-5}\ W/m^2}{10^{-12}\ W/m^2}$$

which can be written as

$$\frac{I_c}{I_0} = 1.2 \times 10^7 = \left(10^{0.08}\right)\left(10^7\right) = 10^{7.08}$$

So the resulting sound level, $SIL_c = 70.8\ dB$. Note that if you just had the table available, you would have to say $71 dB$.

3.1.2. The Use of Ratios Directly

Example 6) We use the intensity ratios directly to obtain the answer, rather than converting to W/m^2.

$$I_1 = 63dB$$

$$I_2 = 70dB$$

The SIL of 63 and 70 dB imply the intensity ratios:

$$\frac{I_1}{I_0} = 10^{6.3} \quad \text{and} \quad \frac{I_2}{I_0} = 10^{7.0}$$

We can express I_1 in terms of I_2 by taking the quotient of these two ratios (where I_0 will divide out), and we get

$$\frac{I_1/I_0}{I_2/I_0} = \frac{10^{6.3}}{10^{7.0}}$$

which gives

$$\frac{I_1}{I_2} = 10^{-0.7}$$

Remember that

$$10^{-0.7} = \frac{1}{10^{0.7}}$$

which we find from the table is

$$\frac{I_1}{I_2} = \frac{1}{5} = 0.2,$$

so we get

$$I_c = I_1 + I_2 = 0.2I_2 + I_2 = 1.2I_2$$

We now go back the the definition of the Sound Intensity Level (β) in eq. (6).

$$\beta_c = 10 \log \frac{I_c}{I_0}$$

and we obtain

$$\beta_c = SIL_c = 10\log\left(\frac{1.2 I_2}{I_0}\right)$$

Now,

$$10\log\left(\frac{1.2 I_2}{I_0}\right) = 10\log\frac{I_2}{I_0} + 10\log\left(1.2\right).$$

The first term on the right hand side is just SIL_2, so we can rewrite the expression for SIL_c as

$$SIL_c = SIL_2 + 10\log\left(1.2\right)$$

Now $\log(1.2) = 0.08$ so we get

$$SIL_c = 70dB + 10\left(.08\right)dB$$

Finally we get

$$SIL_c = 70.8dB$$

as before.

The above example suggests a general procedure which we develop below.

3.2. Most General Formulation for the Addition of SIL

We use ratios. We state the problem as follows.

$$SIL_1 \;=\; \beta_1\; dB \qquad SIL_2 \;=\; \beta_2\; dB$$

we wish to know the combined sound intensity level SIL_c The intensity of the combined sound is given by

$$I_c \;=\; I_1 \;+\; I_2 \;=\; I_1 \;+\; \frac{I_2}{I_1} I_1$$

$$=\; I_1 \left(1 \;+\; \frac{I_2}{I_1}\right)$$

$$\text{Now } \frac{I_2}{I_1} \;=\; \frac{I_2/I_0}{I_1/I_0} \;=\; \frac{10^{\beta_2/10}}{10^{\beta_1/10}} \;=\; 10^{(\beta_2-\beta_1)/10}$$

so $I_c = I_1(1 + 10^{(\beta_2-\beta_1)/10})$ Because the SIL_c is a log we get from the definition

$$\beta_c \;=\; 10\log\frac{I_c}{I_0} \;=\; 10\log\left[\frac{I_1}{I_0}\left(1 + 10^{(\beta_2-\beta_1)/10}\right)\right]$$

remember that the log of a product is the sum of the logs (see eq. (2)), so

$$10\log[\qquad] \;=\; 10\log\left(\frac{I_1}{I_0}\right) \;+\; 10\log\left(1 \;+\; 10^{(\beta_2-\beta_1)/10}\right)$$

The first term is just β_1 so we get

$$\beta_c \;=\; \beta_1 \;+\; 10\log\left(1 \;+\; 10^{(\beta_2-\beta_1)/10}\right)$$

or equivalently

$$\boxed{SIL_c \;=\; SIL_1 \;+\; 10\log\left(1 \;+\; 10^{(\beta_2-\beta_1)/10}\right)} \qquad (10)$$

Note that if the two levels are equal, $\beta_1 = \beta_2$ we get

$$SIL_c = SIL_1 + 10\log(2)$$

which corresponds to

$$SIL_c = SIL_1 + 3dB$$

3.2.1. Coherent Addition

If there is a definite phase relationship between the two and they are the same frequency, then they are <u>coherent</u>. We must **add amplitudes first and then square** to get the resultant intensity. We have to also consider whether they are adding constructively or destructively.

$$\frac{A_1}{A_0} \;=\; \sqrt{\frac{I_1}{I_0}} \qquad \frac{A_2}{A_0} \;=\; \sqrt{\frac{I_2}{I_0}}$$

If we use A_+ for constructive interference

A_- for destructive interference

$$A_+ \;=\; A_1 \;+\; A_2 \qquad I_+ \;=\; {A_+}^2$$

$$A_- \;=\; A_1 \;-\; A_2 \qquad I_- \;=\; {A_-}^2$$

Now

$$\frac{I_\pm}{I_0} \;=\; \left(\frac{A_1 \pm A_2}{A_0}\right)^2 \;=\; \frac{1}{Ao^2}\left({A_1}^2 \;+\; {A_2}^2 \;\pm\; 2A_1A_2\right)$$

$$=\; \frac{{A_1}^2}{{A_0}^2} \;=\; \left(1 \;+\; \frac{{A_2}^2}{{A_1}^2} \;\pm\; \frac{2A_1A_2}{{A_1}^2}\right)$$

$$= \frac{{A_1}^2}{{A_0}^2} = \left(1 + \frac{{A_2}^2}{{A_1}^2} \pm 2\frac{A_2}{A_1}\right)$$

but

$$\left(\frac{A_1}{A_0}\right)^2 = \frac{I_1}{I_0} \ , \ \frac{{A_2}^2}{{A_1}^2} = \frac{I_2}{I_1} \ \text{and} \ \frac{A_2}{A_1} = \sqrt{\frac{I_2}{I_1}}$$

so

$$\frac{I_\pm}{I_0} = \frac{I_1}{I_0}\left(1 + \frac{I_2}{I_1} \pm 2\sqrt{\frac{I_2}{I_1}}\right)$$

This looks like our result from before except we have an additional term. We apply the definition (eq. (6)) of the dB level:

$$10\log\left(\frac{I_\pm}{I_0}\right) = 10\log\left[\frac{I_1}{I_0}\left(1 + \frac{I_2}{I_1} \pm 2\sqrt{\frac{I_2}{I_1}}\right)\right]$$

$$\beta_\pm = 10\log\left(\frac{I_1}{I_0}\right) + 10\log\left\{1 + \frac{I_2}{I_1} \pm 2\sqrt{\frac{I_2}{I_2}}\right\}$$

but as before

$$\frac{I_2}{I_1} = 10^{(\beta_2-\beta_1)/10}$$

and

$$\sqrt{\frac{I_2}{I_1}} = 10^{(\beta_2-\beta_1)/20}$$

so

$$\beta_\pm = \beta_1 + 10\log\left\{1 + 10^{(\beta_2-\beta_1)/10} \pm 2\left(10^{(\beta_2-\beta_1)/20}\right)\right\}$$

$$\boxed{SIL_\pm = SIL_1 + 10\log\left\{1 + 10^{(\beta_2-\beta_1)/10} \pm 2\left(10^{(\beta_2-\beta_1)/20}\right)\right\}} \quad (11)$$

Note that if $\beta_2 = \beta_1$ the term in brackets becomes

$$1 + 10^o \pm 2\left(10^o\right)$$

$$2 \pm 2.$$

For destructive interference the term in brackets is $2 - 2 = 0$ and

$$SIL_c = SIL_1 + 10\log\{0\}$$

Now, as (x) goes to zero, the $\log(x)$ goes to negative infinity, so this result tells us that the SIL_c goes to negative infinity, i.e. it tells us that you cannot hear it. This is what you expect for two equal amplitude coherent sounds adding together for complete destructive interference.

For constructive interference, SIL_+, the term in brackets becomes $2 + 2 = 4$ so

$$SIL_+ = SIL_1 + 10 \log (4)$$

$$SIL_+ = SIL_1 + 6dB$$

Remember that if the two sources are incoherent we get:

$$SIL_c = SIL_1 + 10 \log (2) = SIL_1 + 3dB$$