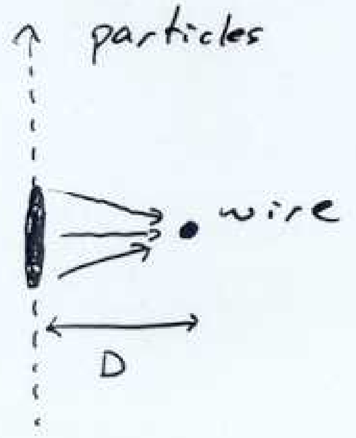
# Real World Measurement

1) Hits on drift chamber wire



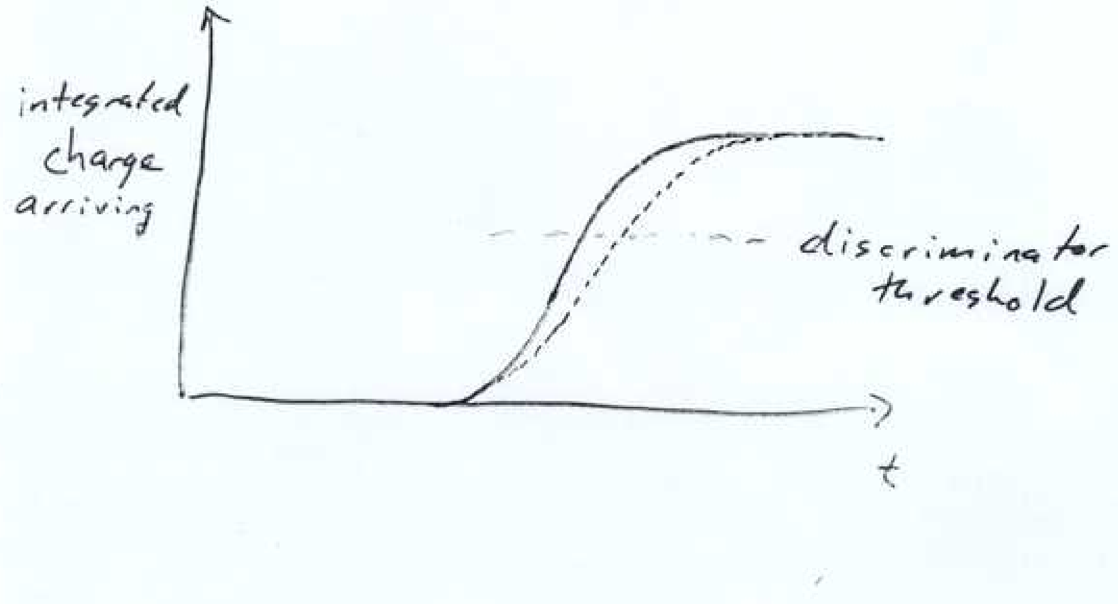

Suppose we can send an infinite number of particles on exactly the same path, and that we know the distance D from the trajectory to the sense wire.

The charged particle passing through the chamber ionizes the gas, and an electric field drifts the electrons towards the wire. An avalanche occurs just before the ~~wire~~ charge cloud reaches the wire.

The time at which the charge arrives at the wire is converted to a distance using the known drift velocity.

Now we fire a huge number of particles along the path, and measure the distance $d$ of the path from the wire

What do we expect?

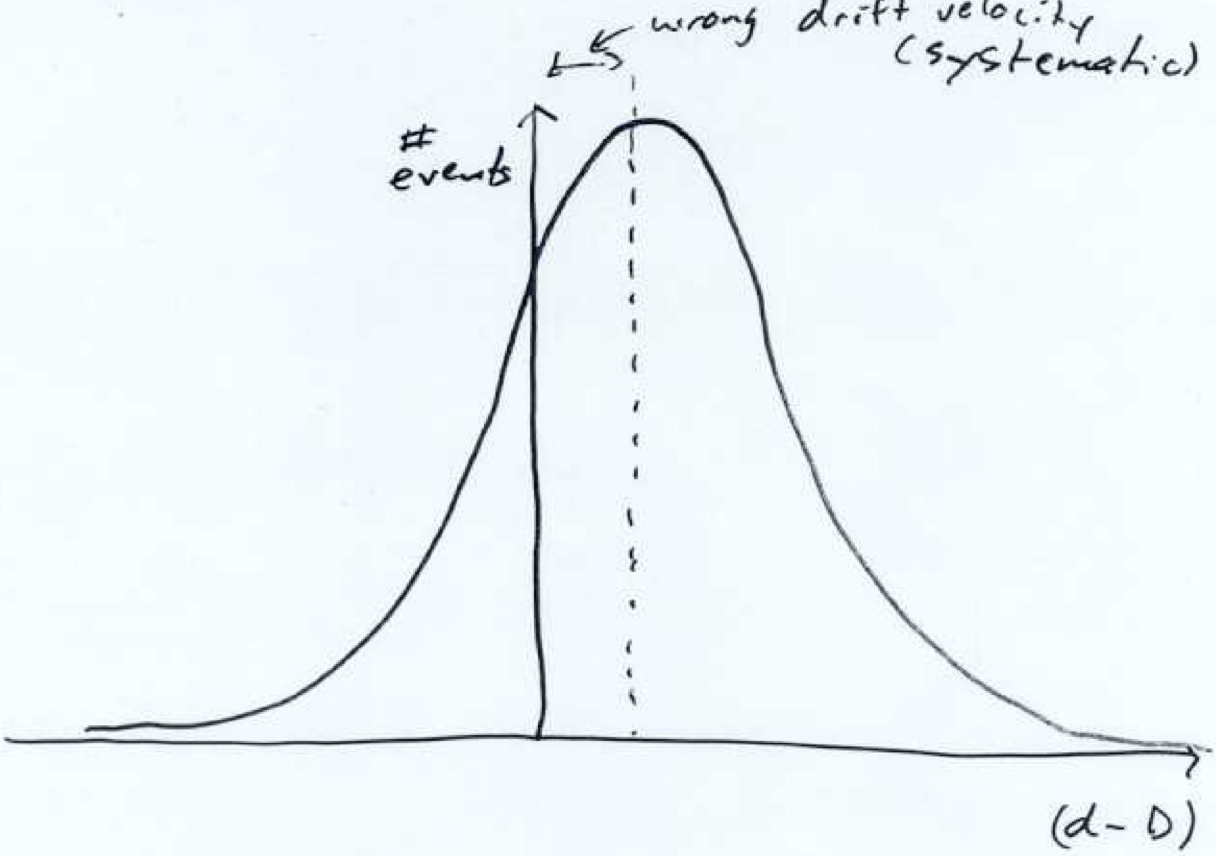


The distance measurement will not be perfect because of:

- fluctuations in the drift length of electron cloud from one track to another
- fluctations in the cloud size $\Rightarrow$ e.g. less charge total $\Rightarrow$ discriminator fires later on curve $\Rightarrow$ time shift
- jitter in the starting point of the cascade changes total charge
- jitter in discriminator threshold shifts time
- drift velocity may wander (gas composition, field strength)
- drift velocity may be wrong

Plot (d − D) for many tracks.

Expect




Ideally we would like to know the full curve, which is the p.d.f. for the measurement d.

In general this is too much information. (Unless the p.d.f. is one of the specials we've seen)

1) We want to know the location of the peak - This allows us to make a drift velocity correction. After this, we will get the correct $d$ on average.

Measure the Mean of the $\text{dis}^{n}$

i.e. $\mu = E(d)$

2) We want to know how wide the distribution is. This characterizes (partially) how well our detector measures the position.

Look at various moments of the distribution. Set $f(d)$ = p.d.f.

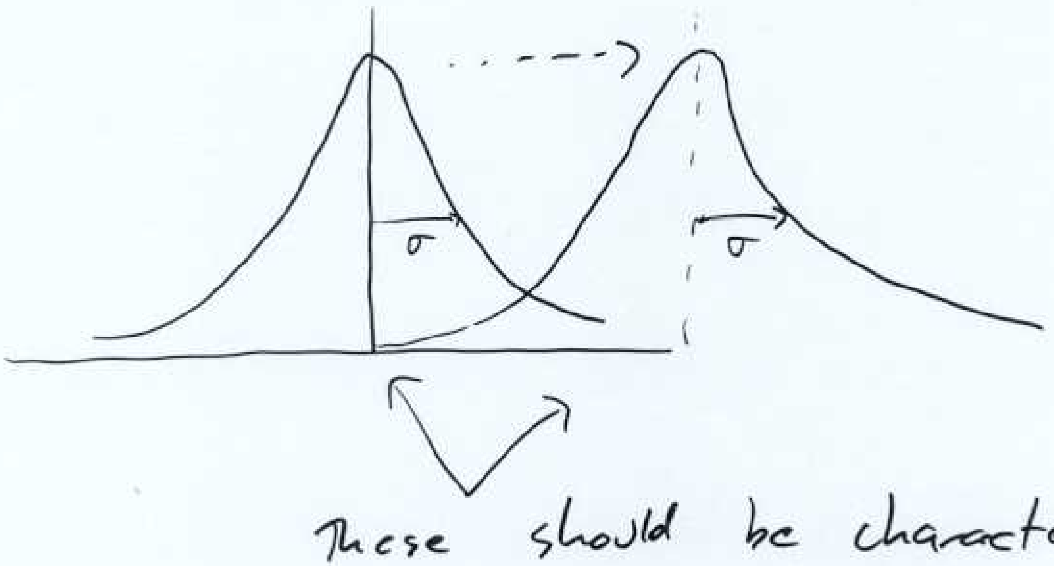
$0^{th}$ : $\int x^0 f(x)\,dx = 1$ not much information

$1^{st}$ : $\int x\, f(x)\,dx = \mu$ mean ✓

$2^{nd}$ : $\int x^2 f(x)\,dx$

- Good, but sensitive to translations of the p.d.f. We want to characterize the width, so we refer to the mean:

$2^{nd}{}'$ : $\int (x-\mu)^2 f(x)\,dx$ variance ✓




These should be characterized by the same <u>width</u>, different <u>means</u>

But the units of $V(d)$ are $[\ell]^2$
so we take the $\sqrt{V} = \sigma$ to
characterize the width

If the p.d.f. is normal the
1st + 2nd moments, or $\mu + \sigma$
completely characterize the detector!

3rd : $\int x^3 f(x) dx$

or $\int (x-\mu)^3 f(x) dx$

- Measures asymmetry of p.d.f.

We conventionally quote only the mean + sigma of a distribution, though the $3^{rd}$ + $4^{th}$ moments might be important (skewness + kurtosis)

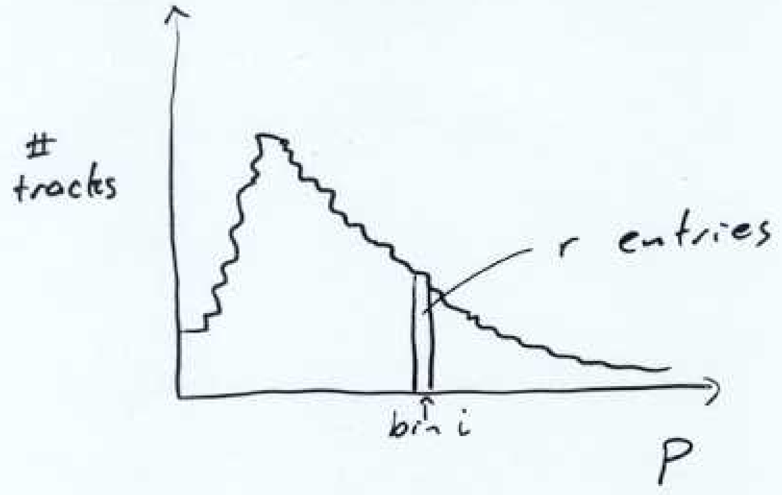
In our case the width of the distribution $\sigma$, which character how well the detector works, is called the <u>resolution</u>.

Note we could have called $2\sigma$ or $\frac{\sigma}{2}$ the resolution. It is just a (natural) convention.

# Histogram Bins

Suppose we histogram the momentum spectrum of tracks in $B\bar{B}$ events

We take a sample of 100 tracks + get a histogram such as:




Now we look at bin $i$ and see it has $r$ entries.
That is for one set of 100 events.
What do we know about what we expect from another 100 similar tracks?

We know that the number of events in that bin should follow a binomial distribution

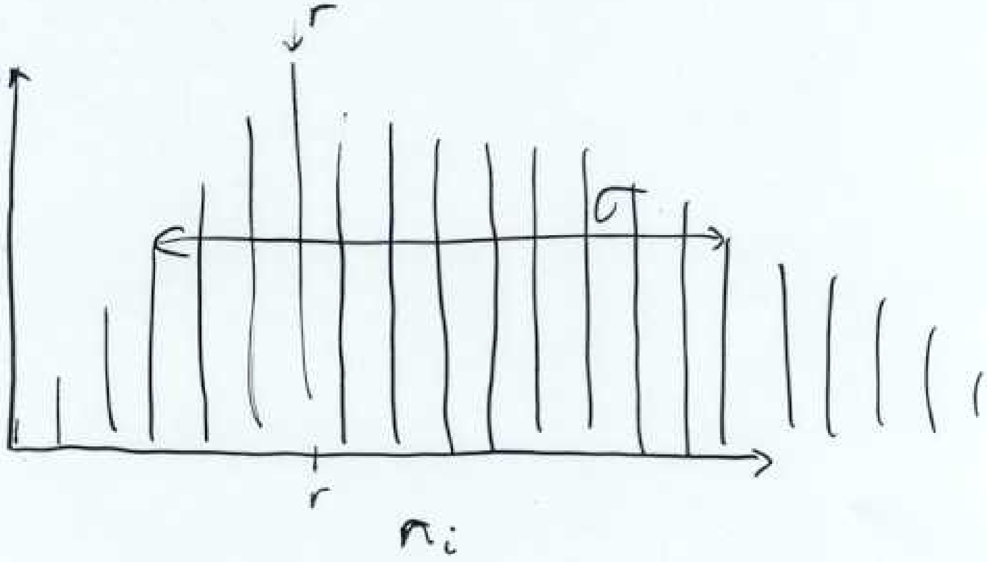
This means that if

$p$ = probability that any given track has a momentum which falls in bin $i$, then the probability of getting $r$ entries in that bin is

$$B(r; 100, p) = \frac{100!}{(100-r)!\,r!} p^r (1-p)^{100-r}$$

We don't know $p$, but let's suppose $p = \frac{r}{100}$. It is unlikely this is too far wrong.

Now I take millions + millions of 100 track samples, and for each sample plot the number of entries in bin $i$. ($n_i$)




<u>If</u> $p$ really was $\frac{r}{100}$, the distribution of $n_i$ would look something like the above

Back to our first histogram with the first 100 tracks. We see from the above plot that if $r$ really was a typical number of events to expect in bin $i$, that we could have easily obtained $r-1$ events, or $r+1$, $r+2$ etc. The probabilities for obtaining these numbers of entries are not substantially different from that at $r$.

Eventually, at very low or large $n_i$ values, the probabilities have dropped significantly.

We need a way to characterize how far we have to go from $r$ (our central value or mean) before we consider the result unlikely.

Conventionally one uses

$\sigma = \sqrt{V(r)}$ as this measure.

We say we have

$r \pm \sigma$ entries in the bin, indicating that any number of entries betwen $r-\sigma$ and $r+\sigma$ should be considered probable values.

$\sigma$ is called the statistical error on the number of entries in the bin.

# Samples

Let $x$ be a random variable with pdf $f(x)$

For eg, suppose $x$ is the measurement of some physical quantity. This means that if we performed an infinite number of measurements, they will be distributed as $f(x)$.

Def^n A set of $n$ measurements $\{x_i\}$ $i=1,n$ of the quantity $x$ is called a sample of size $n$.

# Samples

Let $x$ be a random variable with pdf $f(x)$

For eg, suppose $x$ is the measurement of some physical quantity. This means that if we performed an infinite number of measurements, they will be distributed as $f(x)$.

Def$^n$ A set of $n$ measurements $\{x_i\}$ $i=1,n$ of the quantity $x$ is called a sample of size $n$.

We ask:

what is the distribution of results $\bar{d}$ from a large number of independent samples of size $n$?

As usual, we characterize the properties of this distribution with its mean + vari

$$E(\bar{d}) = \frac{1}{n} \sum_i E(d_i)$$

Define $\left.\begin{array}{l} E(d) = \mu \\ V(d) = \sigma^2 \end{array}\right\}$ we don't know thes

$$E(\bar{d}) = \frac{1}{n} \sum_i \mu = \mu$$

i.e. The mean of the variable $\bar{d}$ is the true mean of the p.d.f. !!

$s^2$ as estimate of $\sigma^2$:

Let's look at the mean of the distribution of $s^2$!

$$E(s^2) = E\left(\frac{1}{n-1} \sum_i \left(d_i - \frac{1}{n}\sum d_j\right)^2\right)$$

$$= \frac{1}{n-1} E\left(\sum_i \left(\frac{n-1}{n} d_i - \frac{1}{n}\sum_{j \neq i} d_j\right)^2\right)$$

$$= \frac{1}{n-1} E\left(\sum_i \left[\left(\frac{n-1}{n}\right)^2 d_i^2 - \frac{2(n-1)}{n^2} d_i \sum_{j\neq i} d_j\right.\right.$$

$$\left.\left. + \frac{1}{n^2} \sum_{j \neq i} \sum_{\substack{k \neq i \\ k \neq j}} d_j d_k + \frac{1}{n^2} \sum_{j \neq i} d_j^2 \right]\right)$$

$$= \frac{1}{n-1} \sum_i \left[ \left(\frac{n-1}{n}\right)^2 E(d_i^2) - \frac{2(n-1)}{n^2} \sum_{j \neq i} E(d_i d\right.$$

$$\left. + \frac{1}{n^2} \sum_{j \neq i} \sum_{\substack{k \neq i \\ k \neq j}} E(d_j d_k) + \frac{1}{n^2} \sum_{j \neq i} E(d_j^2) \right]$$

If the $d_i$ are independent,

$$E(d_i d_j) = E(d_i) E(d_j) = \mu^2$$

Also, $E(d_i^2) = \sigma^2 + \mu^2$

$$\Rightarrow E(s^2) = \frac{1}{n-1} \sum_i \left[ \left(\frac{n-1}{n}\right)^2 (\sigma^2 + \mu^2) - \frac{2(n-1)^2}{n^2} \mu^2 \right.$$

$$\left. + \frac{(n-1)(n-2)}{n^2} \mu^2 + \frac{n-1}{n^2} (\sigma^2 + \mu^2) \right]$$

$$= \frac{n}{n-1} \left( \frac{(n-1)^2 + n - 1}{n^2} (\sigma^2 + \mu^2) \right.$$

$$\left. + \frac{(n-1)(n-2) - 2(n-1)^2}{n^2} \mu^2 \right)$$

$E(s^2) = \sigma^2 + \mu^2 - \mu^2 = \sigma^2$

$\Rightarrow$ The <u>mean</u> of the variable $s^2$ is the true value of the detector resolution, $\sigma^2$

$\bar{d}$ + $s^2$ are called estimators. These particular ones are unbiassed, i.e. their means are the true means.

Now, back to our first sample of n tracks: we have estimated the true distance of the track to wire by $\bar{d} = \frac{1}{n} \Sigma d_i$ from this one sample. How accurate is this estimate? We need to know $V(\bar{d})$, i.e. the width of the probability $dis^n$ of $\bar{d}$.

$$V(\bar{d}) = E(\bar{d}^2) - E(\bar{d})^2$$

$$= E(\bar{d}^2) - \mu^2$$

$$= E\left(\left[\frac{1}{n}\sum_i d_i\right]^2\right) - \mu^2$$

$$= \frac{1}{n^2} E\left( \sum_i d_i^2 + \sum_i \sum_{j \neq i} d_i d_j \right) - \mu^2$$

$$= \frac{1}{n^2} \left( \sum_i E(d_i^2) + \sum_i \sum_{j \neq i} E(d_i d_j) \right) - \mu^2$$

$$= \frac{1}{n^2} \left( \sum_i (\sigma^2 + \mu^2) + \sum_i \sum_{j \neq i} \mu^2 \right) - \mu^2$$

$$= \frac{1}{n^2} \left( n(\sigma^2 + \mu^2) + n(n-1)\mu^2 \right) - \mu^2$$

$$= \frac{1}{n} \left( \sigma^2 + n\mu^2 \right) - \mu^2$$

$$= \frac{\sigma^2}{n}$$

Hence $\sigma_{\bar{d}} = \frac{\sigma}{\sqrt{n}}$ = error on $\bar{d}$