The Dynamics of Persuasion

Sid Redner, Santa Fe Institute (physics.bu.edu/~redner)

CIRM, Luminy France, January 5-9, 2015

T. Antal, E. Ben-Naim, P. Chen, P. L. Krapivsky, M. Mobilia, V. Sood, F. Vazquez, D. Volovik

+ support from NSF

Modeling Consensus:

- introduction to the voter model
- voter model on complex networks
- voting with some confidence
- majority rule

Modeling Discord & Diversity:

- 3-state voter models
- strategic voting
- bounded compromise
- dynamics of social balance
- Axelrod model

lecture 1

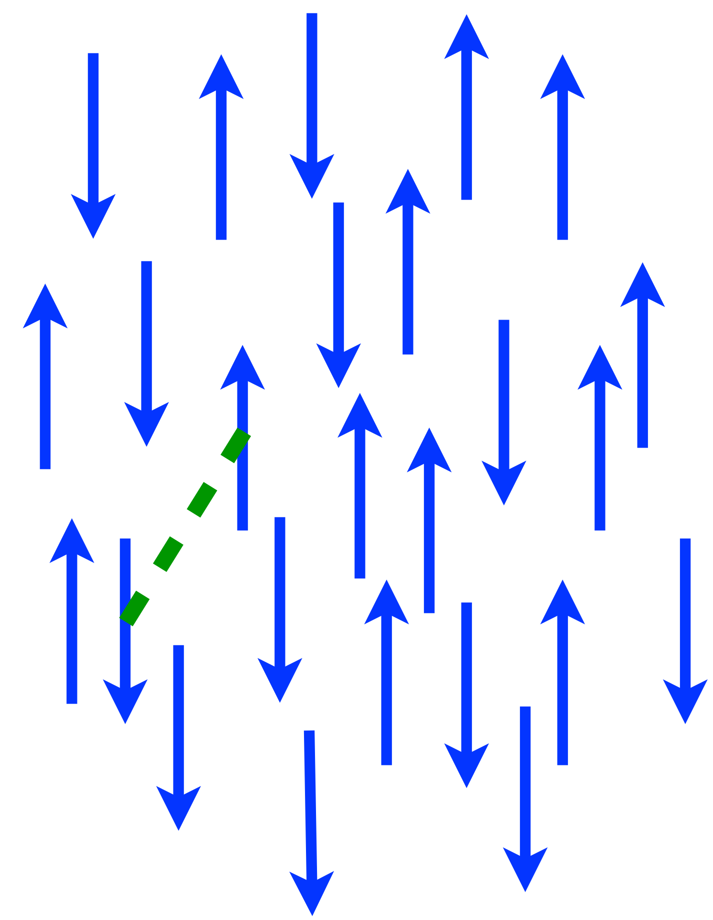
lecture 2

lecture 3

Persuasion Dynamics

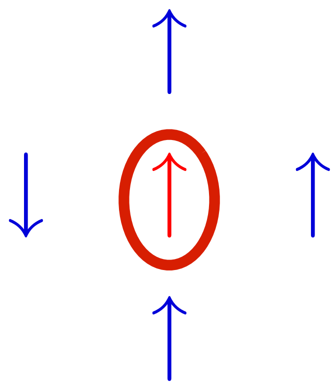
Real People

People as interacting "atoms"



Voter Model

Clifford & Sudbury (1973)
Holley & Liggett (1975)



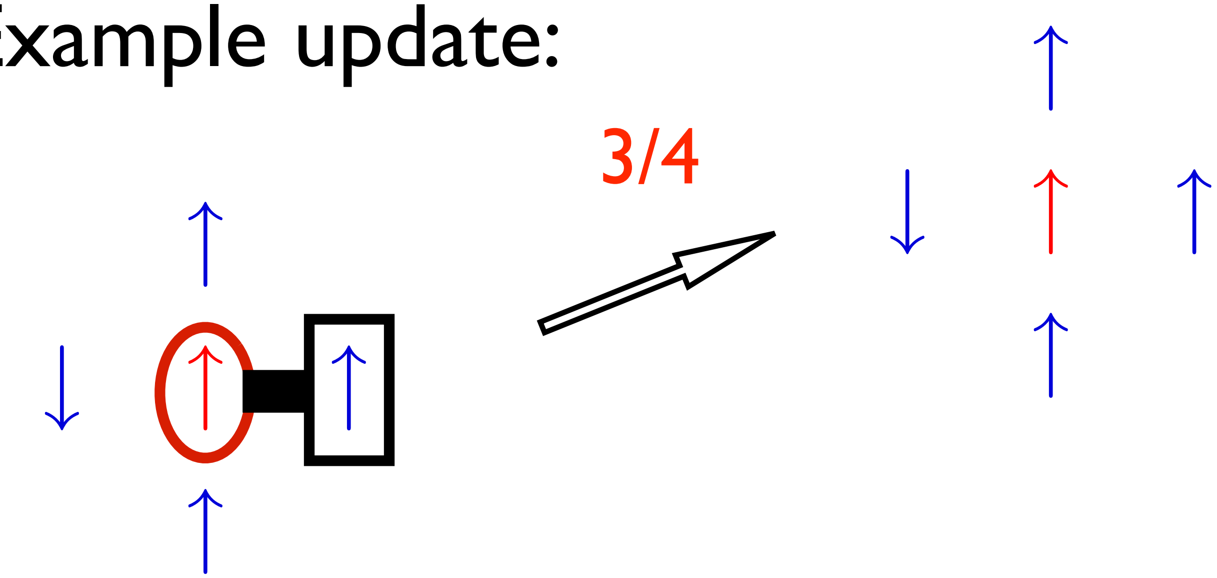
0. Binary voter variable at each site i
1. Pick a random voter
2. Assume state of randomly-selected neighbor

   individual has no self-confidence & adopts neighbor's state

Voter Model

Clifford & Sudbury (1973)
Holley & Liggett (1975)

Example update:



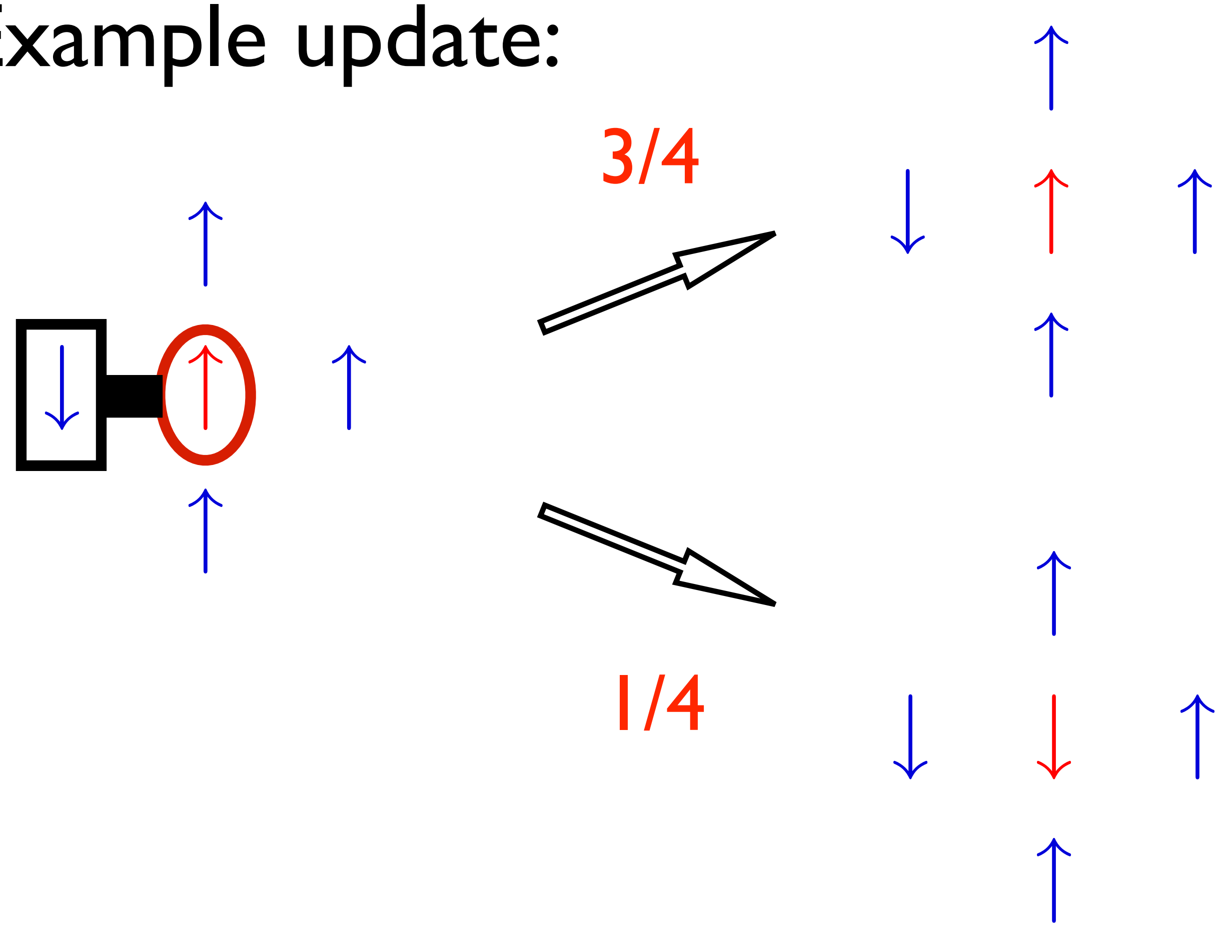
0. Binary voter variable at each site i
1. Pick a random voter
2. Assume state of randomly-selected neighbor

individual has no self-confidence & adopts neighbor's state

Voter Model

Clifford & Sudbury (1973)
Holley & Liggett (1975)

Example update:



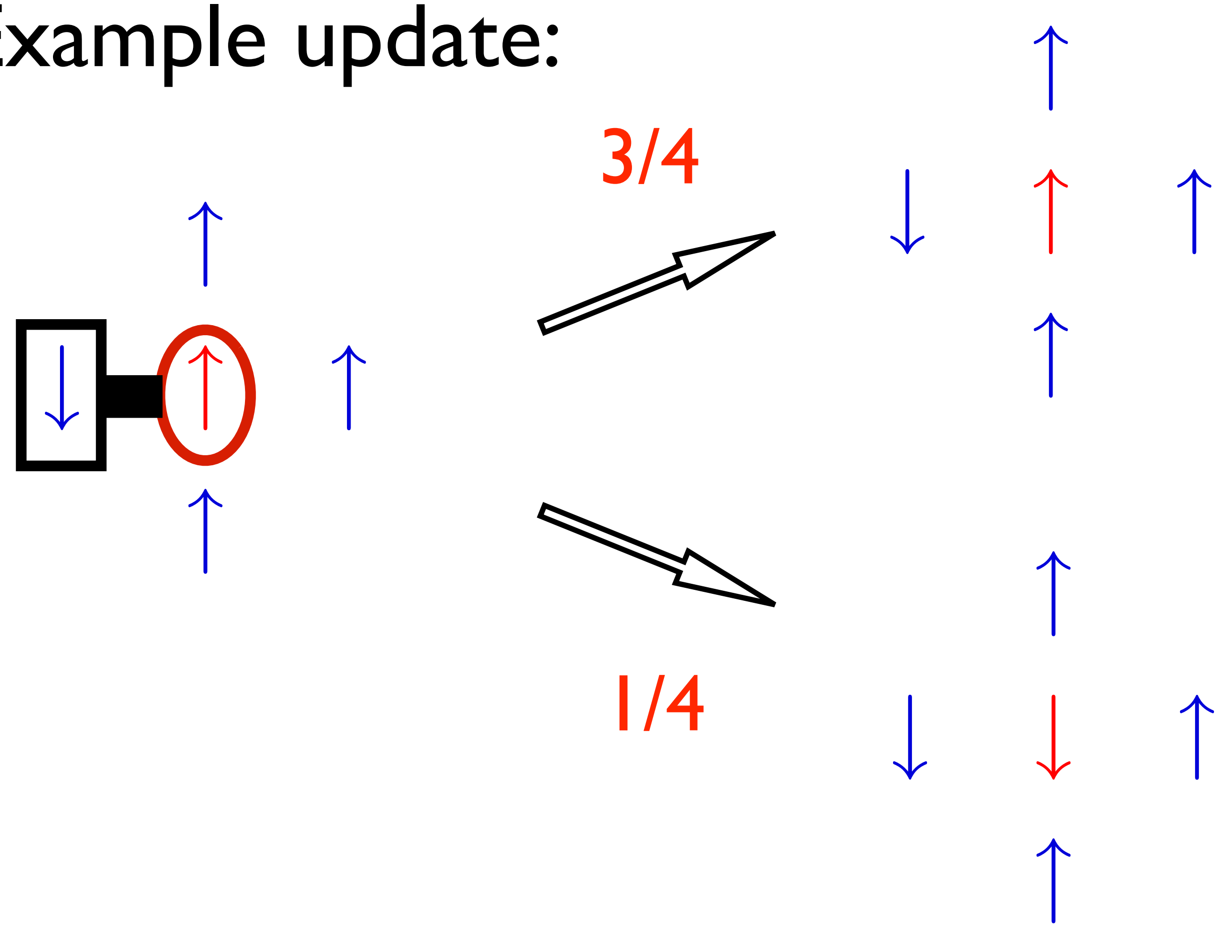
proportional rule

0. Binary voter variable at each site i
1. Pick a random voter
2. Assume state of randomly-selected neighbor
   individual has no self-confidence & adopts neighbor's state

Voter Model

Clifford & Sudbury (1973)
Holley & Liggett (1975)

Example update:



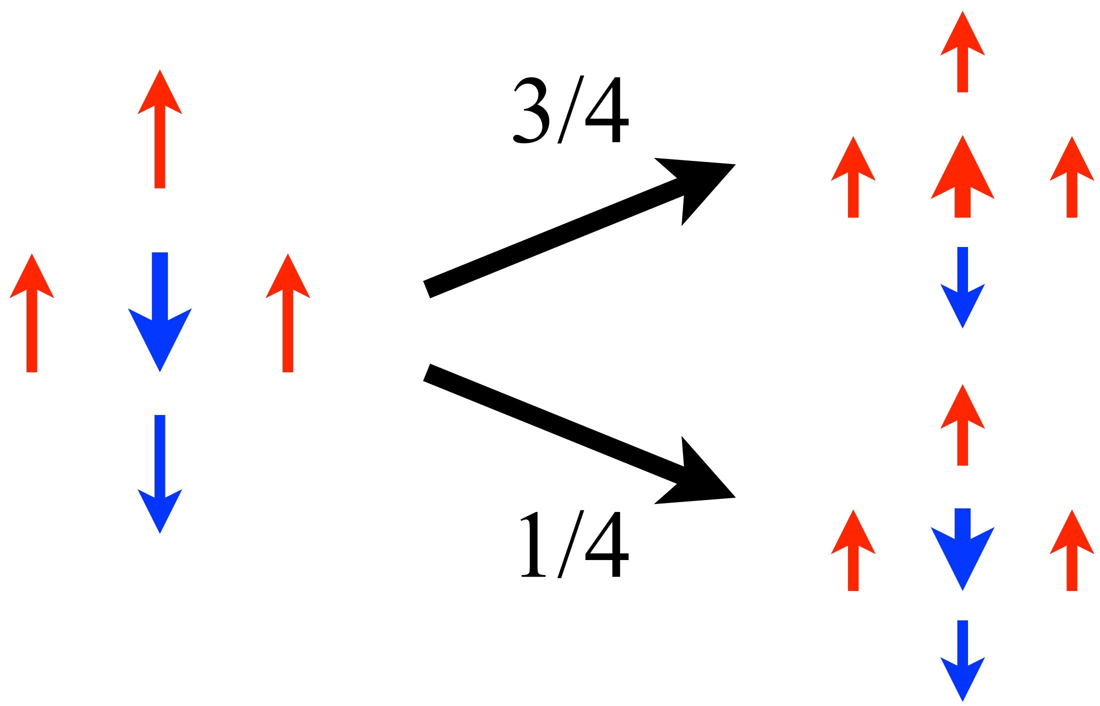
proportional rule

0. Binary voter variable at each site i
1. Pick a random voter
2. Assume state of randomly-selected neighbor
   individual has no self-confidence & adopts neighbor's state
3. Repeat 1 & 2 until consensus *necessarily* occurs in a finite system

Voter vs. Ising Models

Voter model: *proportional rule*

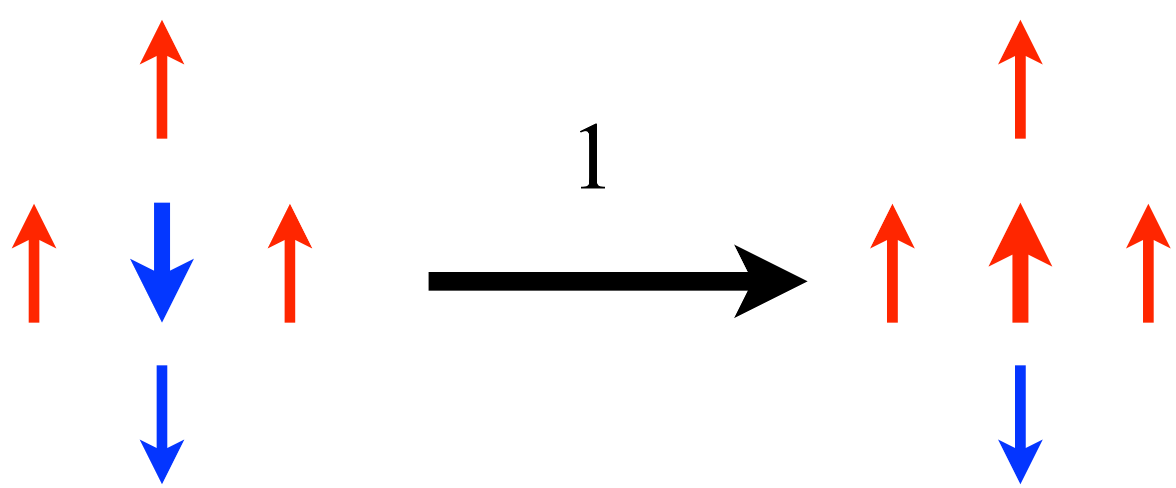
Clifford & Sudbury (1973)
Holley & Liggett (1975)



consensus ***inevitable***
in a finite system

Kinetic Ising model: *majority rule at T=0*

Glauber (1963)

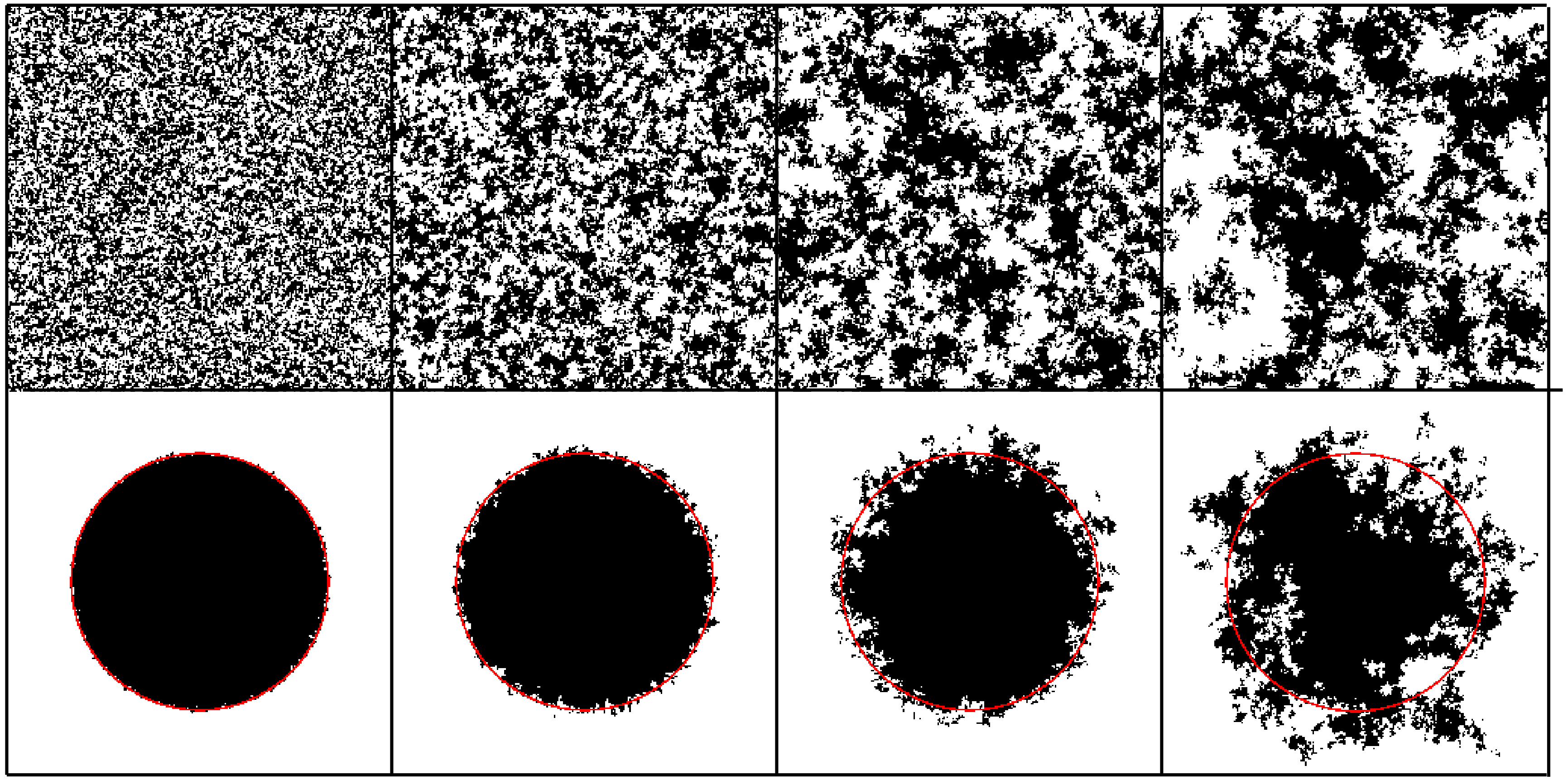


consensus
not inevitable
in a finite system

Voter Evolution vs. Ising Evolution

random initial condition:

droplet initial condition:

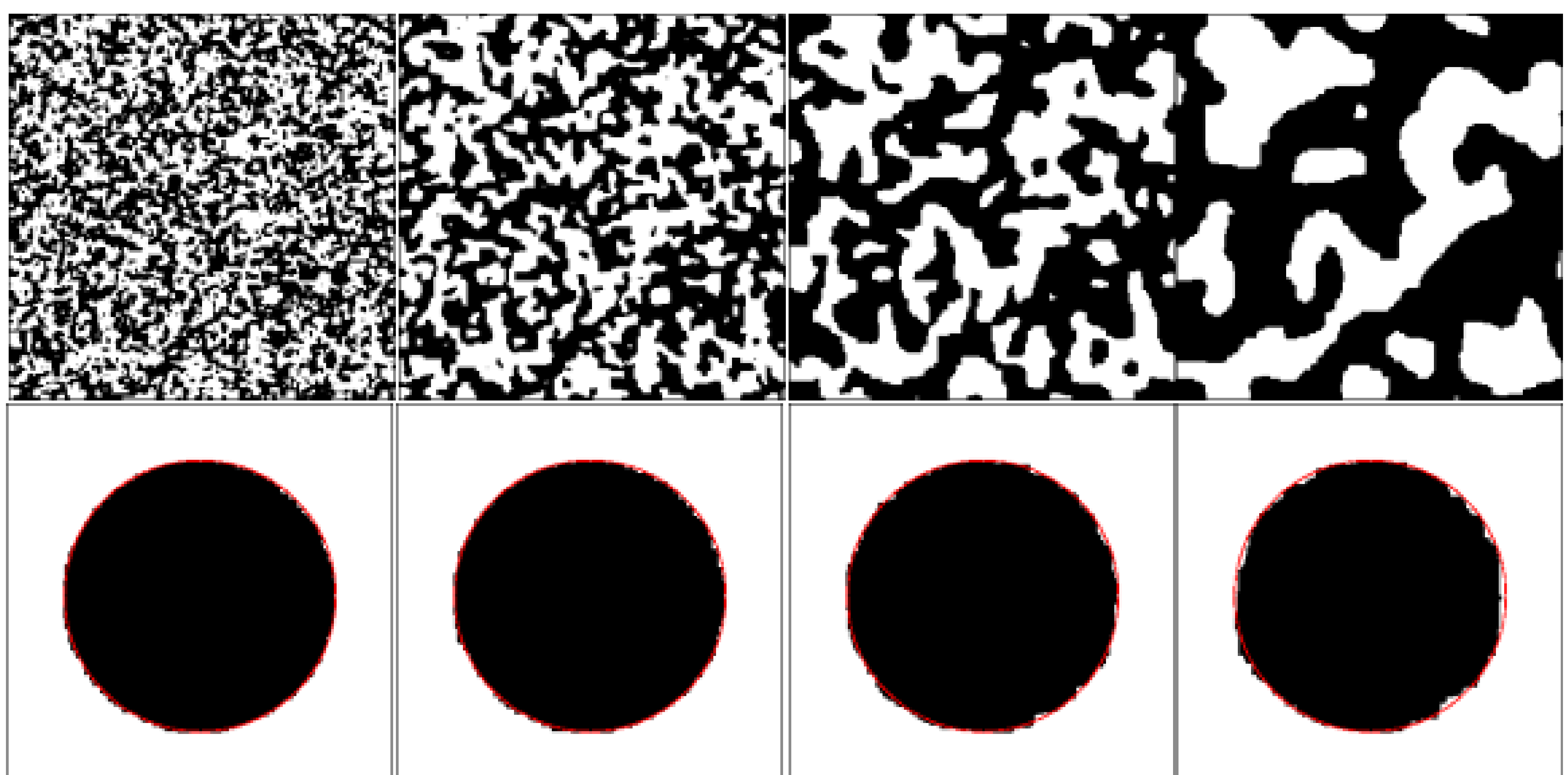


Voter

Dornic et al. (2001)

random initial condition:

droplet initial condition:

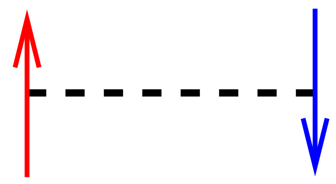


Ising

Lattice Voter Model: 3 Basic Properties

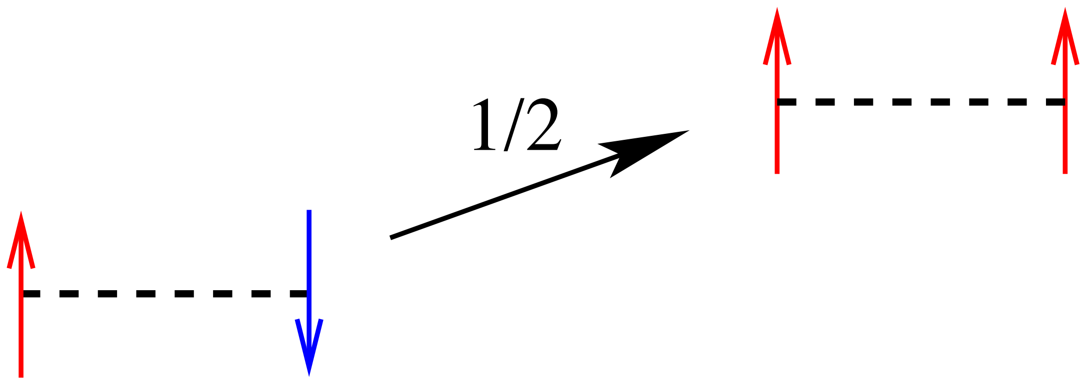
1. Final State (Exit) Probability $\mathcal{E}(\rho_0)$

Evolution of a single active link (homogeneous network):



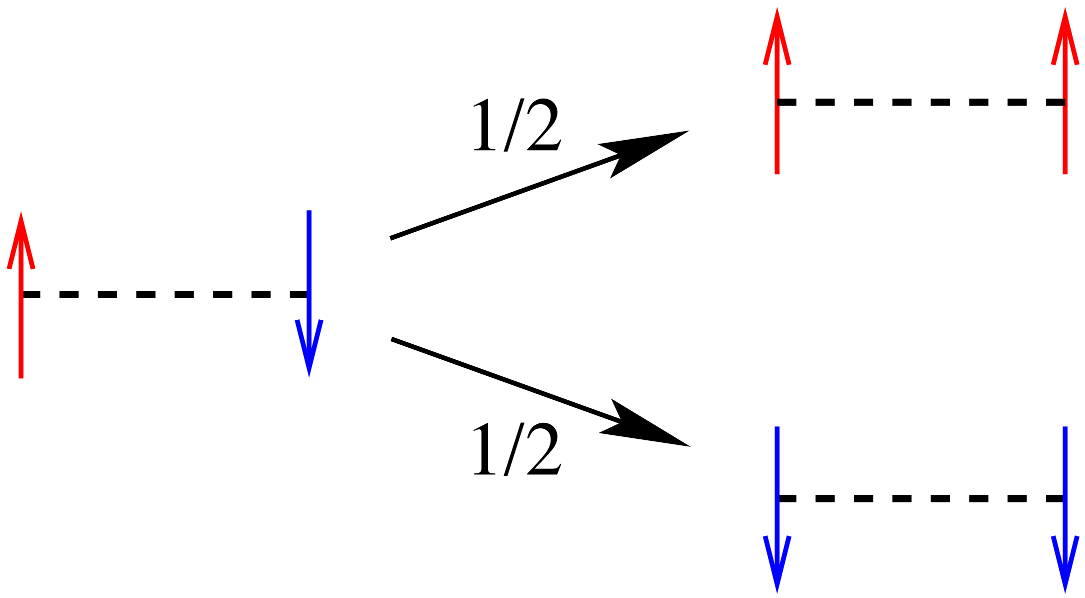
1. Final State (Exit) Probability $\mathcal{E}(\rho_0)$

Evolution of a single active link (homogeneous network):



I. Final State (Exit) Probability $\mathcal{E}(\rho_0) = \rho_0$

Evolution of a single active link (homogeneous network):



average magnetization is conserved!

2. Spatial Dependence of 2-Spin Correlations

flip rate:

$$w_i = \tfrac{1}{2}\Big[1 - \frac{\sigma_i}{z}\sum_{j\in\langle i\rangle}\sigma_j\Big]$$

1-spin correlations:

$$\frac{d\langle\sigma_i\rangle}{dt} = -2\langle\sigma_i w_i\rangle$$

2. Spatial Dependence of 2-Spin Correlations

flip rate:

$$w_i = \tfrac{1}{2}\left[1 - \frac{\sigma_i}{z}\sum_{j\in\langle i\rangle}\sigma_j\right]$$

1-spin correlations:

$$\frac{d\langle\sigma_i\rangle}{dt} = -2\langle\sigma_i w_i\rangle$$
$$= -\langle\sigma_i\rangle + \tfrac{1}{z}\sum_j\langle\sigma_j\rangle$$

$$\langle\sigma_i(t)\rangle = I_i(t)\, e^{-t}$$

for $\langle\sigma_i(t\!=\!0)\rangle = \delta_{i,\mathbf{0}}$

2-spin correlations:

$$\frac{d\langle\sigma_i\sigma_j\rangle}{dt} = -2\langle\sigma_i\sigma_j(w_i + w_j)\rangle$$
$$= -2\langle\sigma_i\sigma_j\rangle + \tfrac{1}{2d}\Big(\sum_{k\in\langle i\rangle}\langle\sigma_k\sigma_j\rangle + \sum_{k\in\langle j\rangle}\langle\sigma_i\sigma_k\rangle\Big)$$

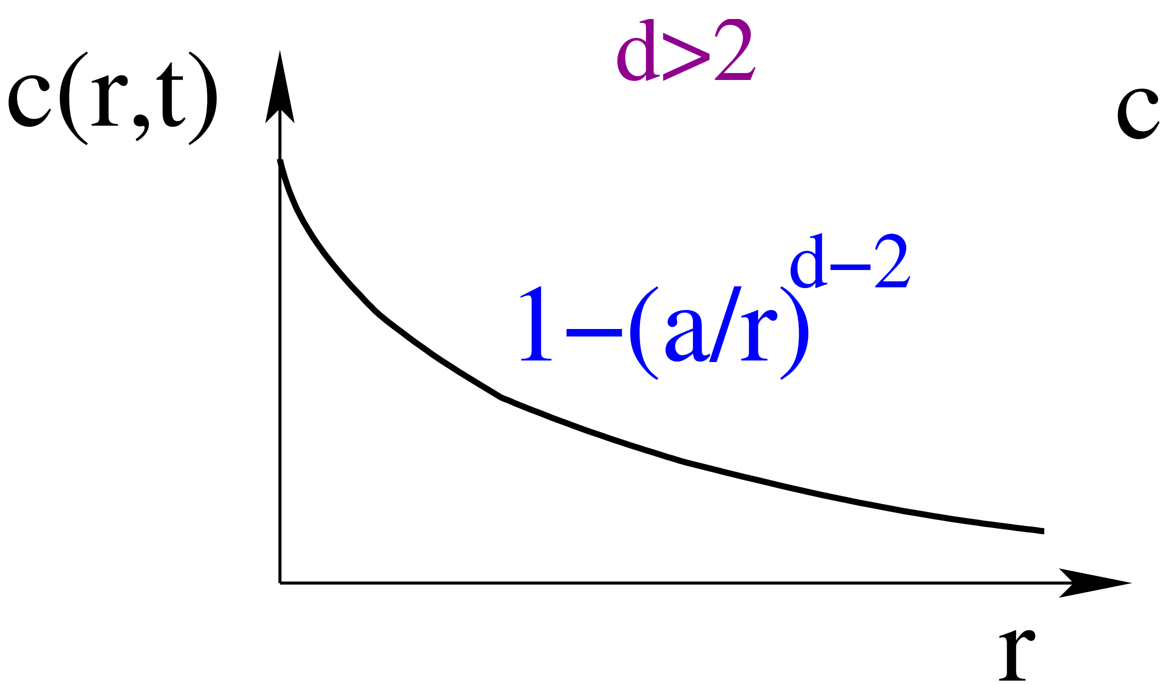
2. Spatial Dependence of 2-Spin Correlations

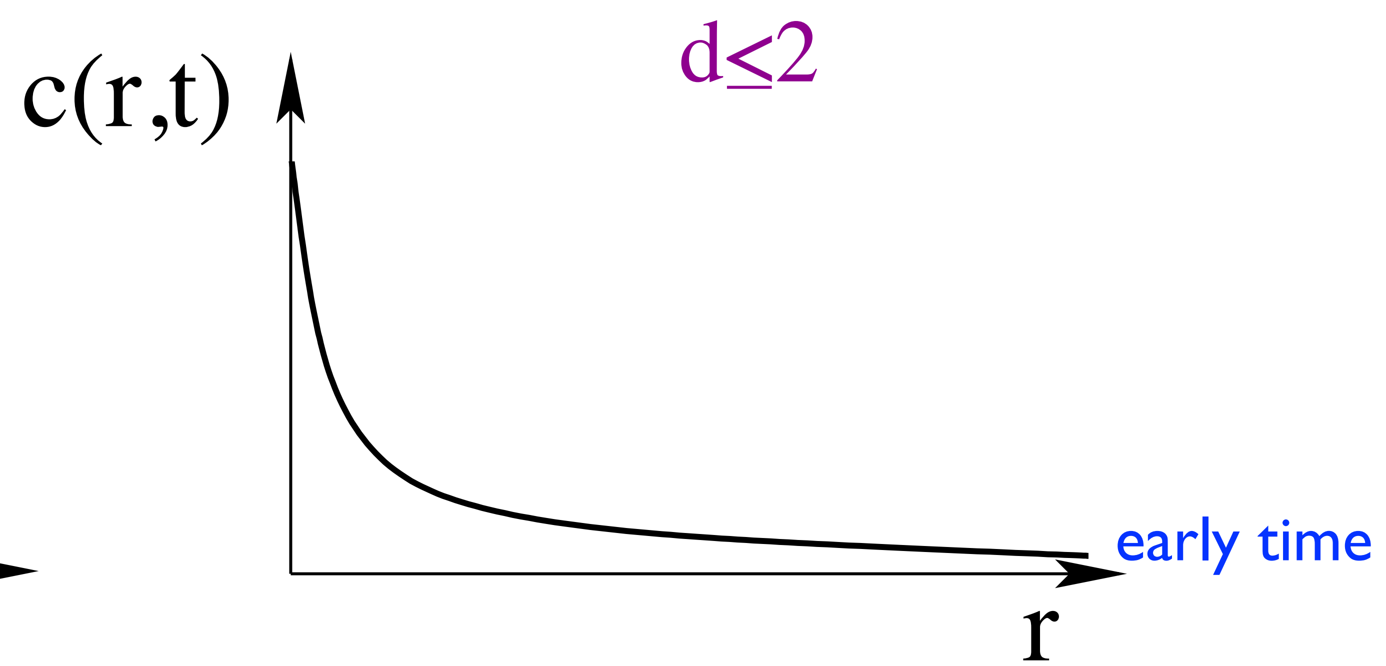
(infinite system)

Equation for 2-spin correlation function:

$$\frac{d\langle\sigma_i\sigma_j\rangle}{dt} = -2\langle\sigma_i\sigma_j(w_i + w_j)\rangle \qquad \frac{\partial c_2(\mathbf{r},t)}{\partial t} = \nabla^2 c_2(\mathbf{r},t)$$

$$c(r=0,t)=1; \quad c(r>0,t=0)=0$$





2. Spatial Dependence of 2-Spin Correlations

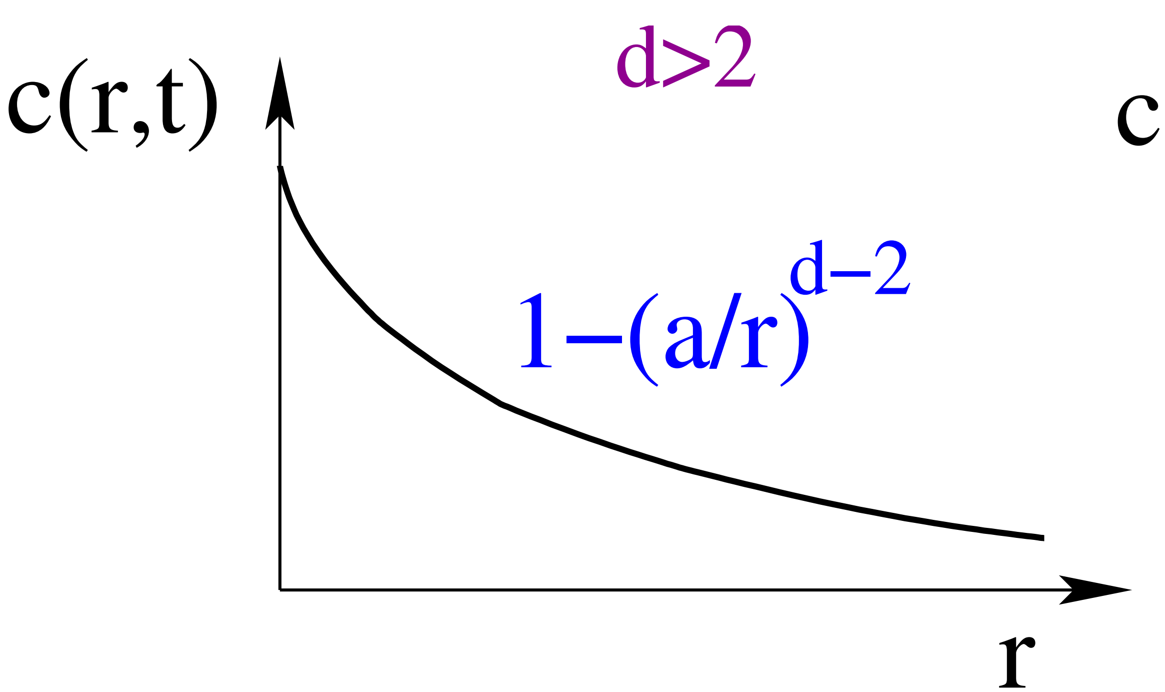
(infinite system)

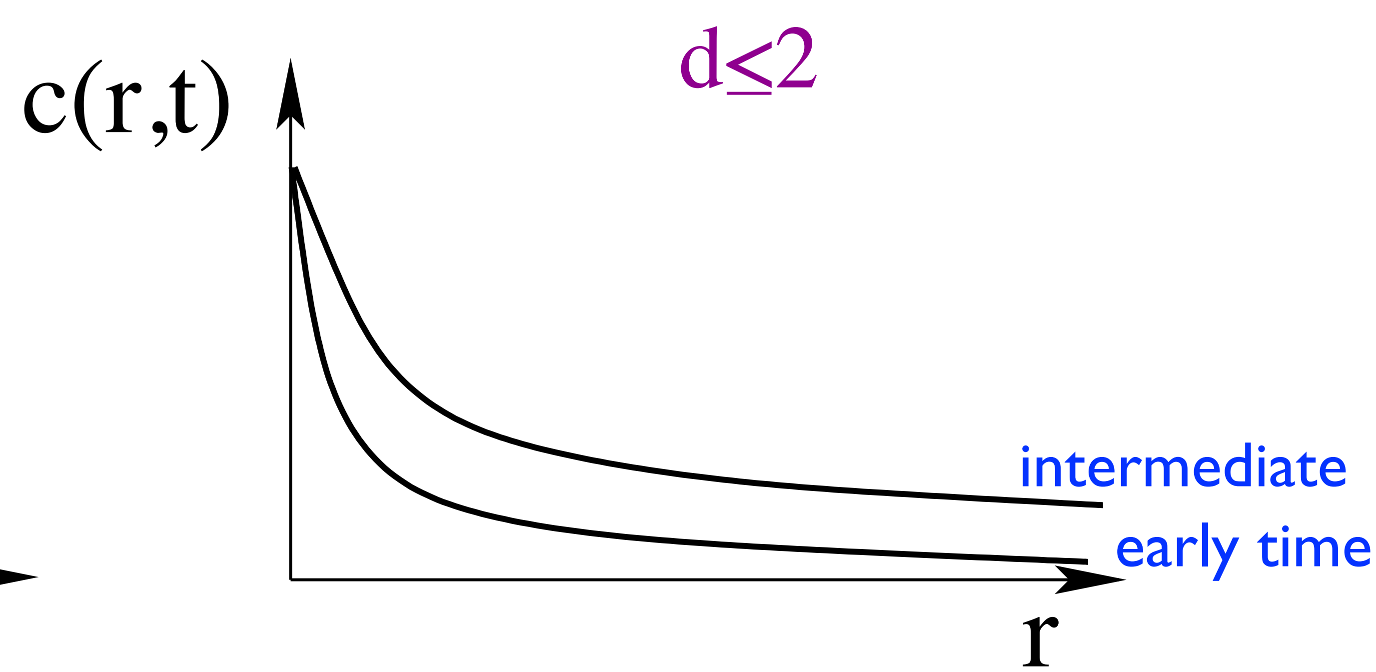
Equation for 2-spin correlation function:

$$\frac{d\langle\sigma_i\sigma_j\rangle}{dt} = -2\langle\sigma_i\sigma_j(w_i+w_j)\rangle$$

$$\frac{\partial c_2(\mathbf{r},t)}{\partial t} = \nabla^2 c_2(\mathbf{r},t)$$

$$c(r=0,t)=1; \quad c(r>0,t=0)=0$$





2. Spatial Dependence of 2-Spin Correlations

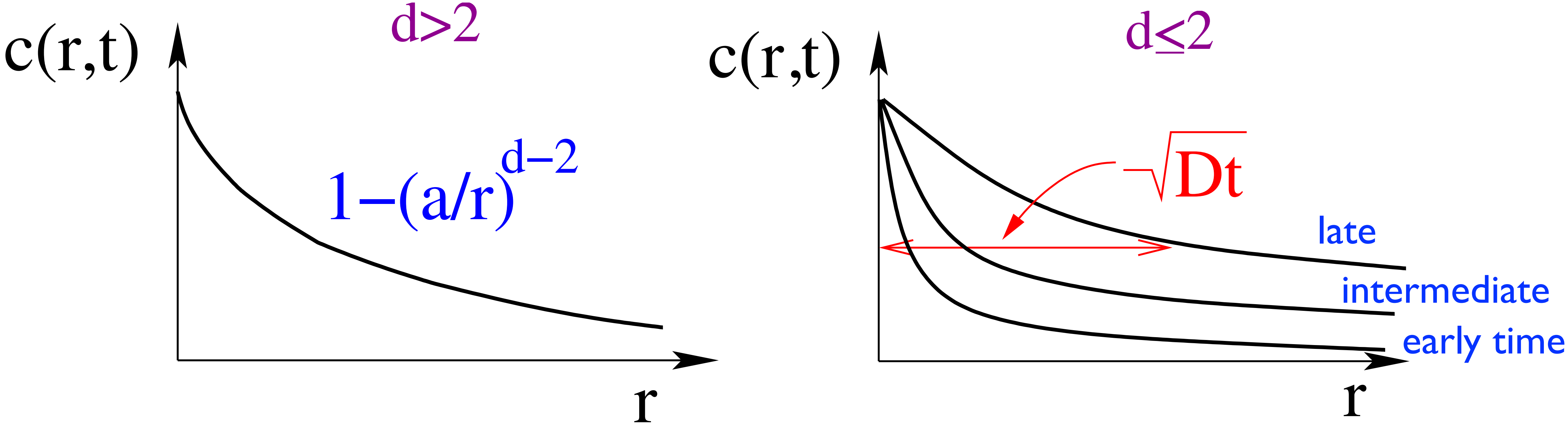
(infinite system)

Equation for 2-spin correlation function:

$$\frac{d\langle\sigma_i\sigma_j\rangle}{dt} = -2\langle\sigma_i\sigma_j(w_i + w_j)\rangle$$

$$\frac{\partial c_2(\mathbf{r},t)}{\partial t} = \nabla^2 c_2(\mathbf{r},t)$$

$$c(r=0,t)=1; \quad c(r>0,t=0)=0$$



2. Spatial Dependence of 2-Spin Correlations

(infinite system)

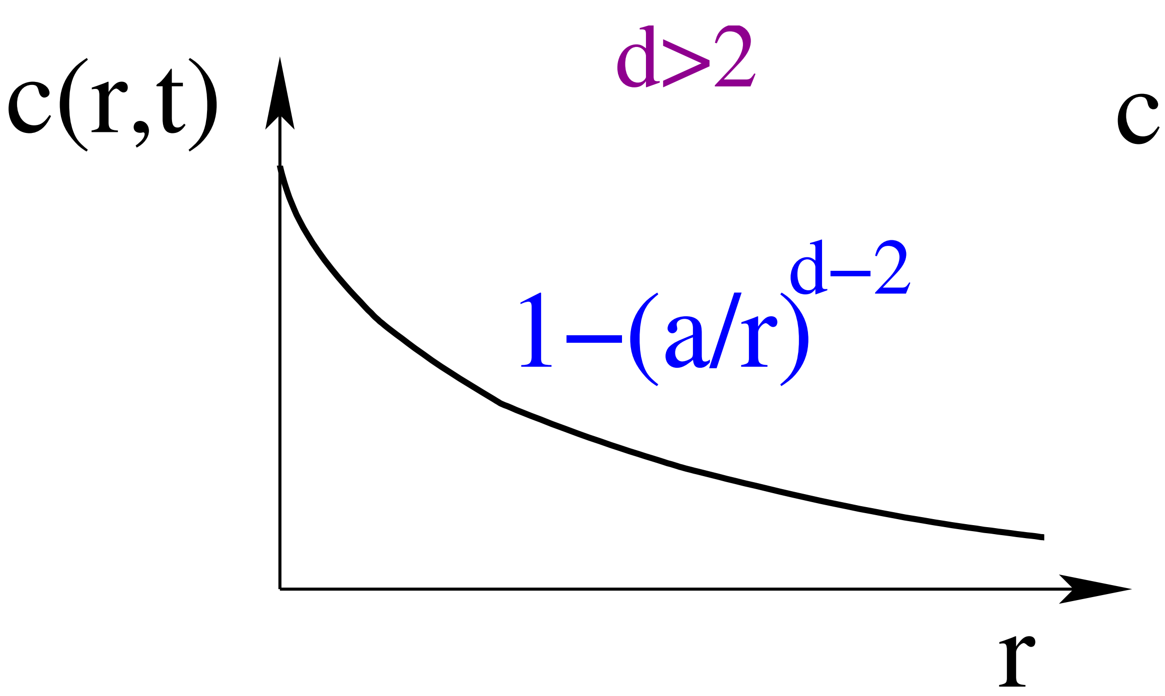
Equation for 2-spin correlation function:

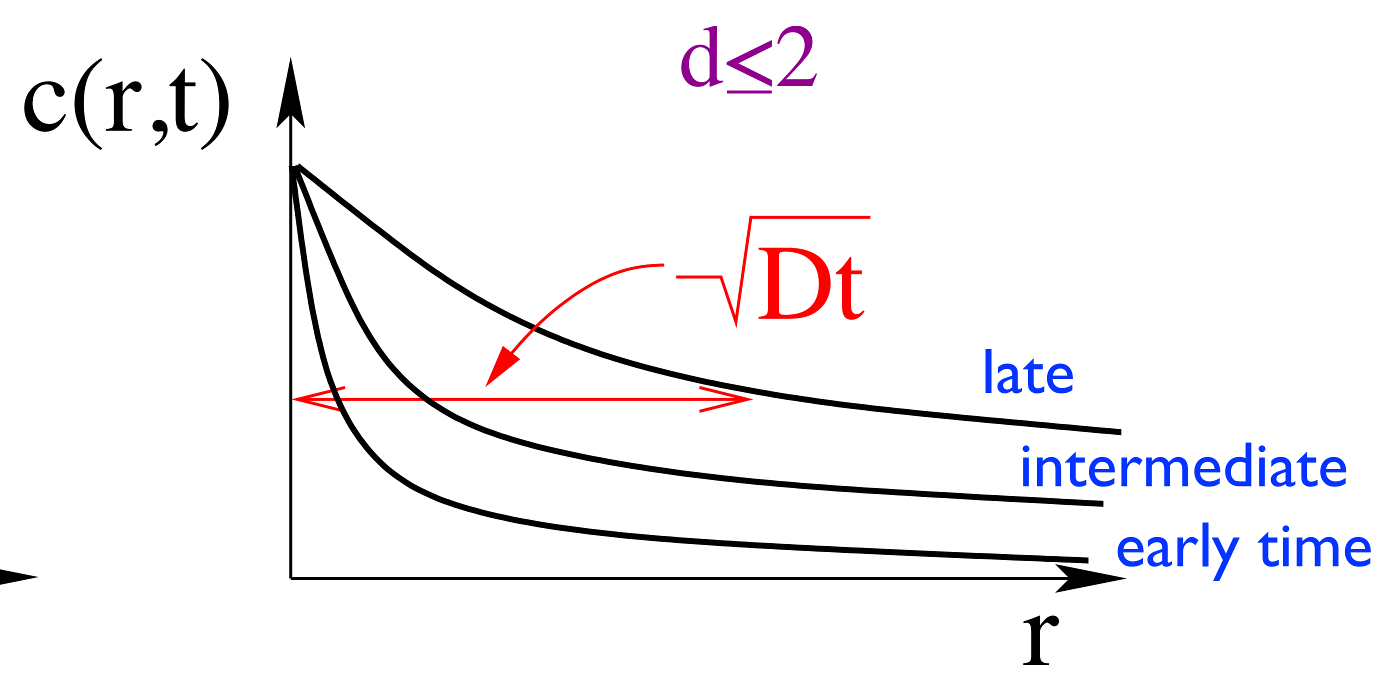
$$\frac{d\langle\sigma_i\sigma_j\rangle}{dt} = -2\langle\sigma_i\sigma_j(w_i + w_j)\rangle \qquad \frac{\partial c_2(\mathbf{r},t)}{\partial t} = \nabla^2 c_2(\mathbf{r},t)$$

$$c(r=0,t)=1; \quad c(r>0,t=0)=0$$

Asymptotic solution:

$$c(r,t) \sim \begin{cases} 1 - \dfrac{1-(\frac{a}{r})^{d-2}}{1-(\frac{a}{\sqrt{Dt}})^{d-2}} & d \neq 2 \\ \dfrac{1-\frac{\ln r}{\ln a}}{1-\frac{\ln\sqrt{Dt}}{\ln a}} & d = 2 \end{cases}$$





3. System Size Dependence of Consensus Time

Liggett (1985), Krapivsky (1992)

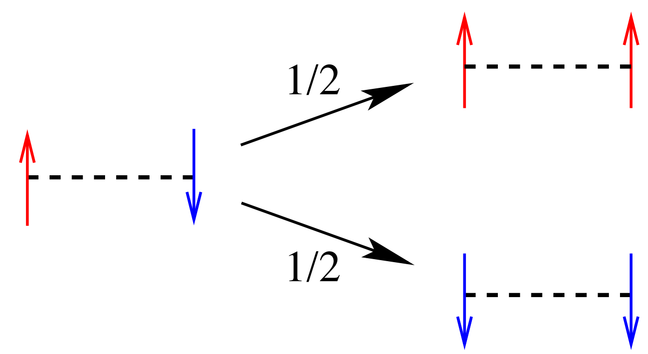
$$\int^{\sqrt{Dt}} c(r,t) r^{d-1}\, dr = N$$

dimension	consensus time
1	N^2
2	N ln N
>2	N

Lattice Voter Model: 3 Basic Properties

1. Final State Probability $\mathcal{E}(\rho_0) = \rho_0$

Evolution of a single active link:

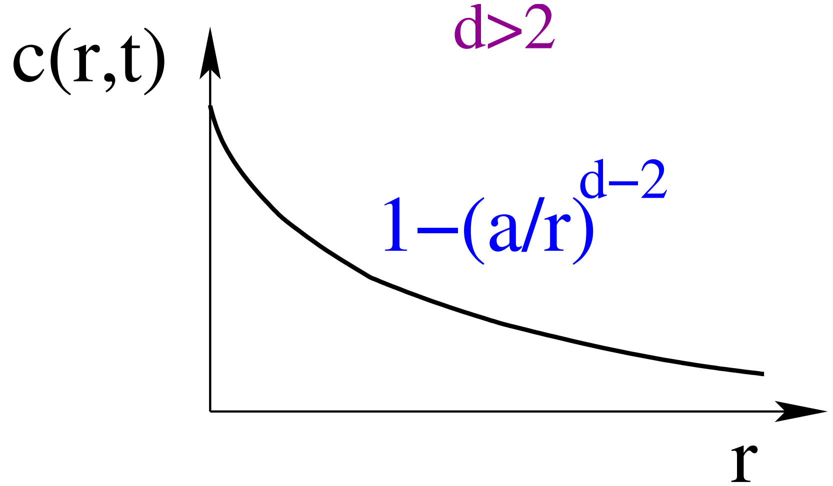


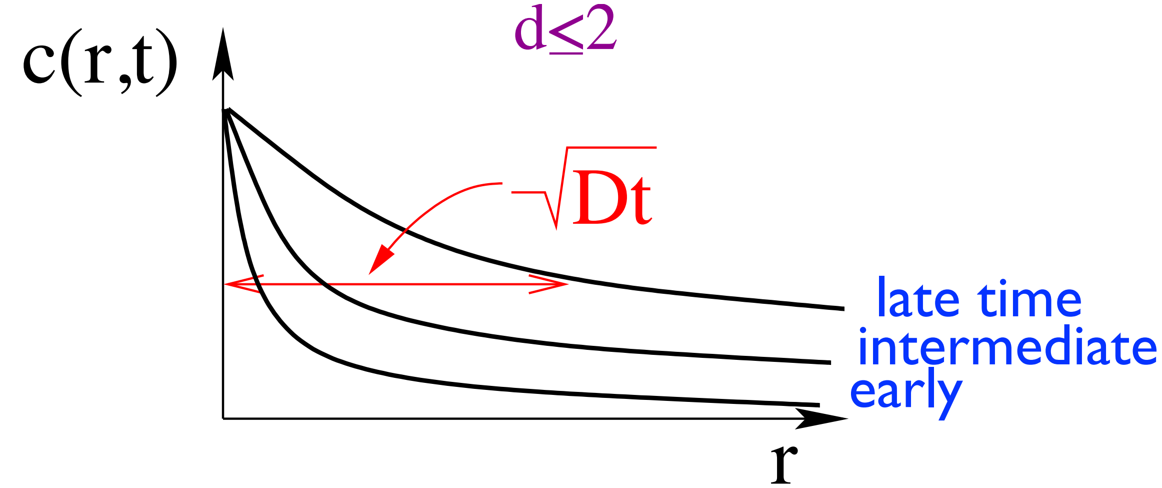
average magnetization conserved

2. Two-Spin Correlations

$$\frac{\partial c(\mathbf{r},t)}{\partial t} = \nabla^2 c(\mathbf{r},t)$$

$c(r=0,t) = 1$
$c(r>0,t=0) = 0$





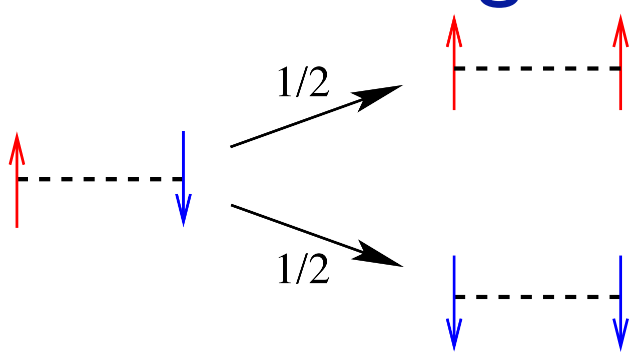
3. Consensus Time

$$\int^{\sqrt{Dt}} c(\mathbf{r},t)\, d\mathbf{r} = N$$

dimension	consensus time
1	N^2
2	N ln N
>2	N

Voter Model on Complex Networks

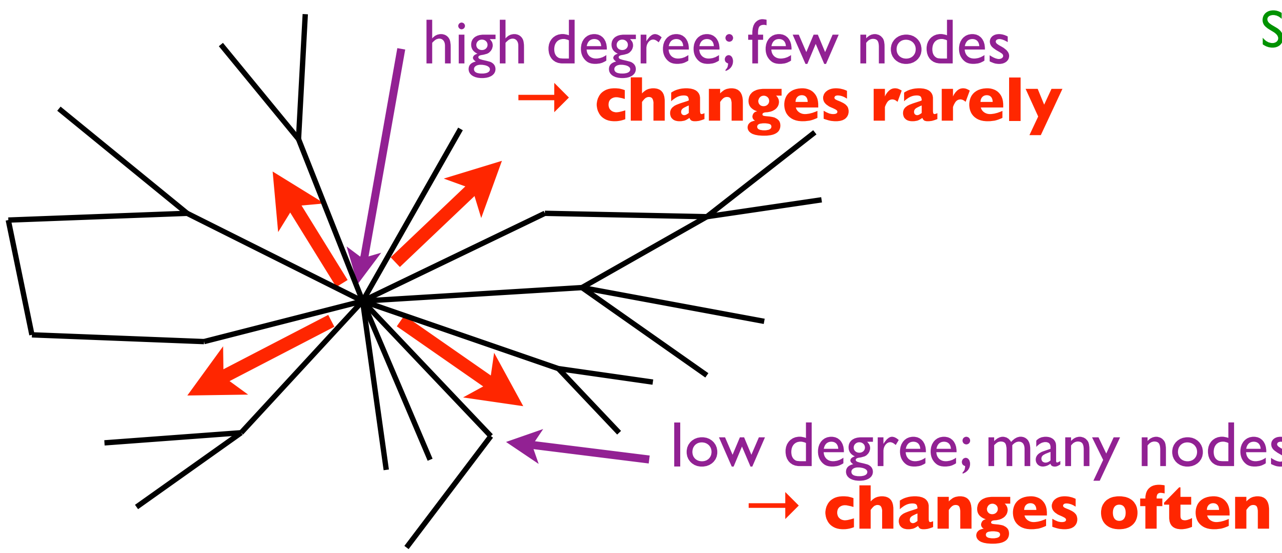
magnetization on regular networks



average magnetization conserved!

magnetization *not conserved* on complex networks

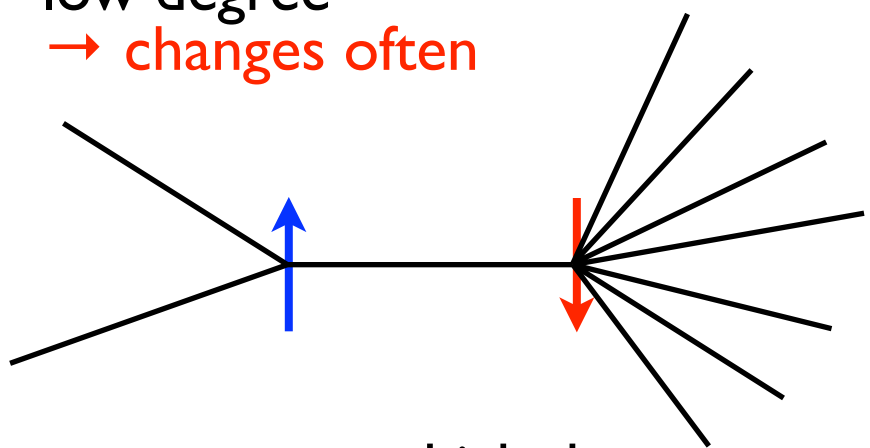
Suchecki, Eguiluz & San Miguel (2005)



*"flow" from **high** degree to **low** degree*

New Conservation Law

Sood & SR (2005)



to compensate the different rates:

degree-weighted 1st moment:

$$\omega \equiv \frac{\sum_k k n_k \rho_k}{\sum_k k n_k} = \frac{\sum_k k n_k \rho_k}{\mu_1}$$

conserved!

$\mu_1 =$ av. degree

$n_k =$ frac. nodes of degree k

$\rho_k =$ frac. $\uparrow$ on nodes of degree k

Conservation Law for Voter Model

Transition probability

$$\mathbf{P}[\eta \to \eta_x] = \sum_y \frac{A_{xy}}{N k_x} [\Phi(x,y) + \Phi(y,x)]$$

adjacency matrix

$\uparrow_x \downarrow_y$ $\downarrow_x \uparrow_y$

voter at x

neighbor at y

$\eta \equiv$ state of system

$\eta_x \equiv$ state after flip at x

$\eta(x) \equiv$ state at x (0,1)

$$\Phi(x,y) \equiv \eta(x)[1-\eta(y)] = \uparrow_x \downarrow_y$$

Density change: $$\langle \Delta\eta(x) \rangle = \underbrace{[1 - 2\eta(x)]}_{\pm 1_{\text{at } x}} \mathbf{P}[\eta \to \eta_x]$$

degree-weighted moments: $$\omega_n = \frac{1}{N\mu_n} \sum_x k_x^n \; \eta(x)$$

change in weighted first moment: $$\langle \Delta\omega_1 \rangle = \sum_{x,y} \frac{A_{xy}}{N k_x} \; k_x \, [\eta(y) - \eta(x)] = 0$$

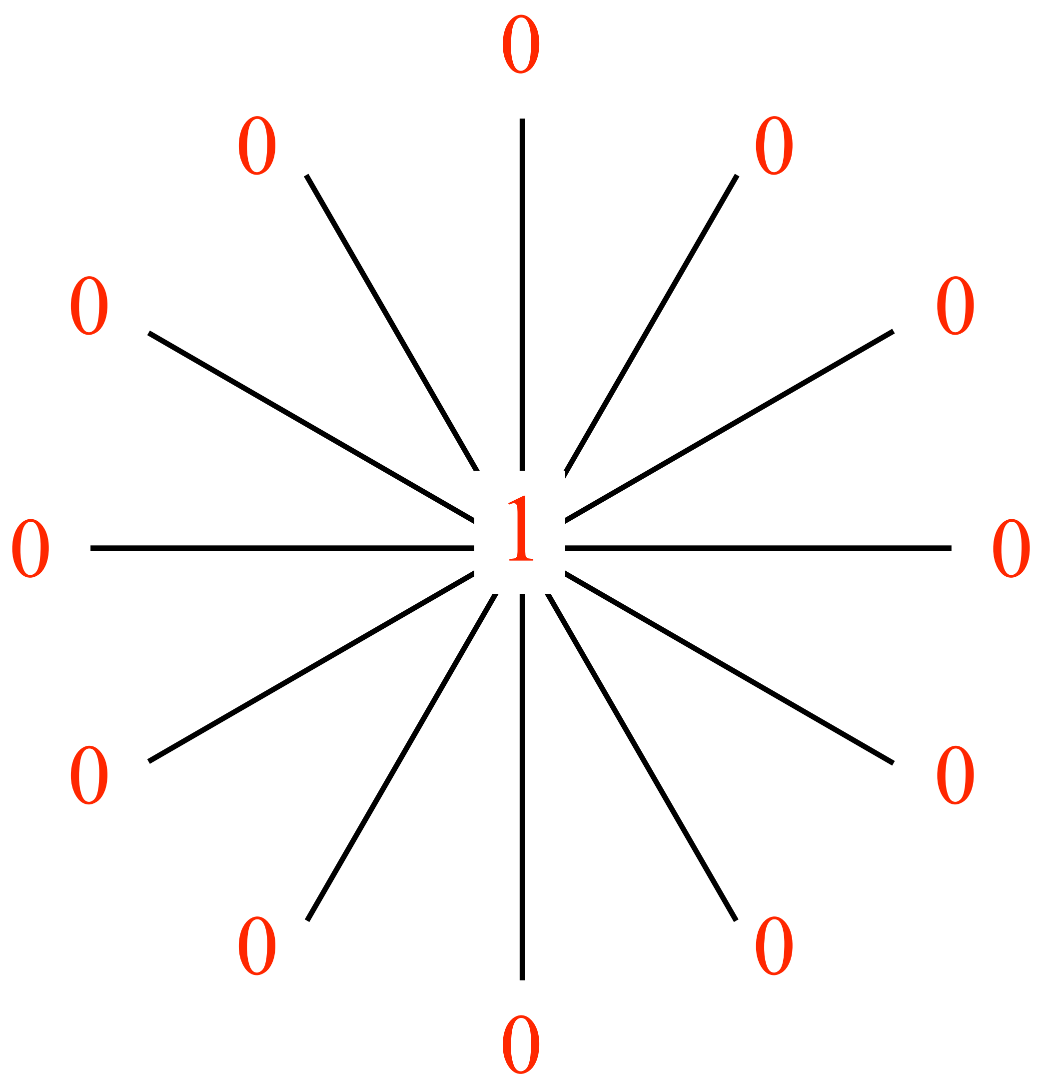
conserved!

Exit Probability on Complex Graphs

$$\mathcal{E}(\omega) = \omega$$

Extreme case: star graph

N nodes: degree 1
1 node: degree N



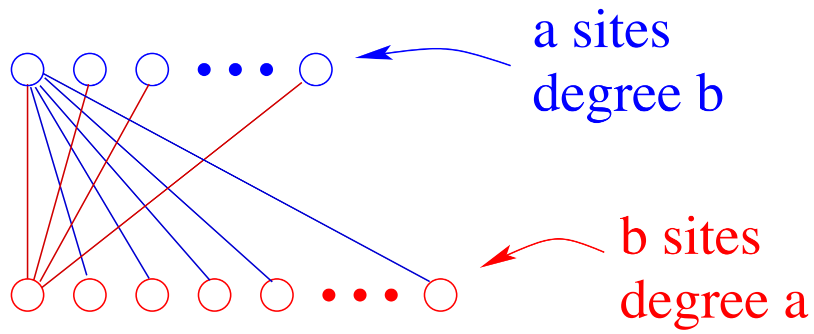
$$\omega = \frac{1}{\mu_1}\sum_k k\, n_k \rho_k = \frac{1}{2}$$

Final state: all 1 with prob. 1/2!

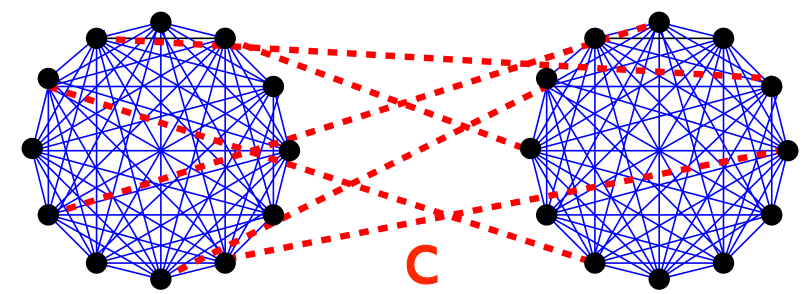
Voter Model on Complex Networks

Sucheki, Eguiluz & San Miguel (2005)
Sood & SR (2005)
Antal, Sood & SR (2005)

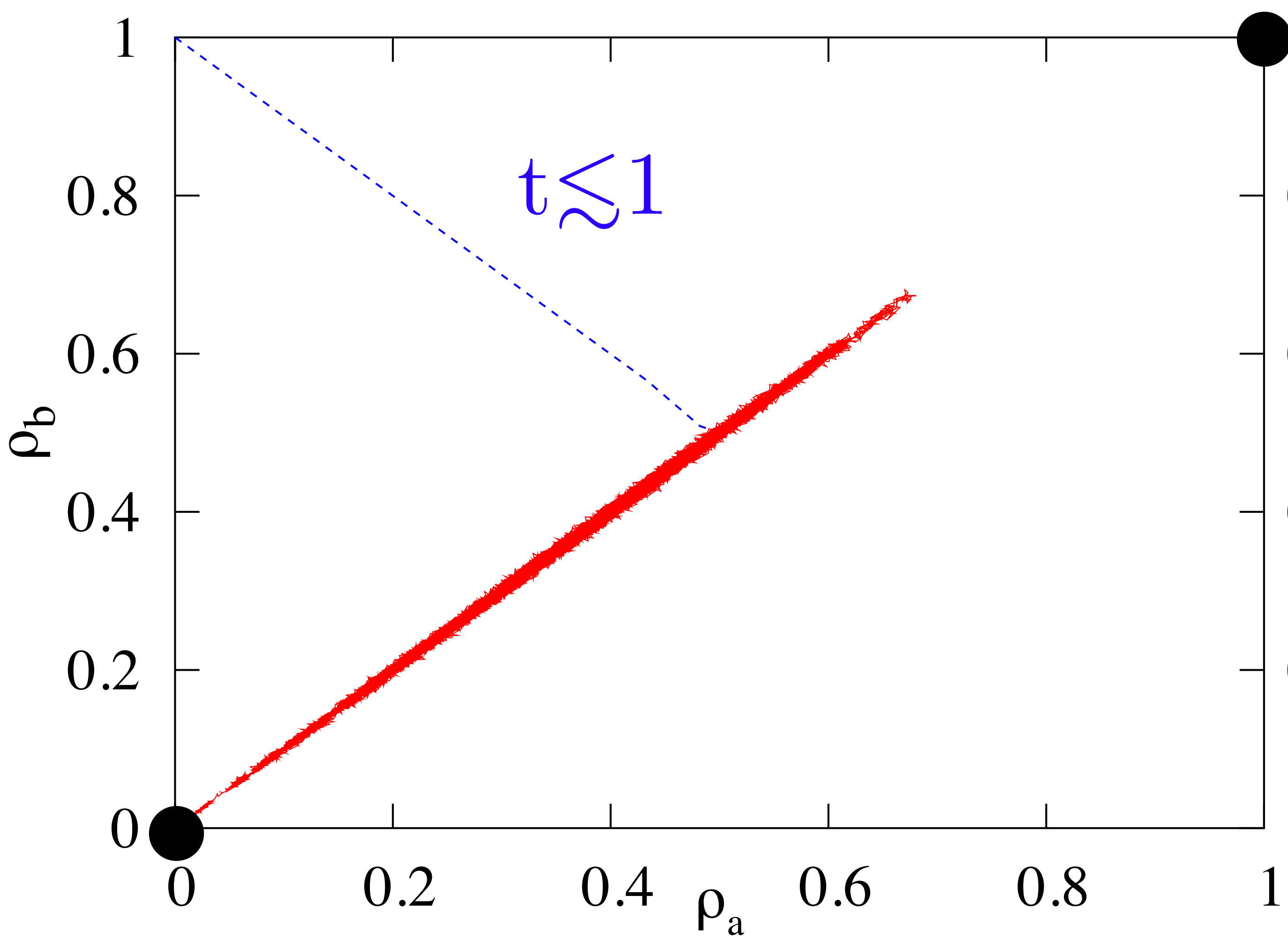
complete bipartite graph

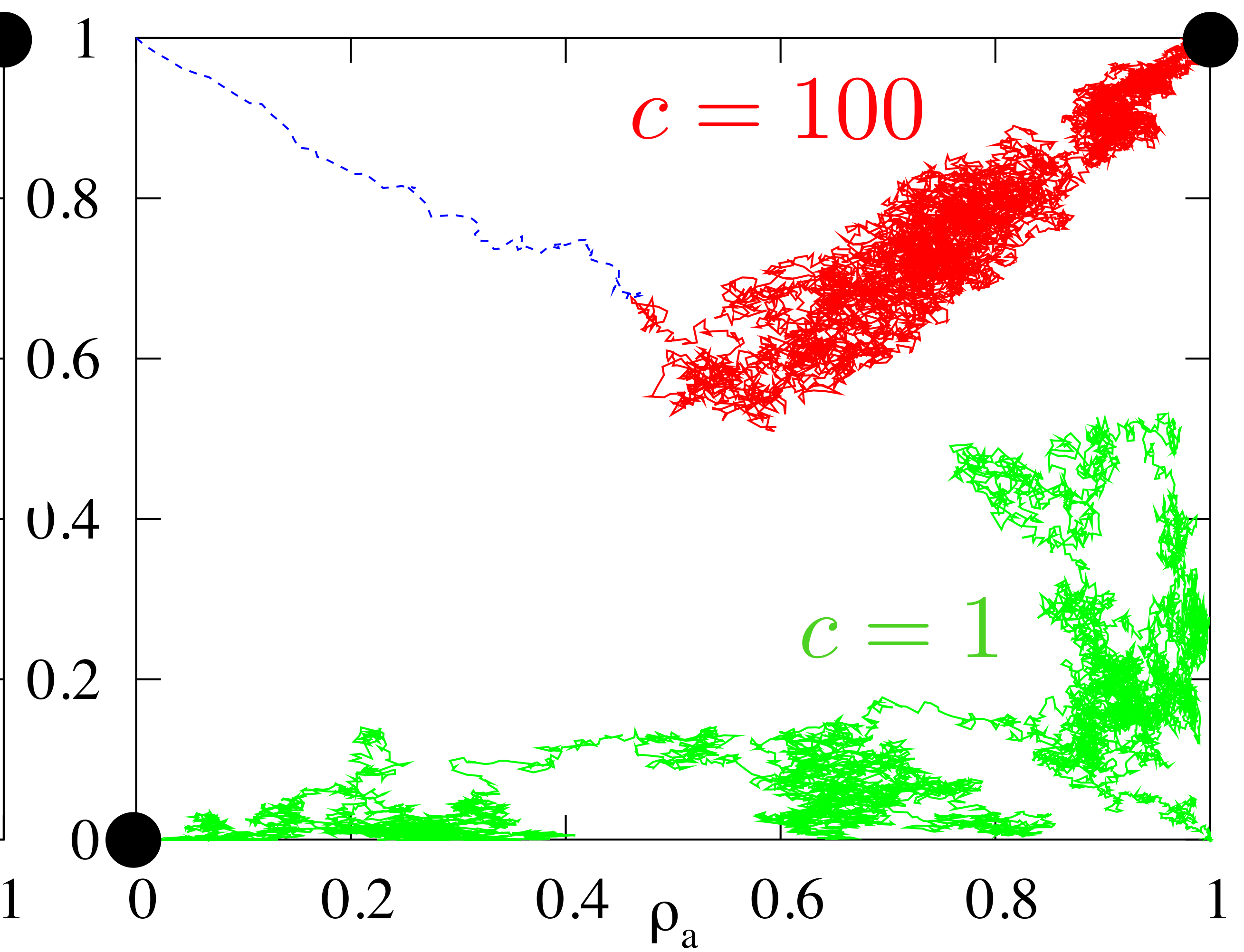


two-clique graph



N=10000, C links/node





Consensus Time Evolution Equation

warmup: complete graph

$T(\rho) \equiv$ av. consensus time starting with density ρ

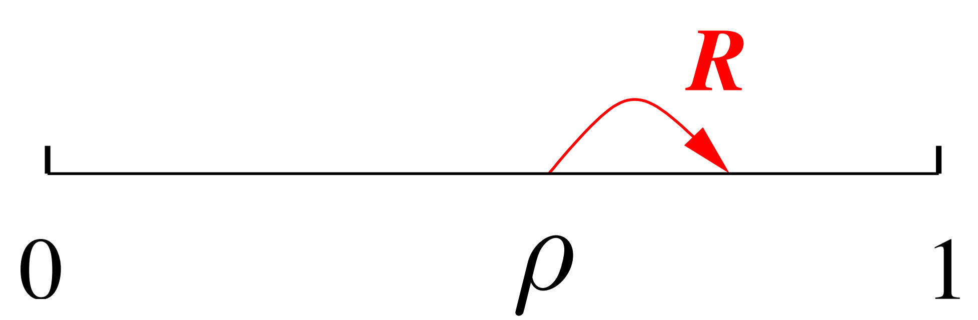
$$
\begin{aligned}
T(\rho) \quad &= \quad \mathcal{R}(\rho)[T(\rho + d\rho) + dt] \\
&\quad +\mathcal{L}(\rho)[T(\rho - d\rho) + dt] \\
&\quad +[1 - \mathcal{R}(\rho) - \mathcal{L}(\rho)][T(\rho) + dt]
\end{aligned}
$$

Consensus Time Evolution Equation

warmup: complete graph

$T(\rho) \equiv$ av. consensus time starting with density ρ

$$
\begin{aligned}
T(\rho) &= \mathcal{R}(\rho)[T(\rho+d\rho)+dt] \\
&\quad +\mathcal{L}(\rho)[T(\rho-d\rho)+dt] \\
&\quad +[1-\mathcal{R}(\rho)-\mathcal{L}(\rho)][T(\rho)+dt]
\end{aligned}
$$



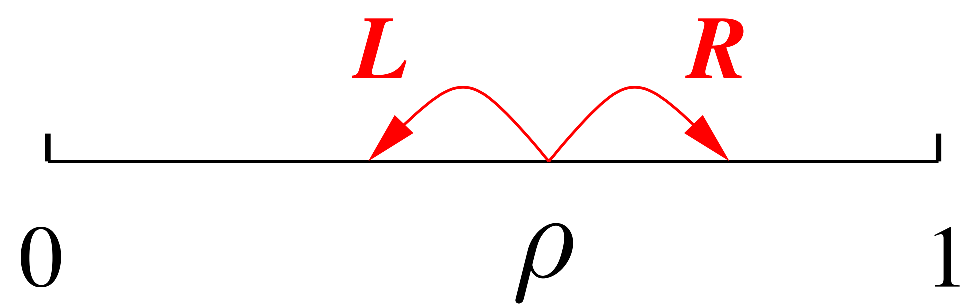
$$
\begin{aligned}
\mathcal{R}(\rho) &\equiv \text{prob}(\downarrow\uparrow\rightarrow\uparrow\uparrow) \\
&= \rho(1-\rho)
\end{aligned}
$$

Consensus Time Evolution Equation

warmup: complete graph

$T(\rho) \equiv$ av. consensus time starting with density ρ

$$
\begin{aligned}
T(\rho) \quad &= \quad \mathcal{R}(\rho)[T(\rho + d\rho) + dt] \\
&\quad\; + \mathcal{L}(\rho)[T(\rho - d\rho) + dt] \\
&\quad\; + [1 - \mathcal{R}(\rho) - \mathcal{L}(\rho)][T(\rho) + dt]
\end{aligned}
$$



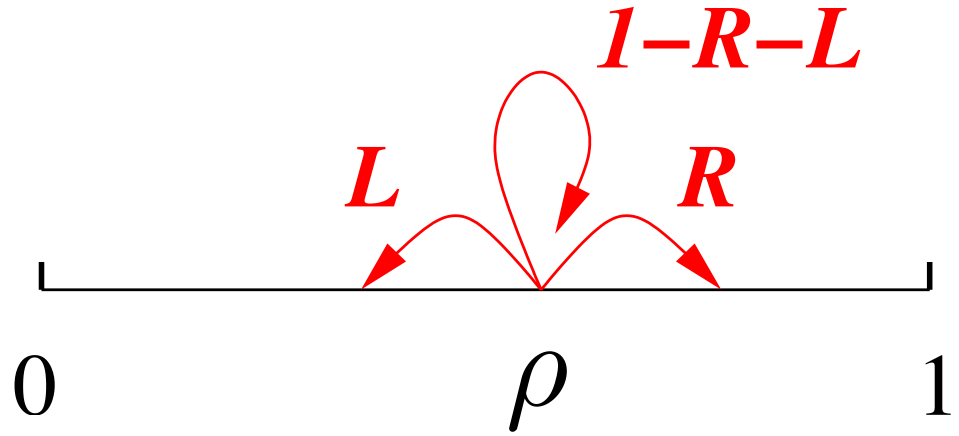
$$
\begin{aligned}
\mathcal{R}(\rho) &\equiv \mathrm{prob}(\downarrow\uparrow \rightarrow \uparrow\uparrow) \\
\mathcal{L}(\rho) &\equiv \mathrm{prob}(\uparrow\downarrow \rightarrow \downarrow\downarrow) \\
&= \rho(1-\rho)
\end{aligned}
$$

Consensus Time Evolution Equation

warmup: complete graph

$T(\rho) \equiv$ av. consensus time starting with density ρ

$$
\begin{aligned}
T(\rho) &= \mathcal{R}(\rho)[T(\rho + d\rho) + dt] \\
&\quad + \mathcal{L}(\rho)[T(\rho - d\rho) + dt] \\
&\quad + [1 - \mathcal{R}(\rho) - \mathcal{L}(\rho)][T(\rho) + dt]
\end{aligned}
$$



$$
\begin{aligned}
\mathcal{R}(\rho) &\equiv \text{prob}(\downarrow\uparrow \to \uparrow\uparrow) \\
\mathcal{L}(\rho) &\equiv \text{prob}(\uparrow\downarrow \to \downarrow\downarrow) \\
&= \rho(1-\rho)
\end{aligned}
$$

Consensus Time on Complete Graph

$$\begin{aligned} T(\rho) &= \mathcal{R}(\rho)[T(\rho+d\rho)+dt] \\ &\quad +\mathcal{L}(\rho)[T(\rho-d\rho)+dt] \\ &\quad +[1-\mathcal{R}(\rho)-\mathcal{L}(\rho)][T(\rho)+dt] \end{aligned}$$

continuum limit: $$T'' = -\frac{N}{\rho(1-\rho)}$$

solution:

$$T(\rho) = -N\left[\rho\ln\rho + (1-\rho)\ln(1-\rho)\right]$$

Consensus Time on Heterogeneous Networks

$$T(\{\rho_k\}) \equiv \text{av. consensus time starting with density } \rho_k$$
$$\text{on nodes of degree } k$$

$$T(\{\rho_k\}) = \sum_k \mathcal{R}_k(\{\rho_k\})[T(\{\rho_k^+\}) + dt]$$
$$+ \sum_k \mathcal{L}_k(\{\rho_k\})[T(\{\rho_k^-\}) + dt]$$
$$+ \left[1 - \sum_k [\mathcal{R}_k(\{\rho_k\}) + \mathcal{L}_k(\{\rho_k\})]\right][T(\{\rho_k\}) + dt]$$

$$\mathcal{R}_k(\{\rho_k\}) = \text{prob}(\rho_k \to \rho_k^+)$$
$$= \frac{1}{N} \sum_x{}' \frac{1}{k_x} \sum_y P(\downarrow, —, \uparrow)$$
$$= n_k \omega (1 - \rho_k)$$

$$\mathcal{L}_k(\{\rho_k\}) = n_k \rho_k (1 - \omega)$$

Consensus Time on Heterogeneous Networks

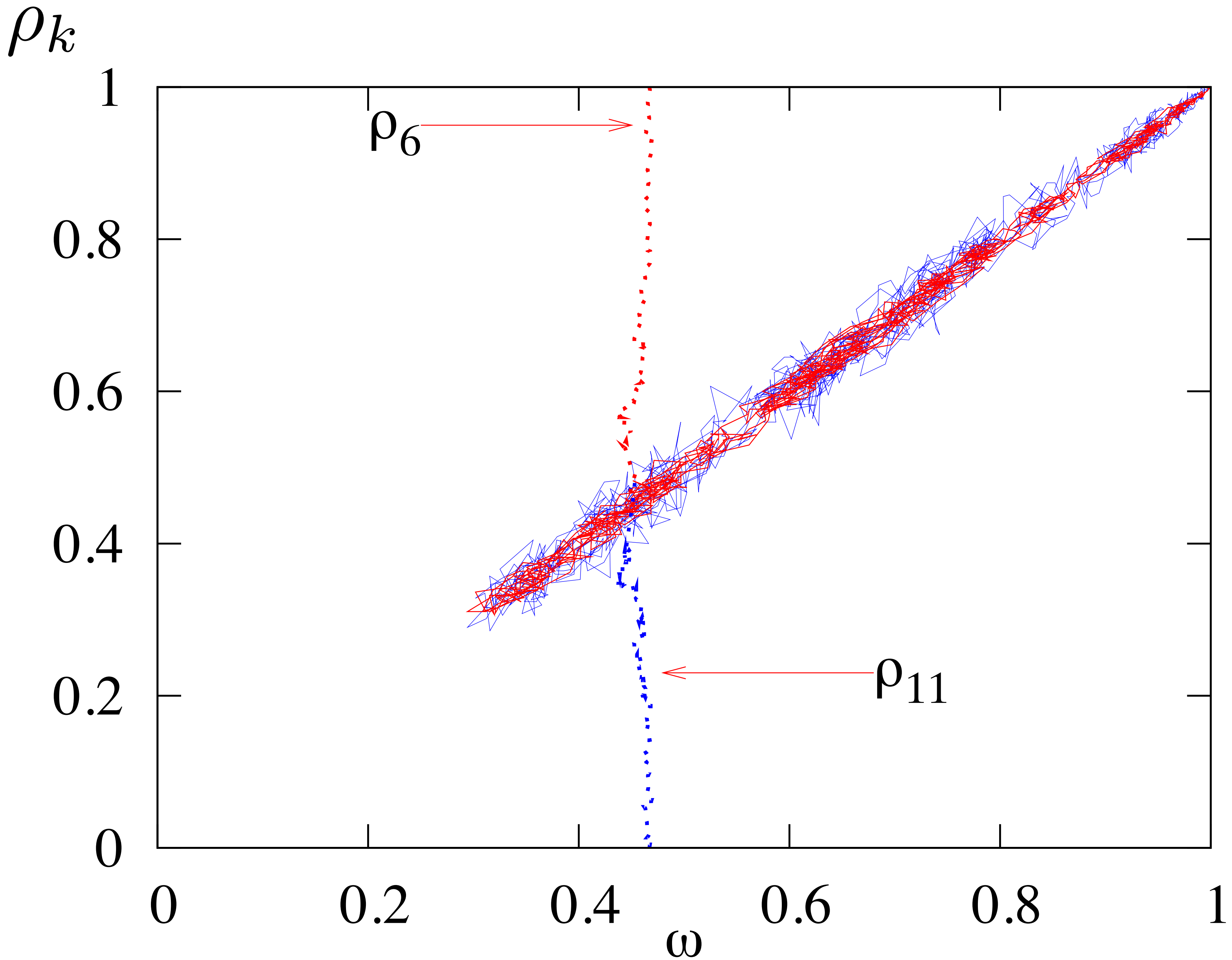
continuum limit:

$$\sum_k \left[(\omega - \rho_k) \frac{\partial T}{\partial \rho_k} + \frac{\omega + \rho_k - 2\omega\rho_k}{2Nn_k} \frac{\partial^2 T}{\partial \rho_k^2} \right] = -1$$

Voter Model on Complex Networks

configuration model

$$n_k \sim k^{-2.5}, \quad \mu_1 = 8$$



Consensus Time on Heterogeneous Networks

continuum limit:

$$\sum_k \left[(\omega - \rho_k) \frac{\partial T}{\partial \rho_k} + \frac{\omega + \rho_k - 2\omega\rho_k}{2Nn_k} \frac{\partial^2 T}{\partial \rho_k^2} \right] = -1$$

now use $\rho_k \longrightarrow \omega \quad \forall k$

and $\frac{\partial}{\partial \rho_k} = \frac{\partial \omega}{\partial \rho_k} \frac{\partial}{\partial \omega} = \frac{k n_k}{\mu_1} \frac{\partial}{\partial \omega}$

to give $\frac{\partial^2 T}{\partial \omega^2} = -\frac{N \mu_1^2 / \mu_2}{\omega(1-\omega)}$ same as $T'' = -\frac{N}{\rho(1-\rho)}$

with *effective* size $N_{\text{eff}} = N \, \mu_1^2 / \mu_2$

$$\mu_n = \sum_k k^n \, n_k$$

Consensus Time for Complex Networks with $n_k \sim k^{-\nu}$

$$T_N \propto N_{\rm eff} = N\frac{\mu_1^2}{\mu_2}$$

Consensus Time for Complex Networks with $n_k \sim k^{-\nu}$

$$T_N \propto N_{\mathrm{eff}} = N\frac{\mu_1^2}{\mu_2} \sim \begin{cases} N & \nu > 3 \\ & \\ & \\ & \end{cases}$$

Consensus Time for Complex Networks with $n_k \sim k^{-\nu}$

$$T_N \propto N_{\rm eff} = N\frac{\mu_1^2}{\mu_2} \sim \begin{cases} N & \nu > 3 \\ N/\ln N & \nu = 3 \end{cases}$$

Consensus Time for Complex Networks with $n_k \sim k^{-\nu}$

$$T_N \propto N_{\rm eff} = N\frac{\mu_1^2}{\mu_2} \sim \begin{cases} N & \nu > 3 \\ N/\ln N & \nu = 3 \\ N^{2(\nu-2)/(\nu-1)} & 2 < \nu < 3 \\ (\ln N)^2 & \nu = 2 \\ \mathcal{O}(1) & \nu < 2 \end{cases}$$

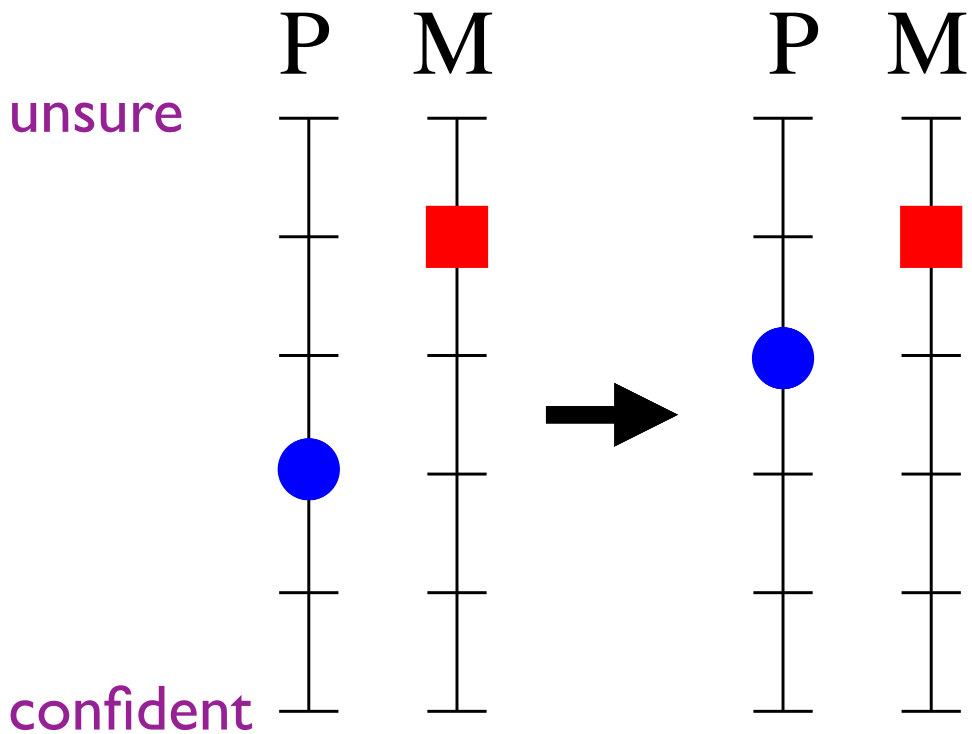
fast consensus

Invasion process:

$$T_N \sim \begin{cases} N & \nu > 2, \\ N \ln N & \nu = 2, \\ N^{2-\nu} & \nu < 2. \end{cases}$$

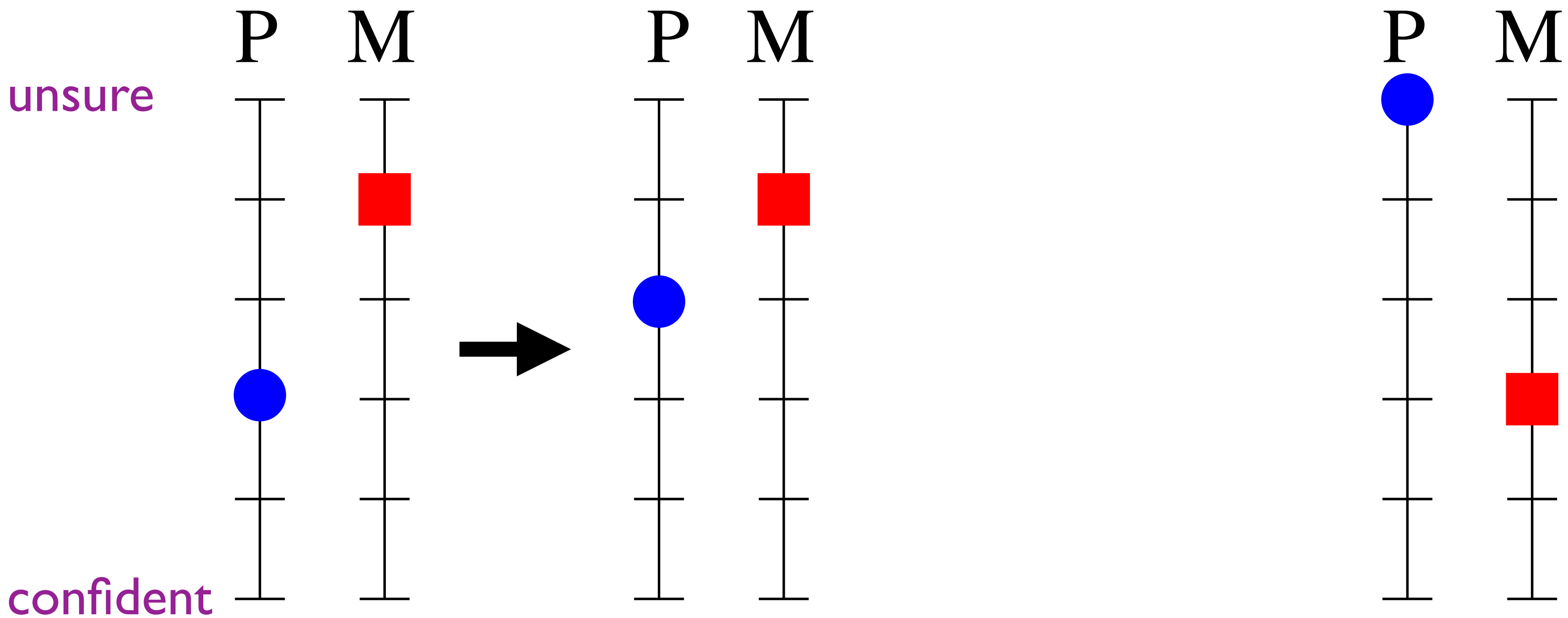
“Confident” Voter Model

motivation: Centola (2010)
related work: Dall’Asta & Castellano (2007)



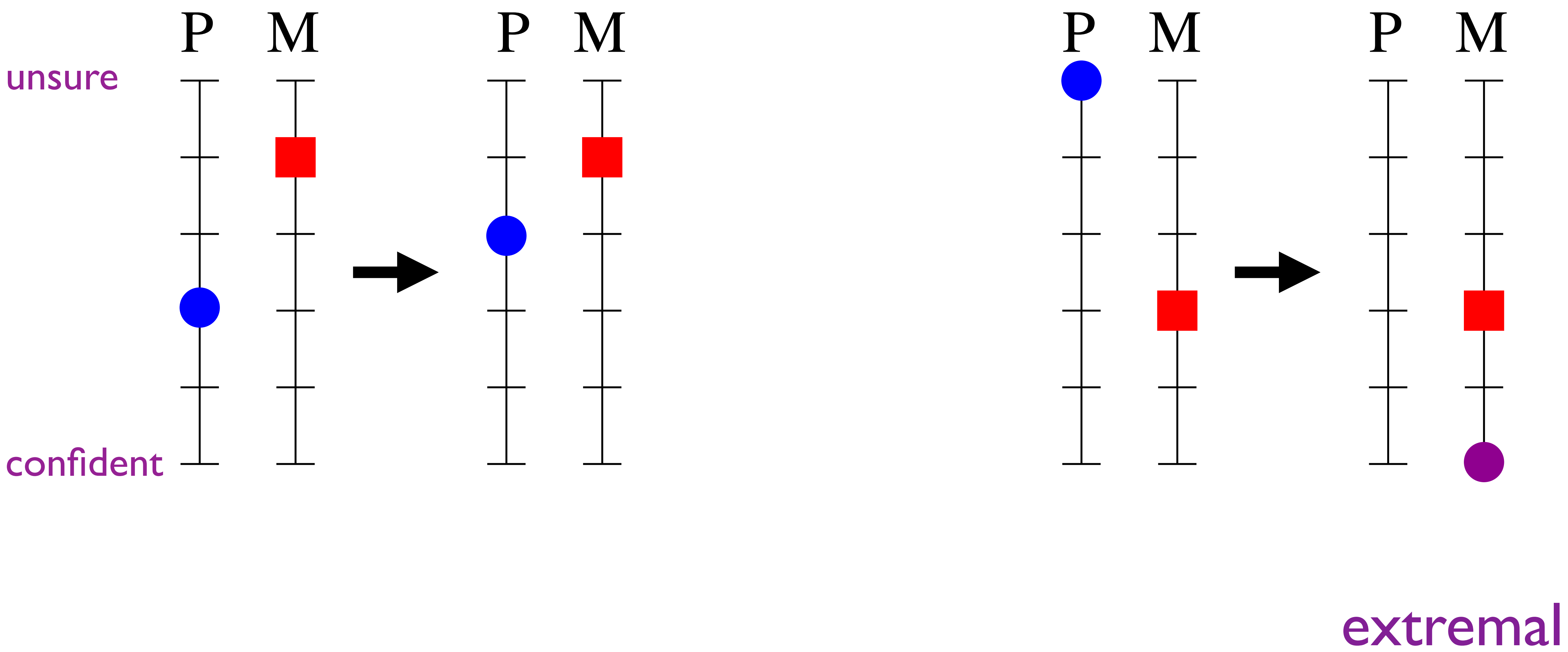
“Confident” Voter Model

motivation: Centola (2010)
related work: Dall’Asta & Castellano (2007)



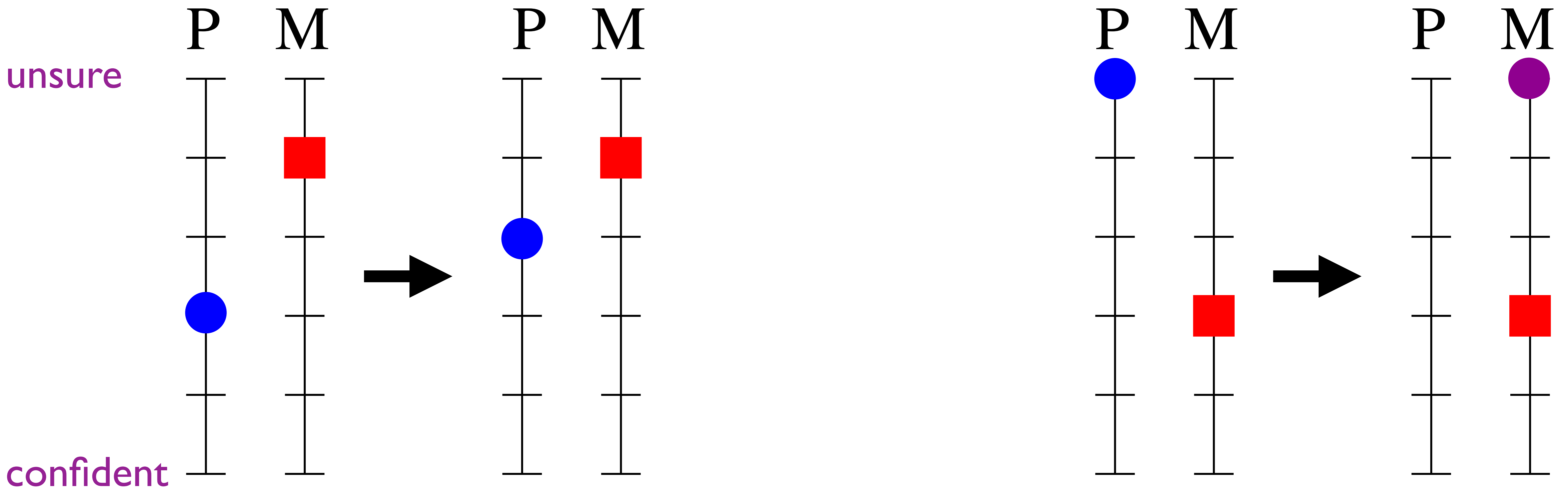
“Confident” Voter Model

motivation: Centola (2010)
related work: Dall’Asta & Castellano (2007)



“Confident” Voter Model

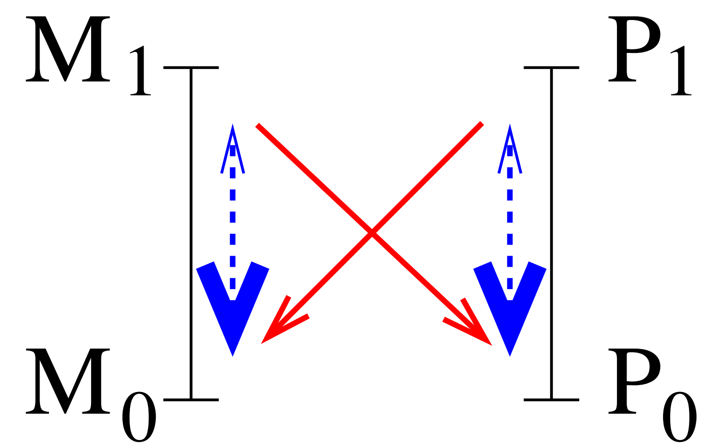
motivation: Centola (2010)
related work: Dall’Asta & Castellano (2007)



marginal

Simplest case: 2 internal states

densities P_0, P_1, M_0, M_1, with $P_0+P_1+M_0+M_1=1$



basic processes:

$$M_1\,P_1 \;\rightarrow\; P_0P_1 \text{ or } M_0M_1 \qquad M_0\,P_0 \;\rightarrow\; M_0P_1 \text{ or } M_1P_0$$

$$P_0\,P_1 \;\rightarrow\; P_0P_1 \text{ or } P_0P_0 \qquad M_0M_1 \;\rightarrow\; M_0M_1 \text{ or } M_0M_0$$

$$M_1\,P_0 \;\rightarrow\; M_1P_1 \text{ or } P_0P_0 \qquad M_0\,P_1 \;\rightarrow\; M_1P_1 \text{ or } M_0M_0$$

rate equations/mean-field limit:

$$\dot{P}_0 = -M_0P_0 + M_1P_1 + P_0P_1$$

$$\dot{P}_1 = M_0P_0 - M_1P_1 - P_0P_1 + (M_1P_0 - M_0P_1)$$

similarly for M_0, M_1

special soluble case: symmetric limit

$$P_0 + P_1 = M_0 + M_1 = \tfrac{1}{2}$$

$$\dot{P}_0 = -M_0 P_0 + M_1 P_1 + P_0 P_1$$

$$\dot{P}_1 = \quad M_0 P_0 - M_1 P_1 - P_0 P_1 + (M_1 P_0 - M_0 P_1)$$

$$\rightarrow \quad \dot{P}_0 = -\dot{P}_1 = P_0^2 + \tfrac{1}{2} P_0 - \tfrac{1}{4}$$

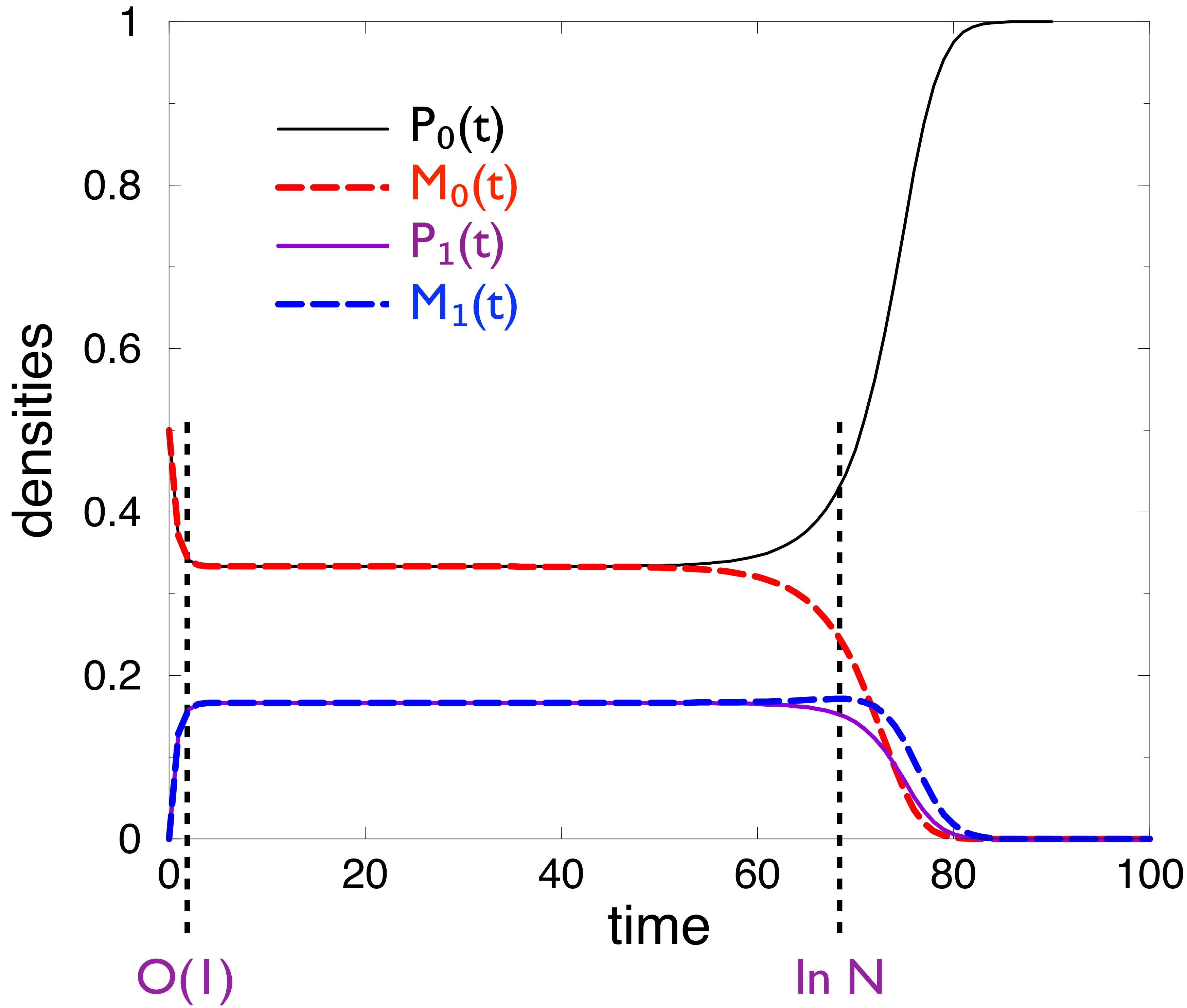
$$= -(P_0 - \lambda_+)(P_0 - \lambda_-)$$

$$\lambda_\pm = \tfrac{1}{4}(-1 \pm \sqrt{5}) \approx 0.309, -0.809$$

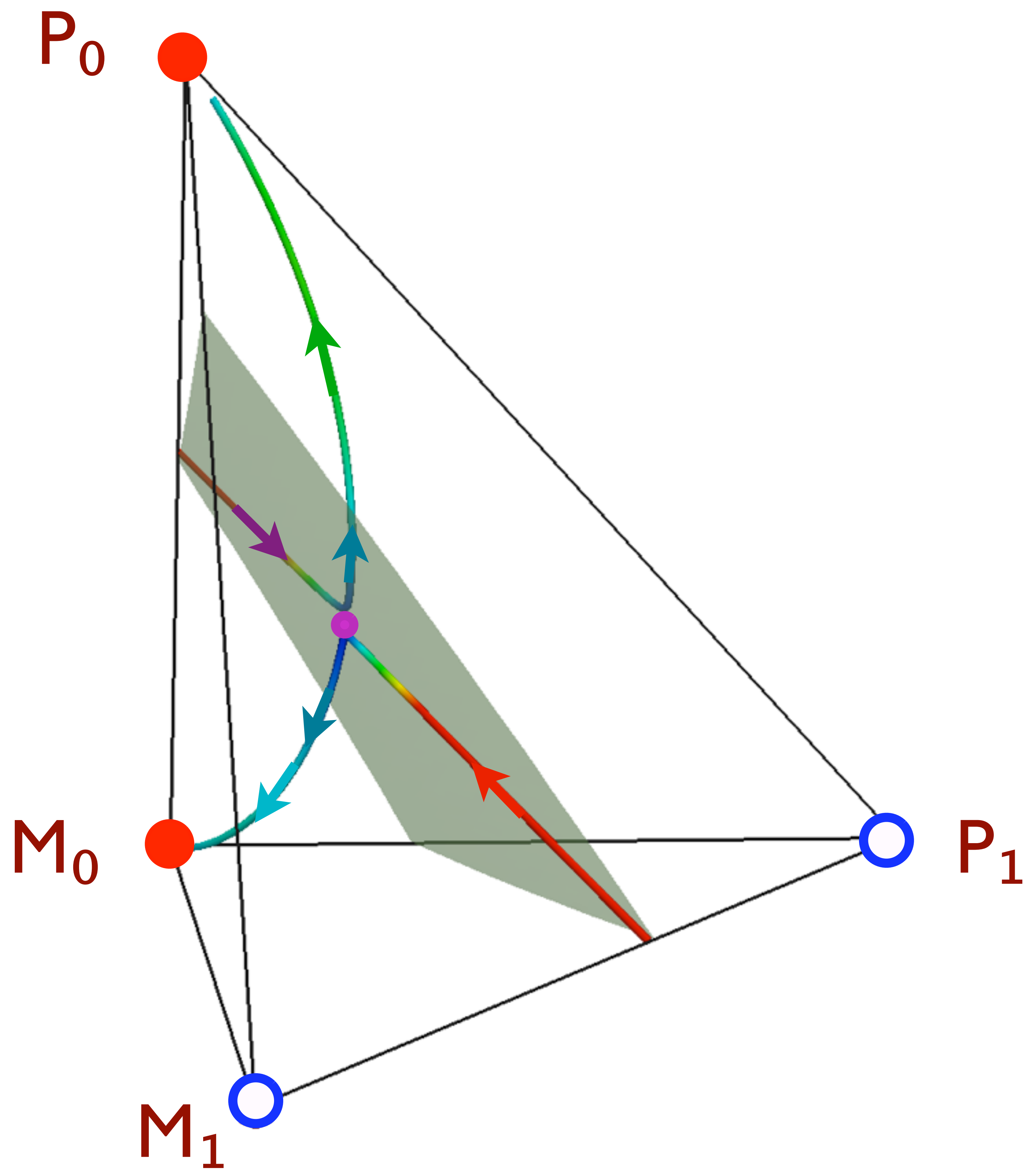
solution: $$\frac{P_0(t) - \lambda_+}{P_0(t) - \lambda_-} = \frac{P_0(0) - \lambda_+}{P_0(0) - \lambda_-} \, e^{-(\lambda_+ - \lambda_-)t}$$

near symmetric limit:

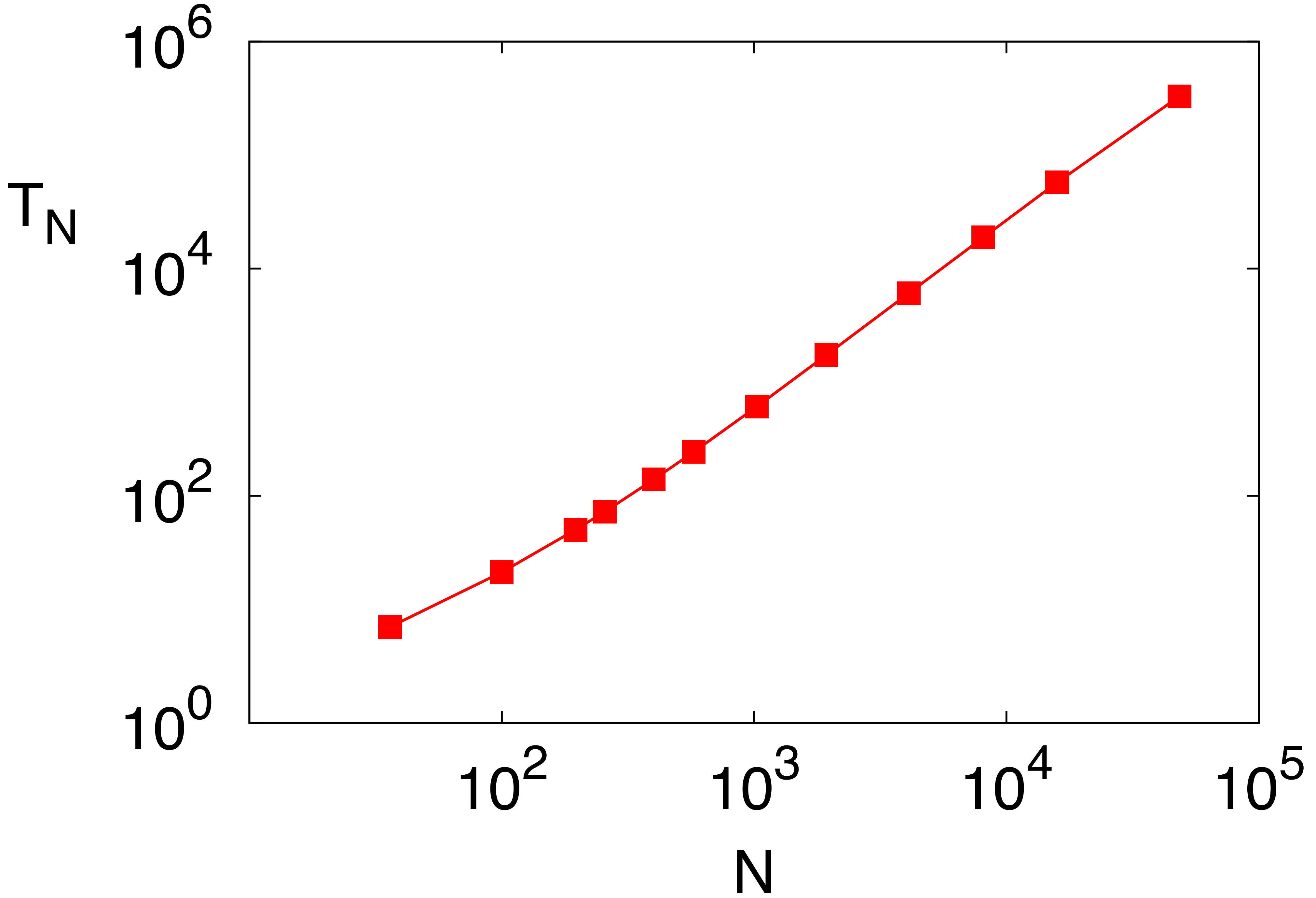
$$P_0 = \frac{1}{2} + 10^{-5},\ M_0 = \frac{1}{2} - 10^{-5},\ P_1 = M_1 = 0$$



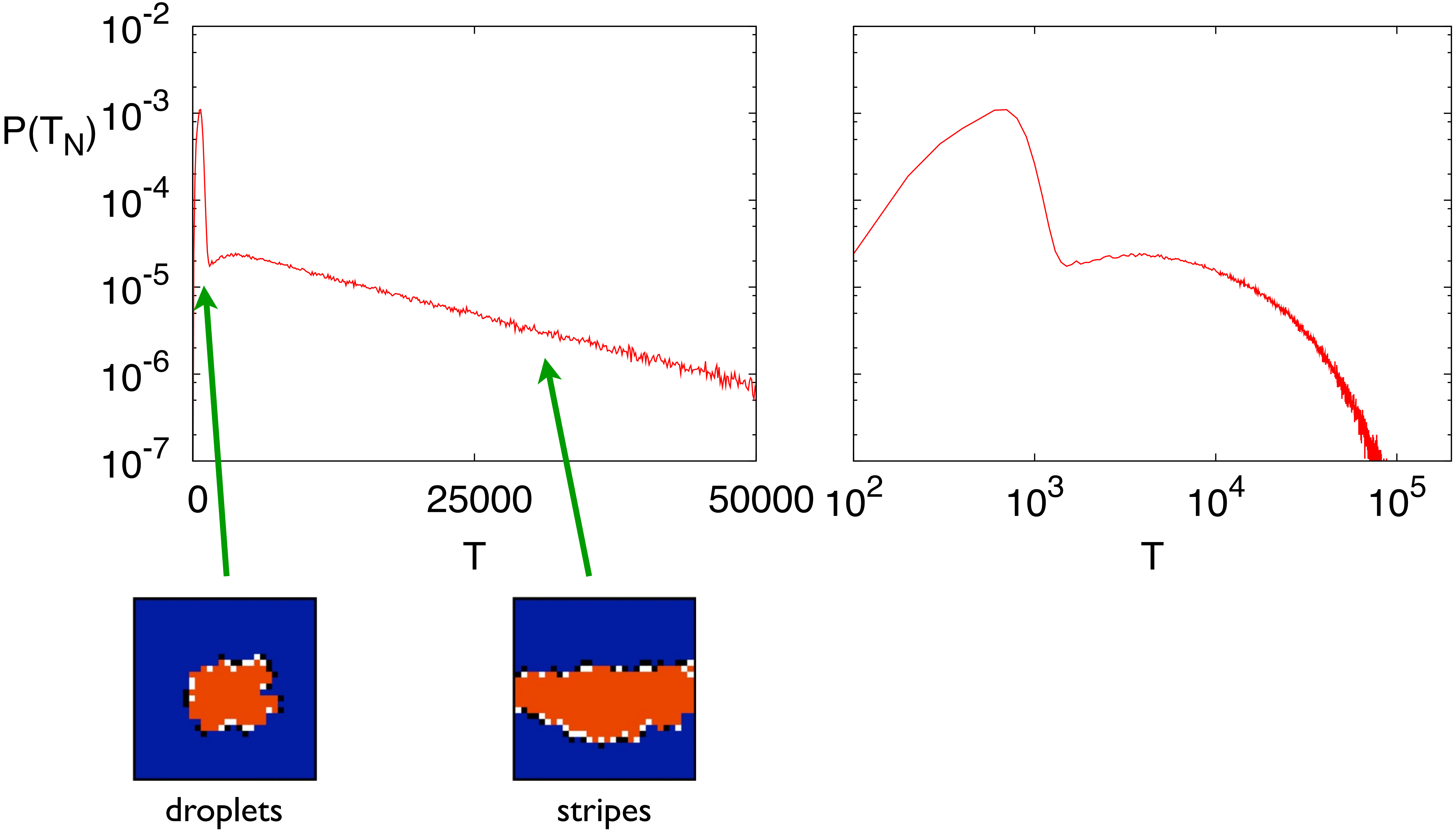
near symmetric limit: composition tetrahedron

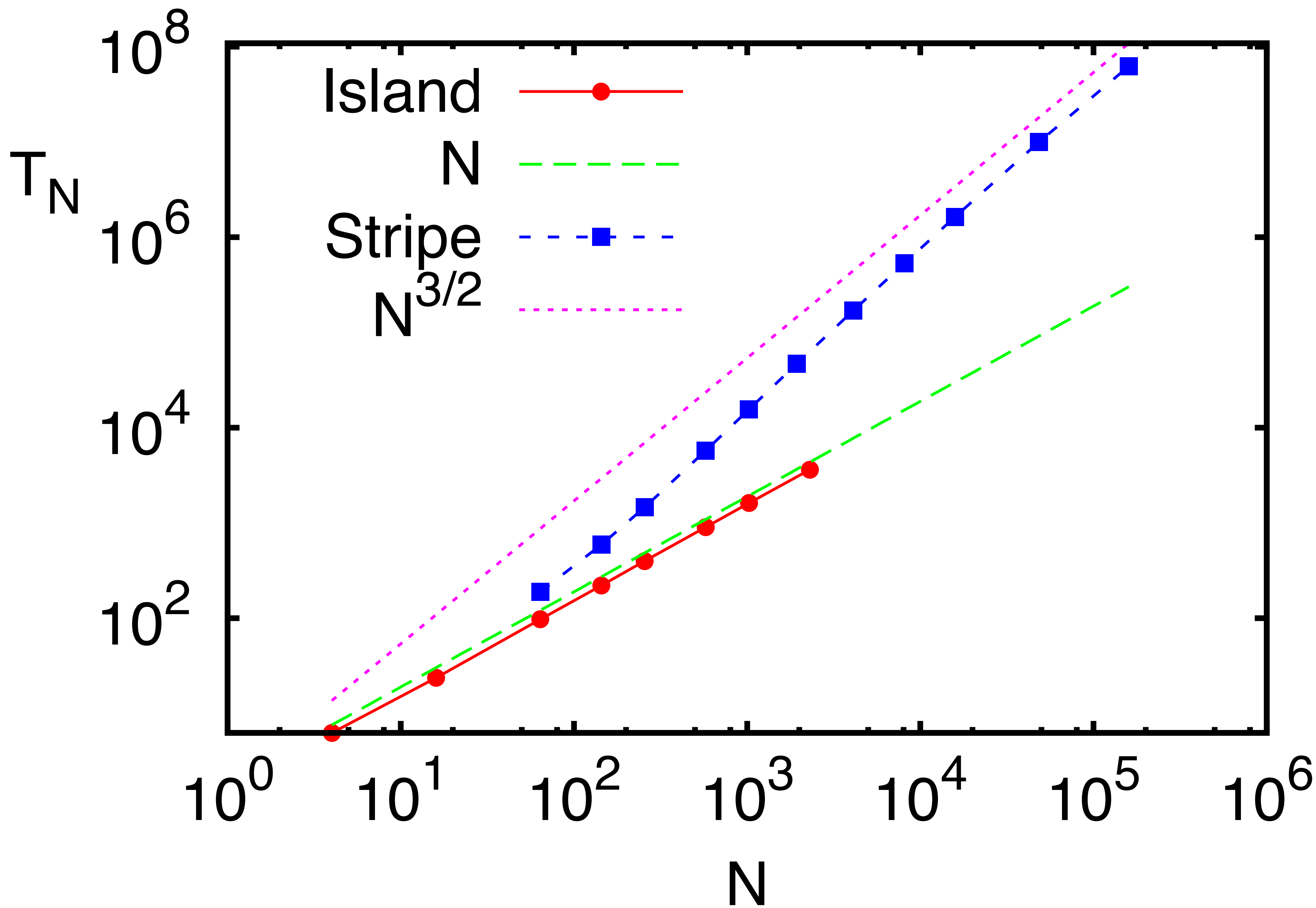


Consensus Time in Two Dimensions



Consensus Time Distribution





two time scales control approach to consensus

see also Spirin, Krapivsky, SR (2001), Chen & SR (2005)

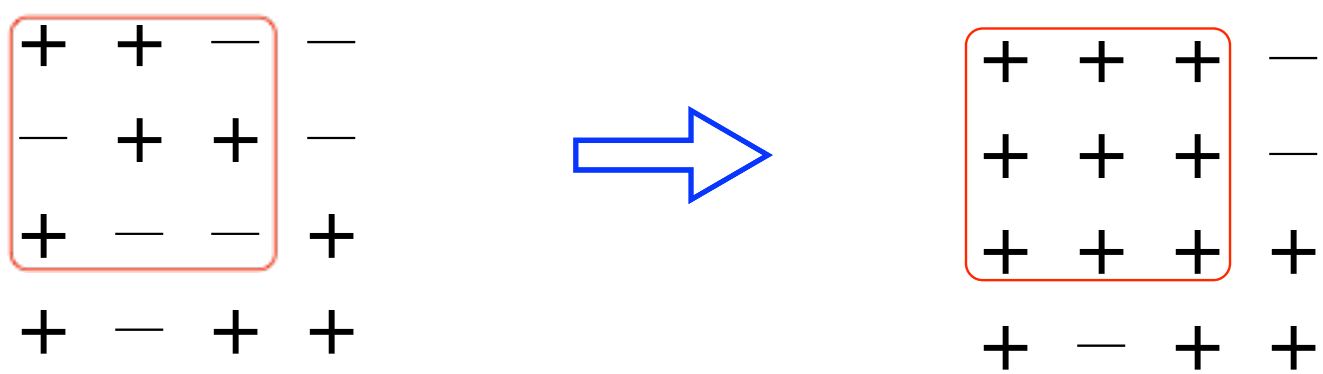
Ising model

Majority vote model

Majority rule

Galam (1999), Krapivsky & SR (2003), Slanina & Lavicka (2003), Chen & SR (2005)

1. Pick a random group of G spins (with G odd).
2. All spins in G adopt the majority state.
3. Repeat until consensus necessarily occurs.



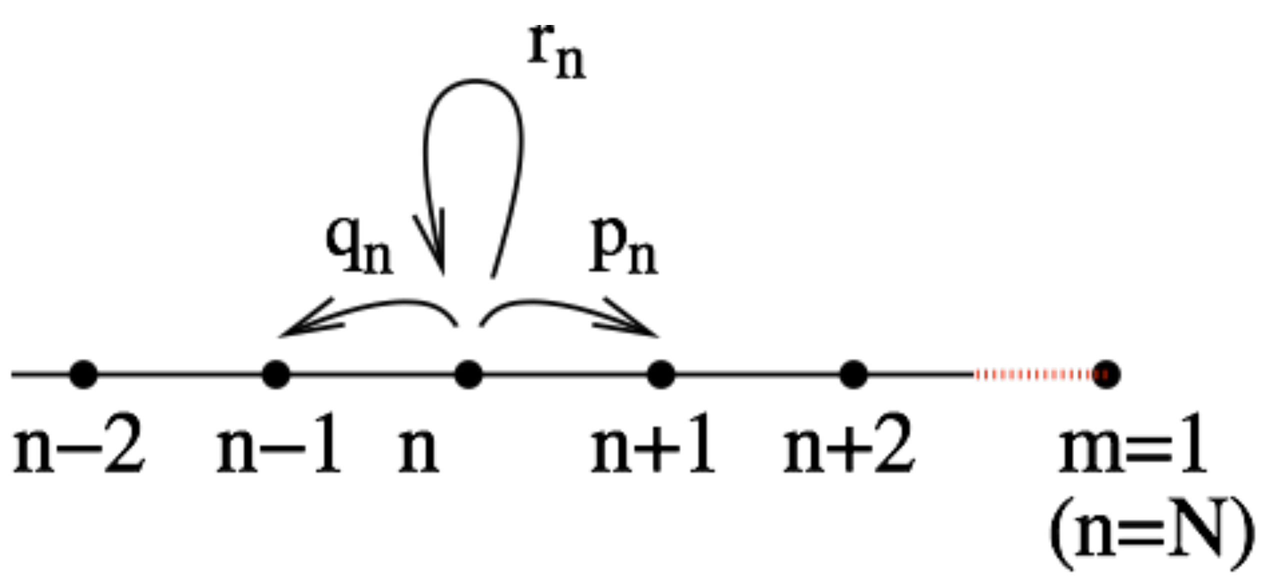
Basic questions: *1. Which final state is reached?*

2. What is the time until consensus?

Mean-field theory (for G=3)

$$E_n \equiv \text{exit probability to } m = 1 \text{ starting from } n \text{ plus spins}$$

$$= p_n E_{n+1} + q_n E_{n-1} + r_n E_n$$



where $p_n = \binom{3}{2}\binom{N-3}{n-2} / \binom{N}{n}$

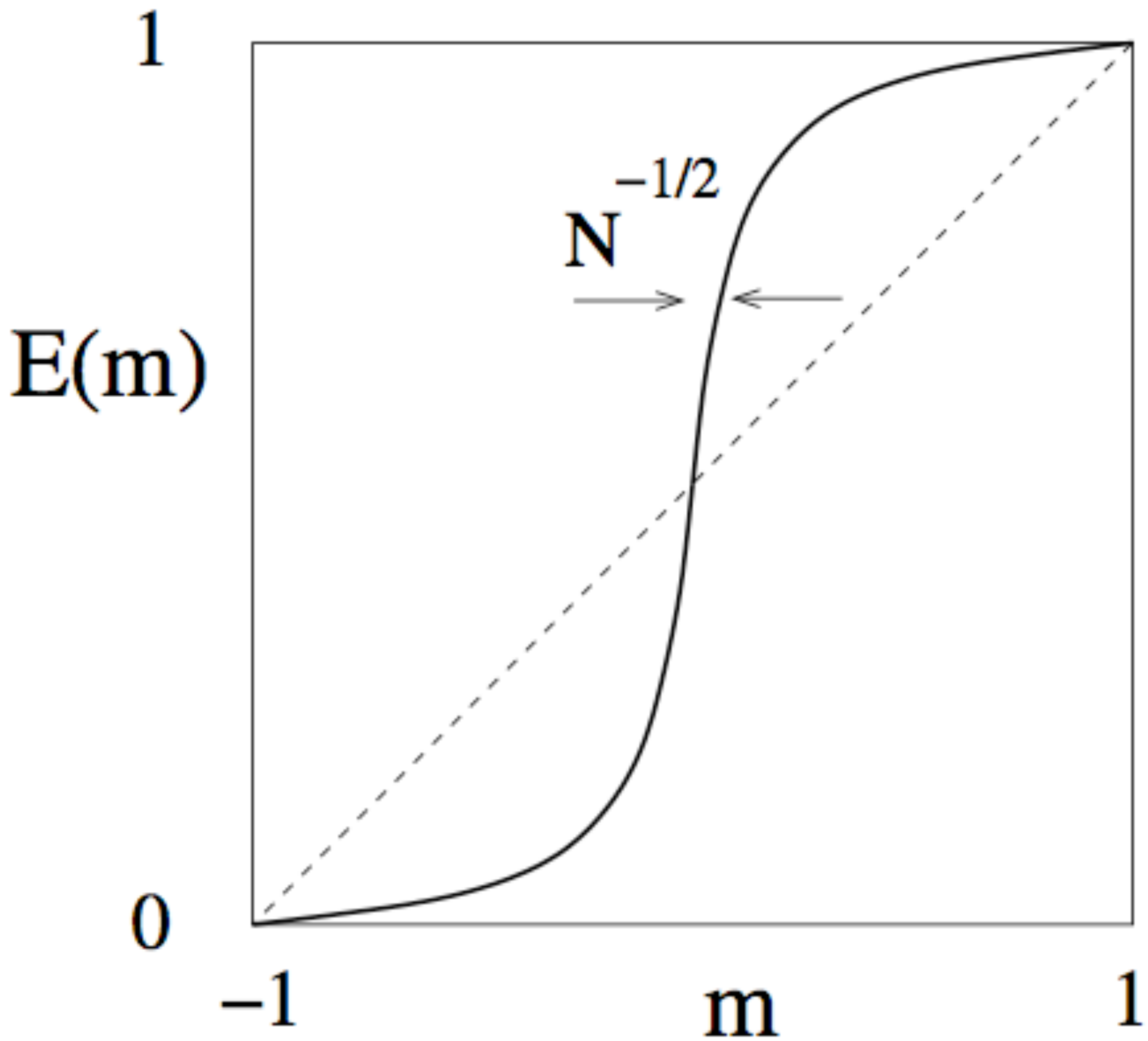
$q_n = \binom{3}{1}\binom{N-3}{n-1} / \binom{N}{n}$

$r_n = 1 - p_n - q_n$

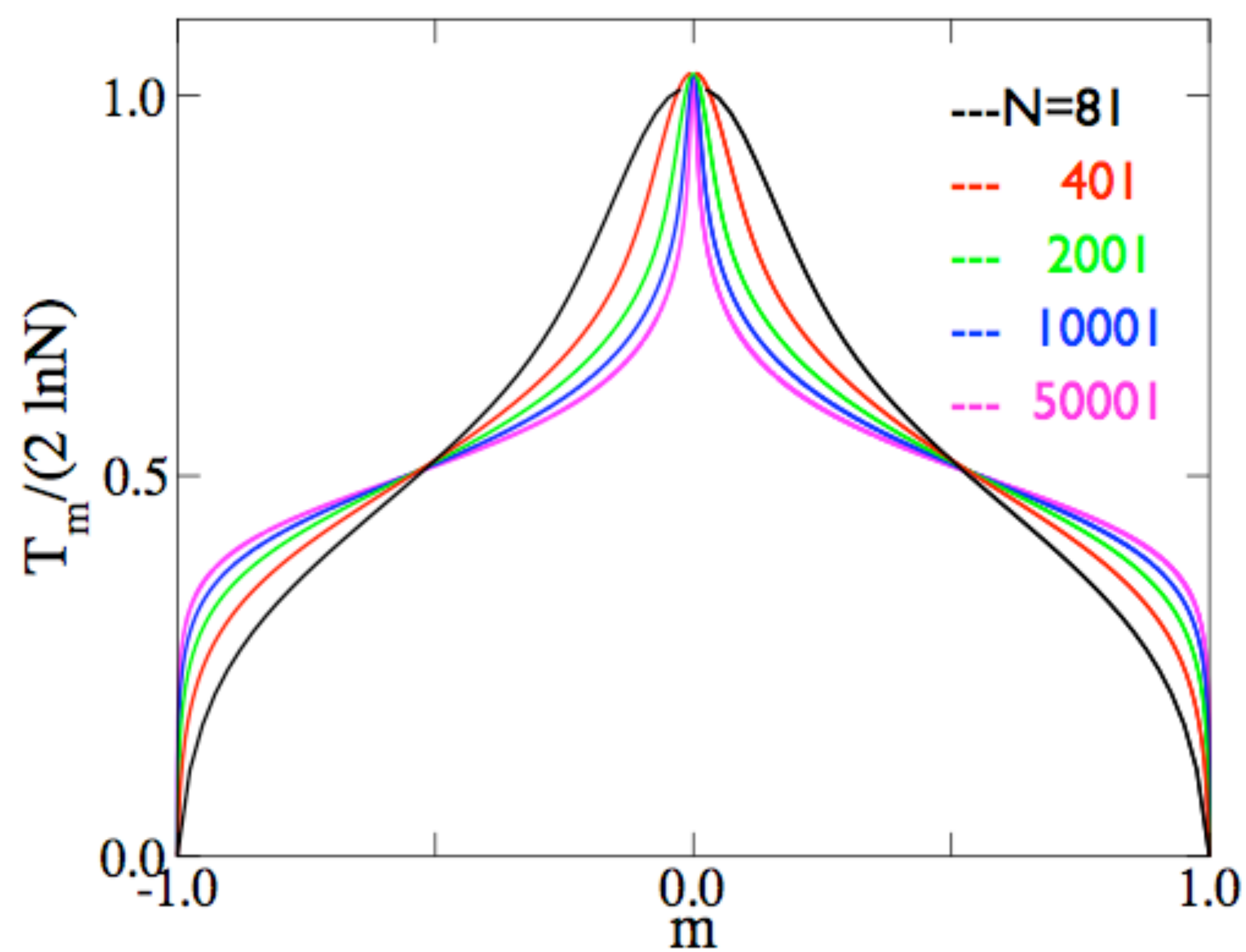
$$T_n \equiv \text{mean time to } m = 1 \text{ starting from } n \text{ plus spins}$$

$$= p_n(T_{n+1} + \delta t) + q_n(T_{n-1} + \delta t) + r_n(T_n + \delta t)$$

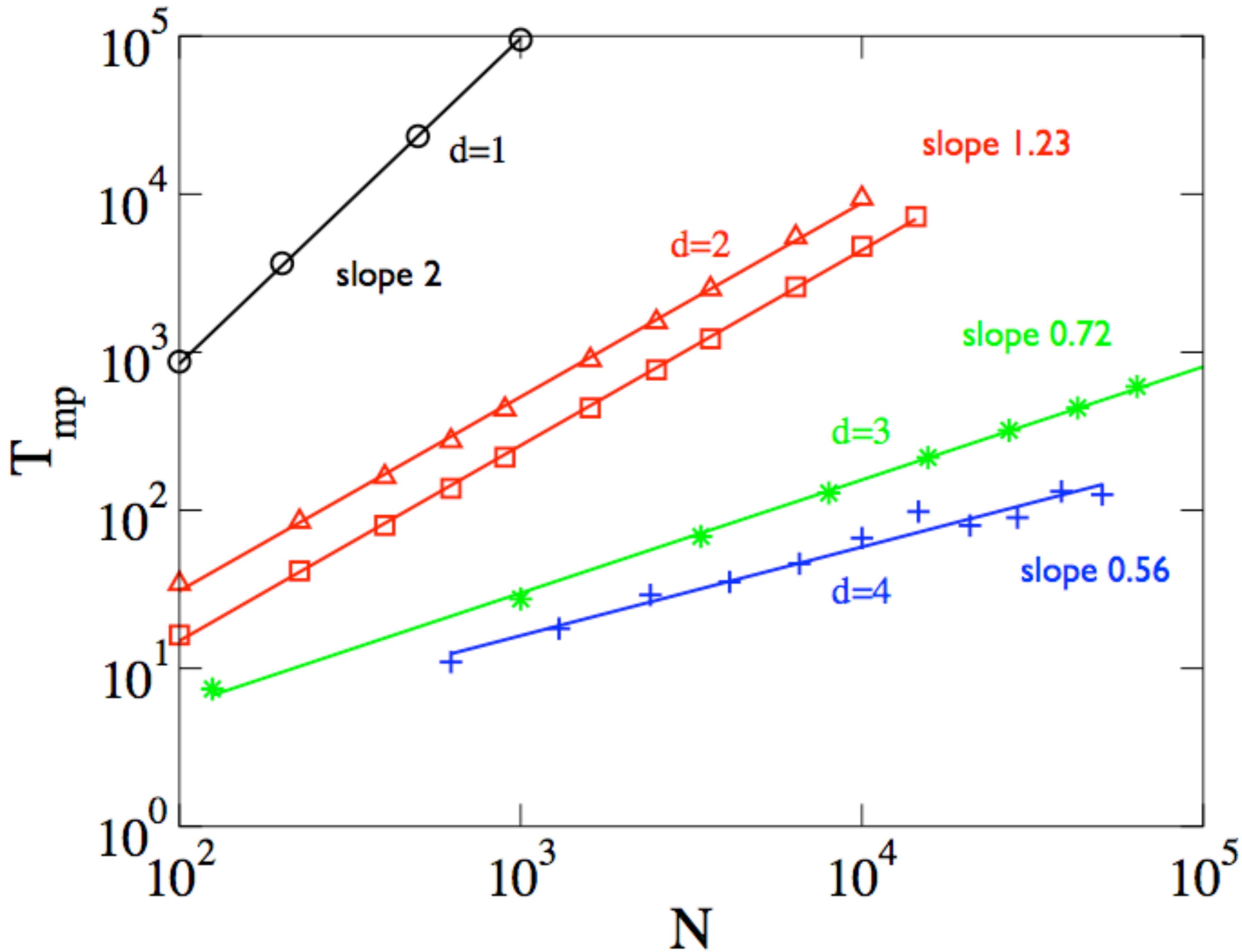
Exit probability
(schematic)



Consensus time
(data)

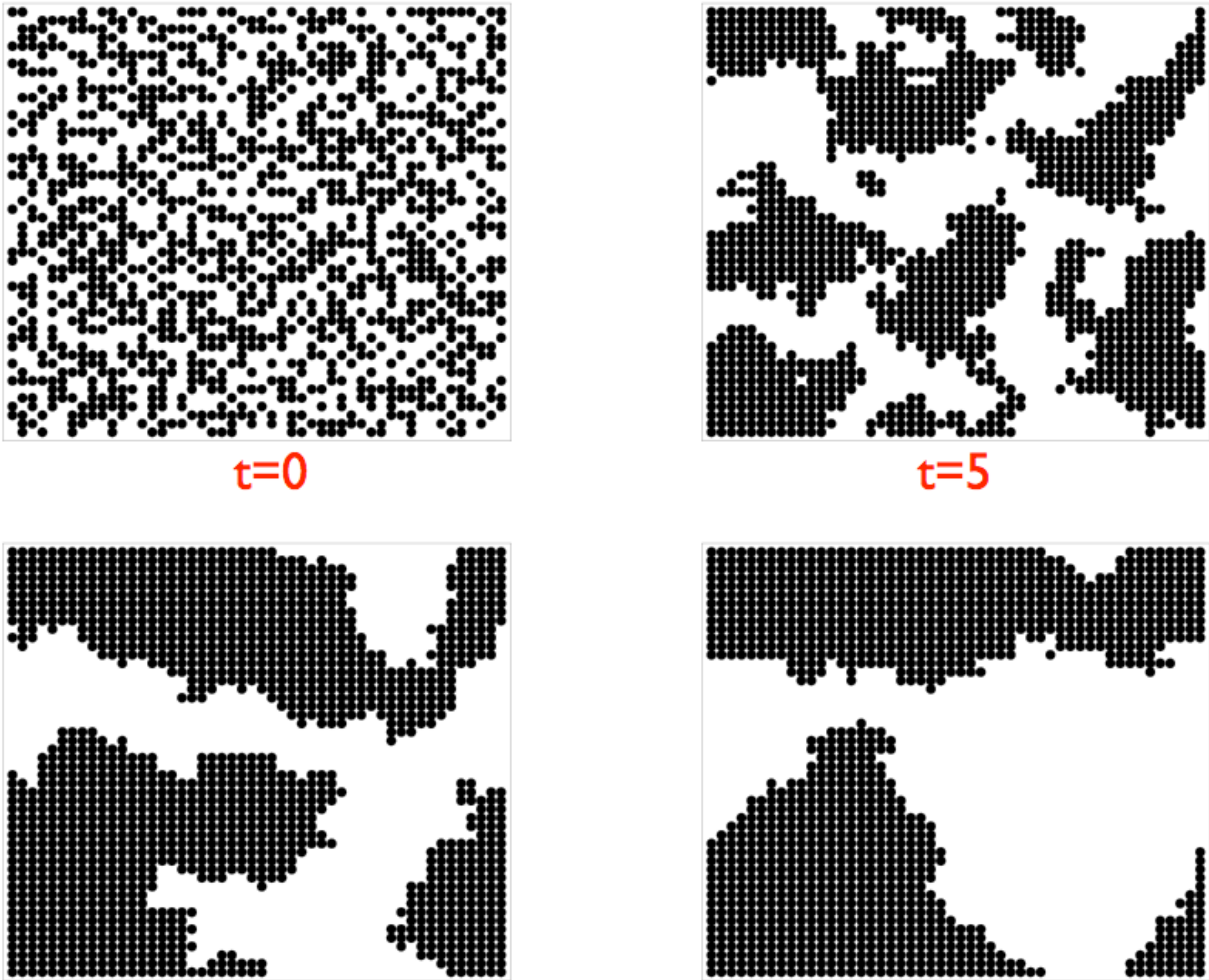


Consensus time for finite spatial dimensions



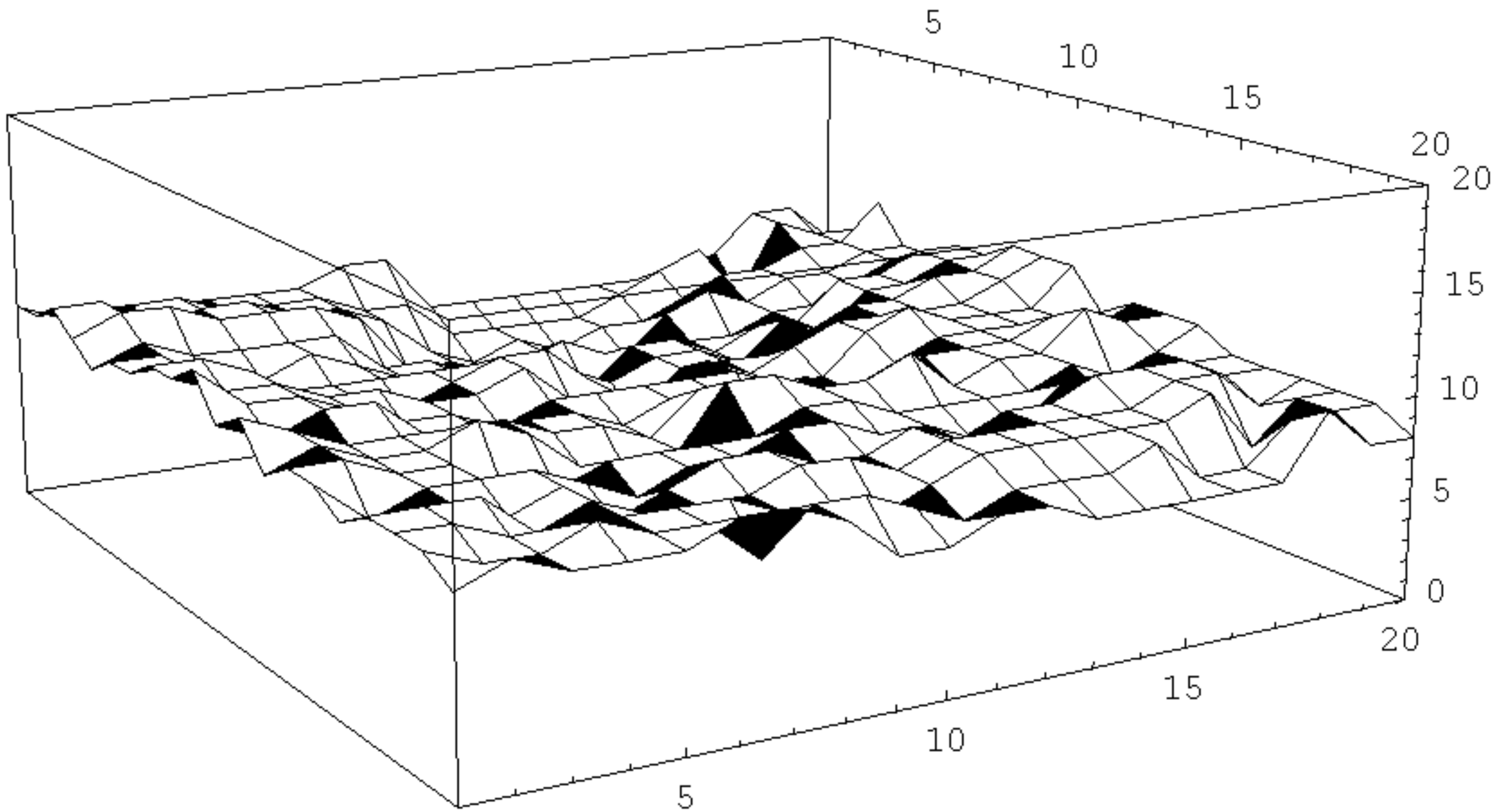
Critical dimension appears to be >4!

Anomalous dynamics in 2d: stripes ~33% of the time!

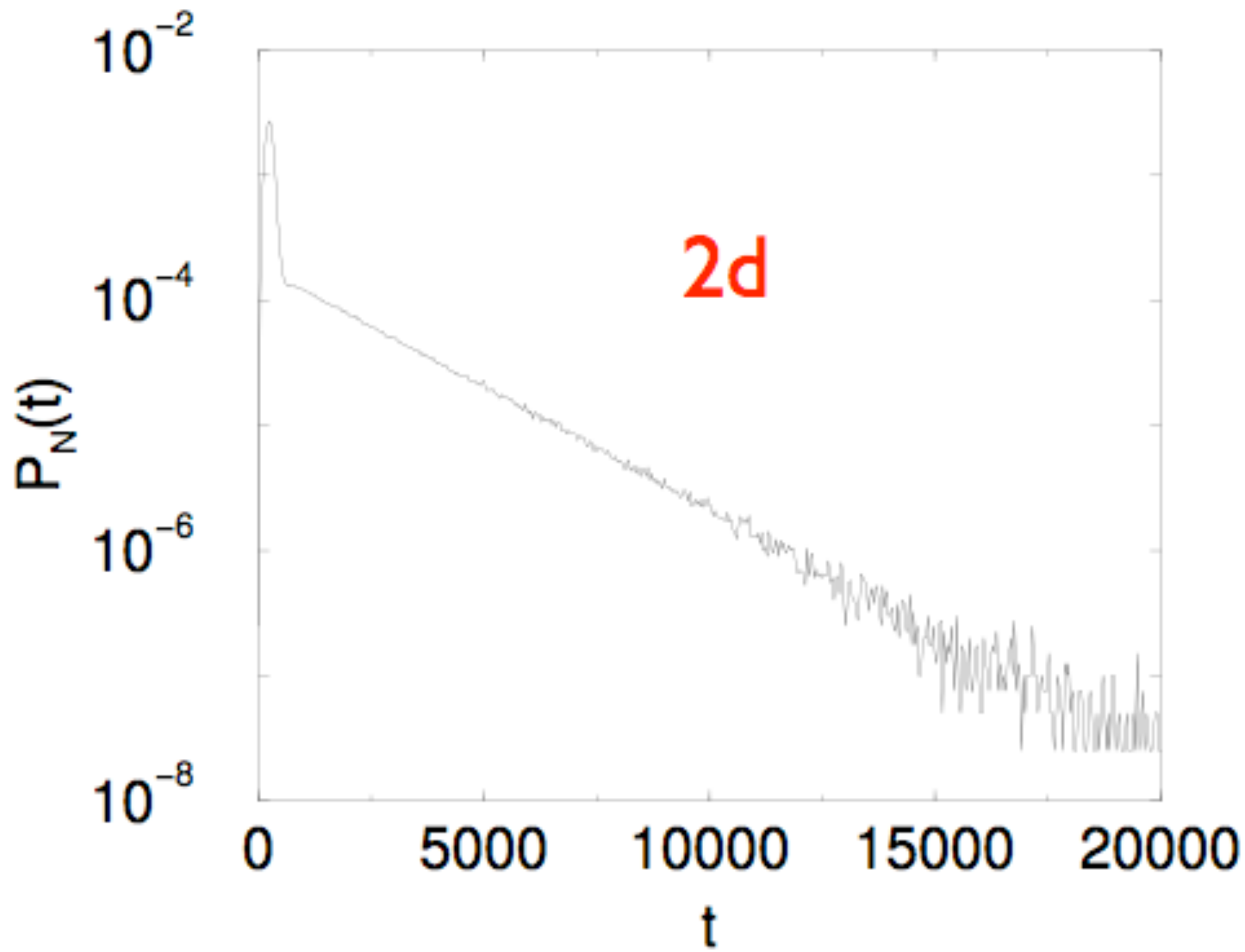


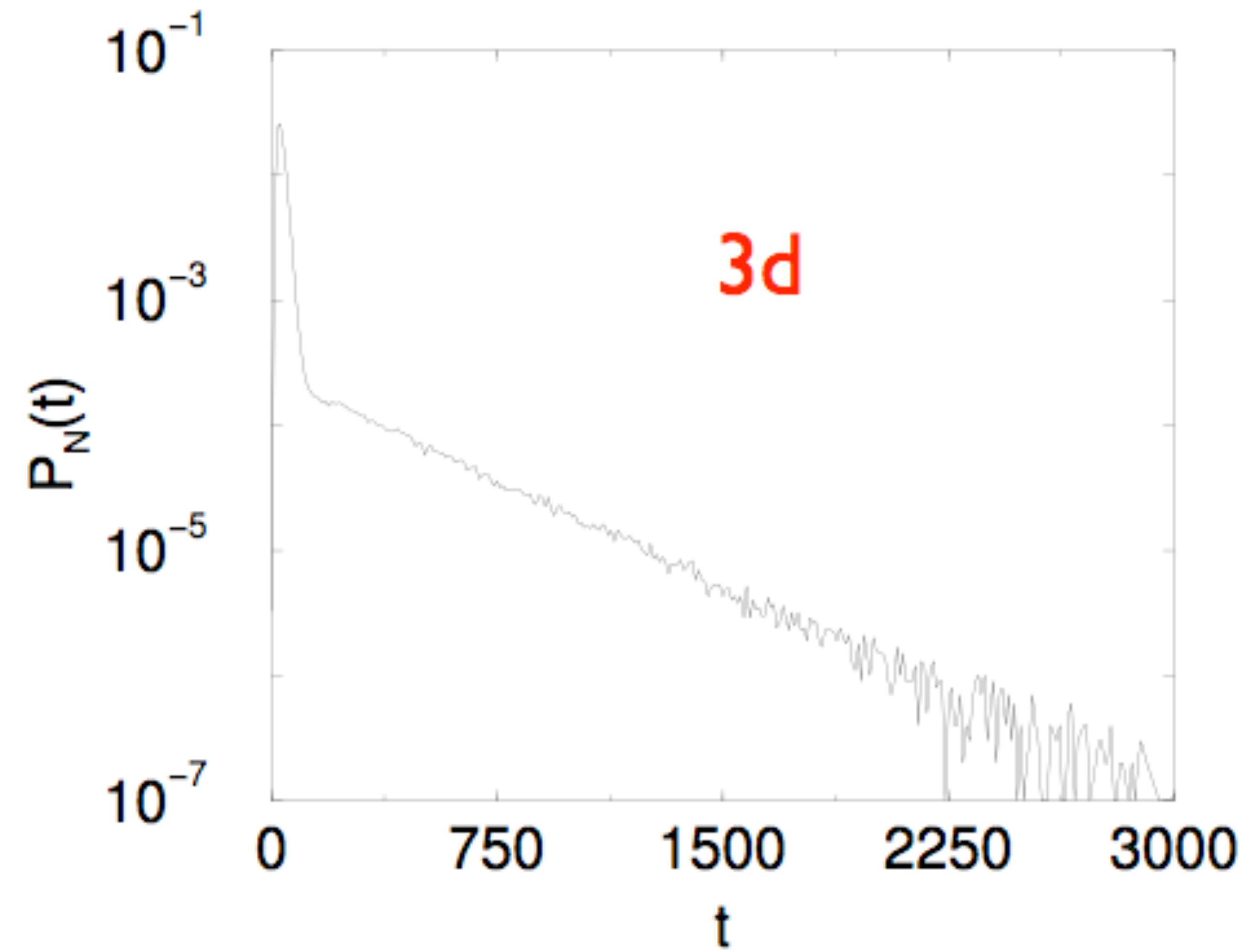
t=0 t=5 t=20 t=80

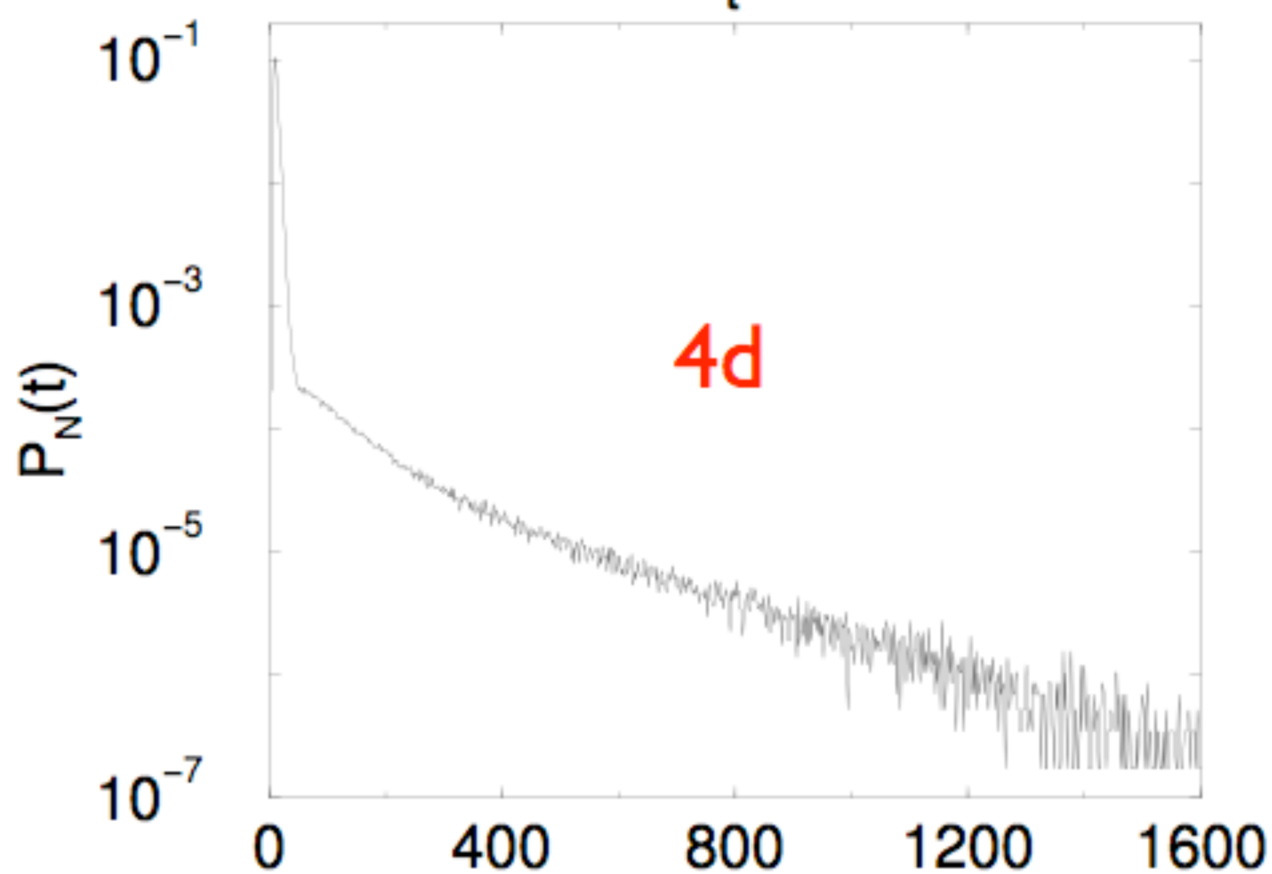
Slab formation in 3d ~8% of the time



Consensus time distribution







multiscale relaxation
to final consensus

The Dynamics of Persuasion

Sid Redner, Santa Fe Institute (physics.bu.edu/~redner)

CIRM, Luminy France, January 5-9, 2015

T. Antal, E. Ben-Naim, P. Chen, P. L. Krapivsky, M. Mobilia, V. Sood, F. Vazquez, D. Volovik

+ support from NSF

Modeling Consensus:

- introduction to the voter model
- voter model on complex networks
- voting with some confidence
- majority rule

lecture 1

Modeling Discord & Diversity:

- 3-state voter models
- strategic voting
- bounded compromise
- dynamics of social balance
- Axelrod model

lecture 2

lecture 3

Discord & Diversity

If people are reasonable, why is consensus hard to reach?

Possibilities:
- insufficient communication
- appreciable diversity
- stubbornness
- …. your favorite mechanism

Models:

Three voting states: − 0 +; − + noninteracting — Vazquez & SR (2004)

Strategic voting: ideology vs. strategy — Volovik, Mobilia & SR (2009)

Bounded confidence: compromise only when close
Deffuant, Neau, Amblard & Weisbuch (2000)
Hegselmann & Krause (2002)
Ben Naim, Krapivsky & SR (2003)

Social balance: dynamics of positive/negative links
Antal, Krapivsky & SR (2005, 2006)

Axelrod model: many features, many traits
Axelrod (1997)
Castellano, Marsili & Vespignani (2000)
Vazquez & SR (2007)

Three Voting States: − 0 +

0. 3-state voter at each site: $- \quad 0 \quad +$
1. Pick a random voter
2. Assume state of neighbor if compatible
3. Repeat until either consensus or frozen final state

− 0 → − −
− 0 → 0 0

+0 → ++
+0 → 0 0

compatible

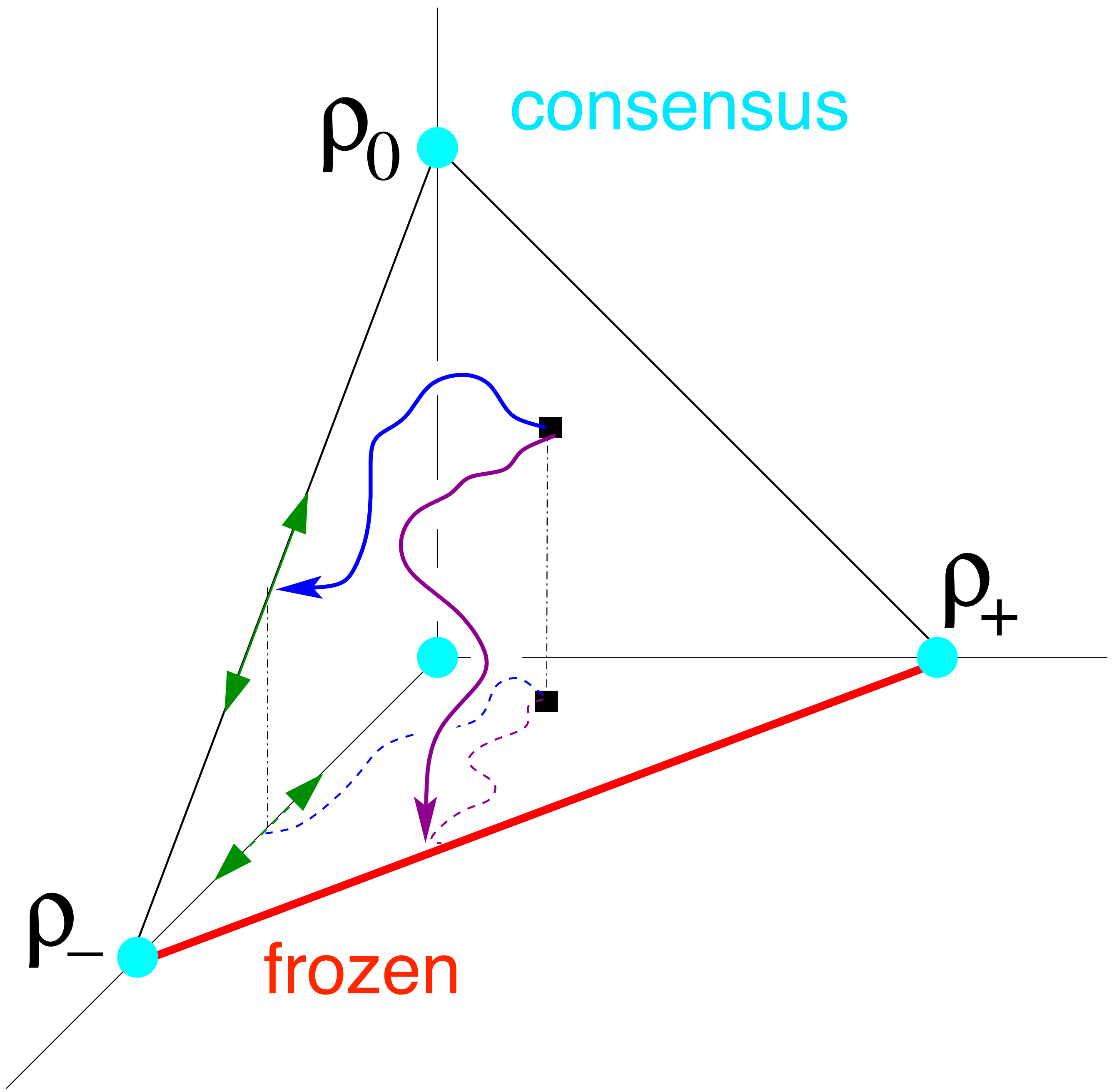
$$(N_0, N_-) \rightarrow (N_0 \pm 1, N_- \mp 1) \quad \text{prob.} \quad \frac{N_0 N_-}{N^2} = \rho_0 \rho_-$$

$$(N_0, N_+) \rightarrow (N_0 \pm 1, N_+ \mp 1) \quad \text{prob.} \quad \frac{N_0 N_+}{N^2} = \rho_0 \rho_+$$

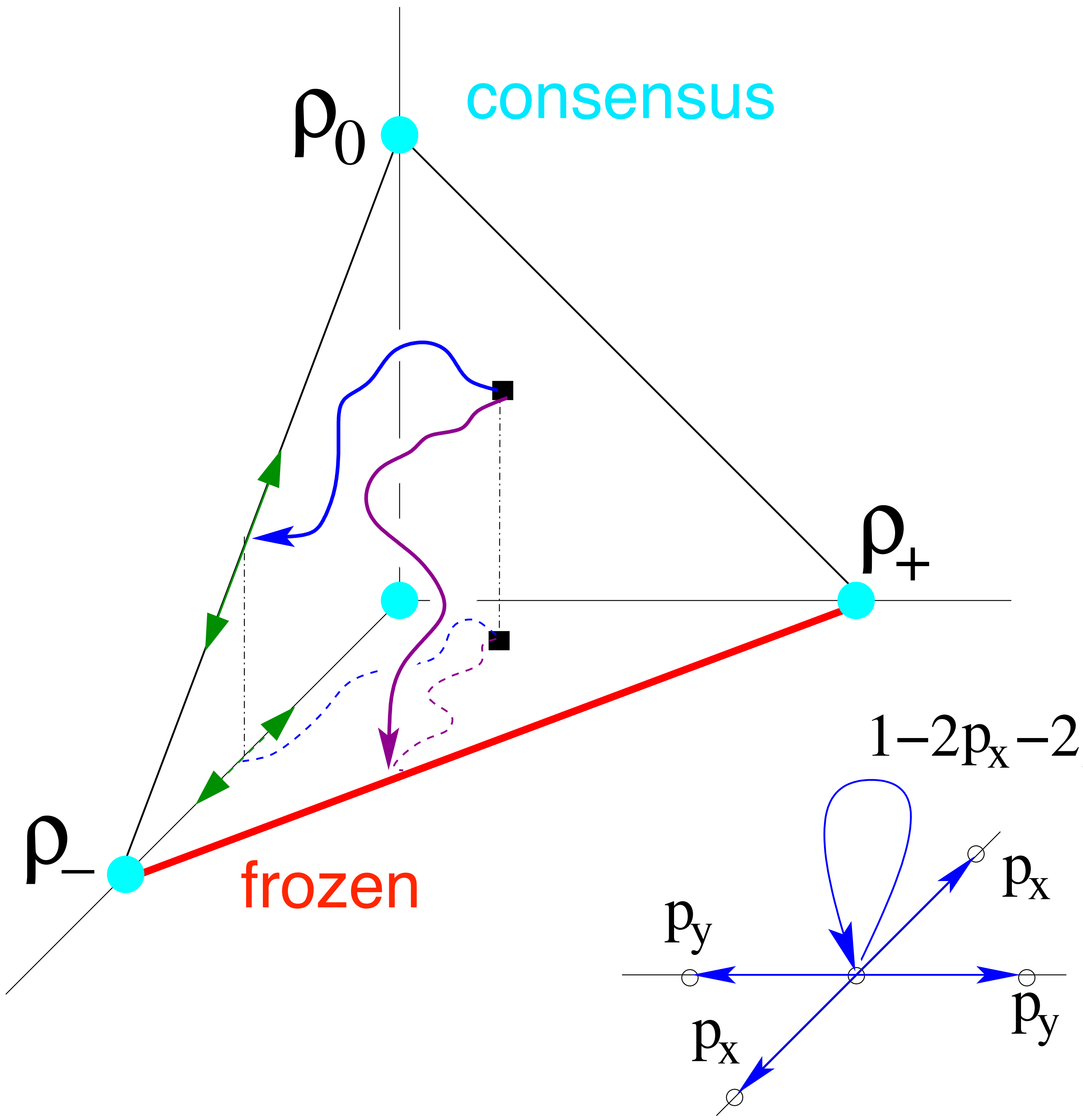
+ − → + −

incompatible

Evolution in Composition Triangle



Evolution in Composition Triangle



$$p_x = \rho_0 \rho_-$$

$$p_y = \rho_0 \rho_+$$

The Phase Diagram

$$F(\rho_-, \rho_+) = \text{prob. to reach frozen state starting from } (\rho_-, \rho_+)$$

recursion:
$$\begin{aligned} F(\rho_-, \rho_+) &= p_x \left[F(\rho_- - \delta, \rho_+) + F(\rho_- + \delta, \rho_+)\right] \\ &= p_y \left[F(\rho_-, \rho_+ - \delta) + F(\rho_-, \rho_+ + \delta)\right] \\ &= \left[1 - 2(p_x + p_y)\right] F(\rho_-, \rho_+) \end{aligned}$$

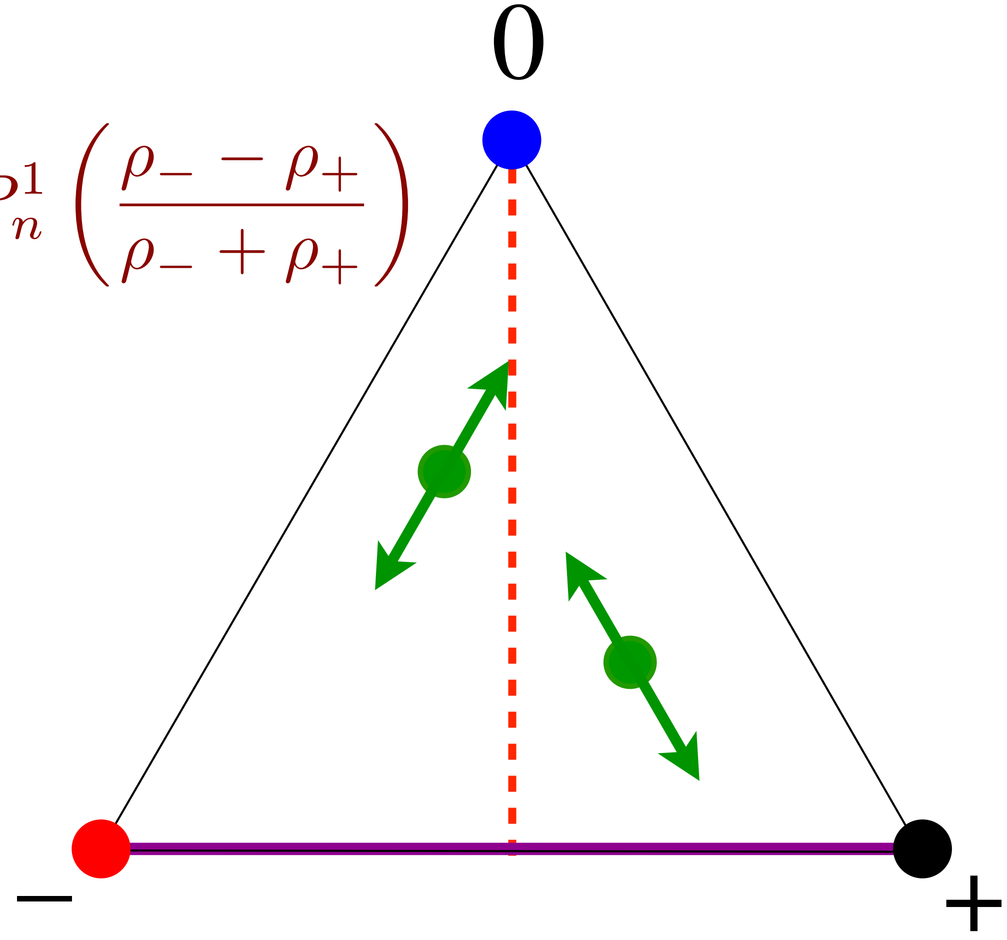
continuum limit:
$$\rho_- \frac{\partial^2 F(\rho_-, \rho_+)}{\partial \rho_-^2} + \rho_+ \frac{\partial^2 F(\rho_-, \rho_+)}{\partial \rho_+^2} = 0$$

$$F(\rho_-, 0) = 0$$
$$F(0, \rho_+) = 0$$
$$F(\rho_+, 1-\rho_+) = 1$$

$$F(\rho_-, \rho_+) = \sum_{n \text{ odd}} \frac{2(2n+1)}{n(n+1)} \sqrt{\rho_- \rho_+}\, (\rho_- + \rho_+)^n\, P_n^1\left(\frac{\rho_- - \rho_+}{\rho_- + \rho_+}\right)$$

$$F(\rho_0) = 1 - \frac{1 - (1-\rho_0)^2}{\sqrt{1 + (1-\rho_0)^2}}$$

symmetric limit



The Phase Diagram

$$F(\rho_-,\rho_+) = \text{prob. to reach frozen state starting from } (\rho_-,\rho_+)$$

recursion:
$$\begin{aligned} F(\rho_-,\rho_+) &= p_x\,[F(\rho_- - \delta,\rho_+) + F(\rho_- + \delta,\rho_+)] \\ &= p_y\,[F(\rho_-,\rho_+ - \delta) + F(\rho_-,\rho_+ + \delta)] \\ &= [1 - 2(p_x + p_y)]\,F(\rho_-,\rho_+) \end{aligned}$$

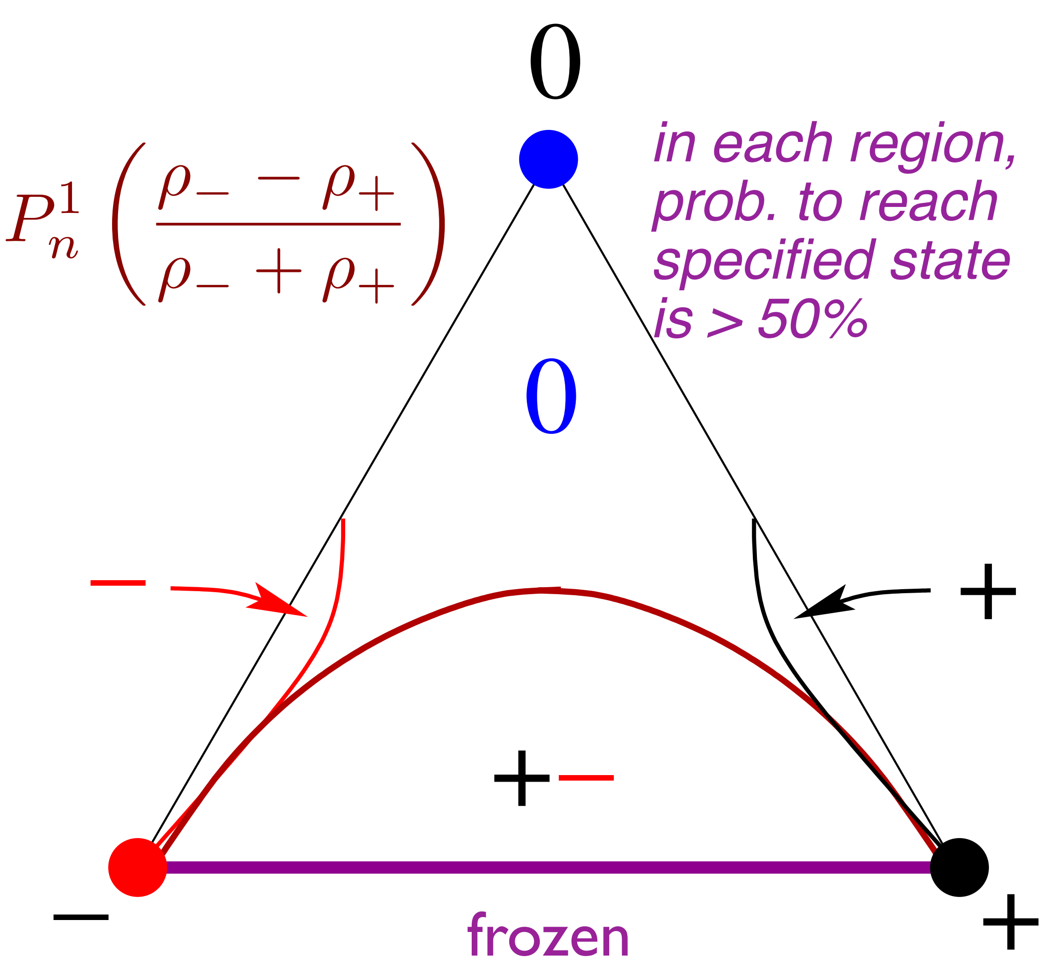
continuum limit:
$$\rho_- \frac{\partial^2 F(\rho_-,\rho_+)}{\partial \rho_-^2} + \rho_+ \frac{\partial^2 F(\rho_-,\rho_+)}{\partial \rho_+^2} = 0$$

$$F(\rho_-,0) = 0$$
$$F(0,\rho_+) = 0$$
$$F(\rho_+,1-\rho_+) = 1$$

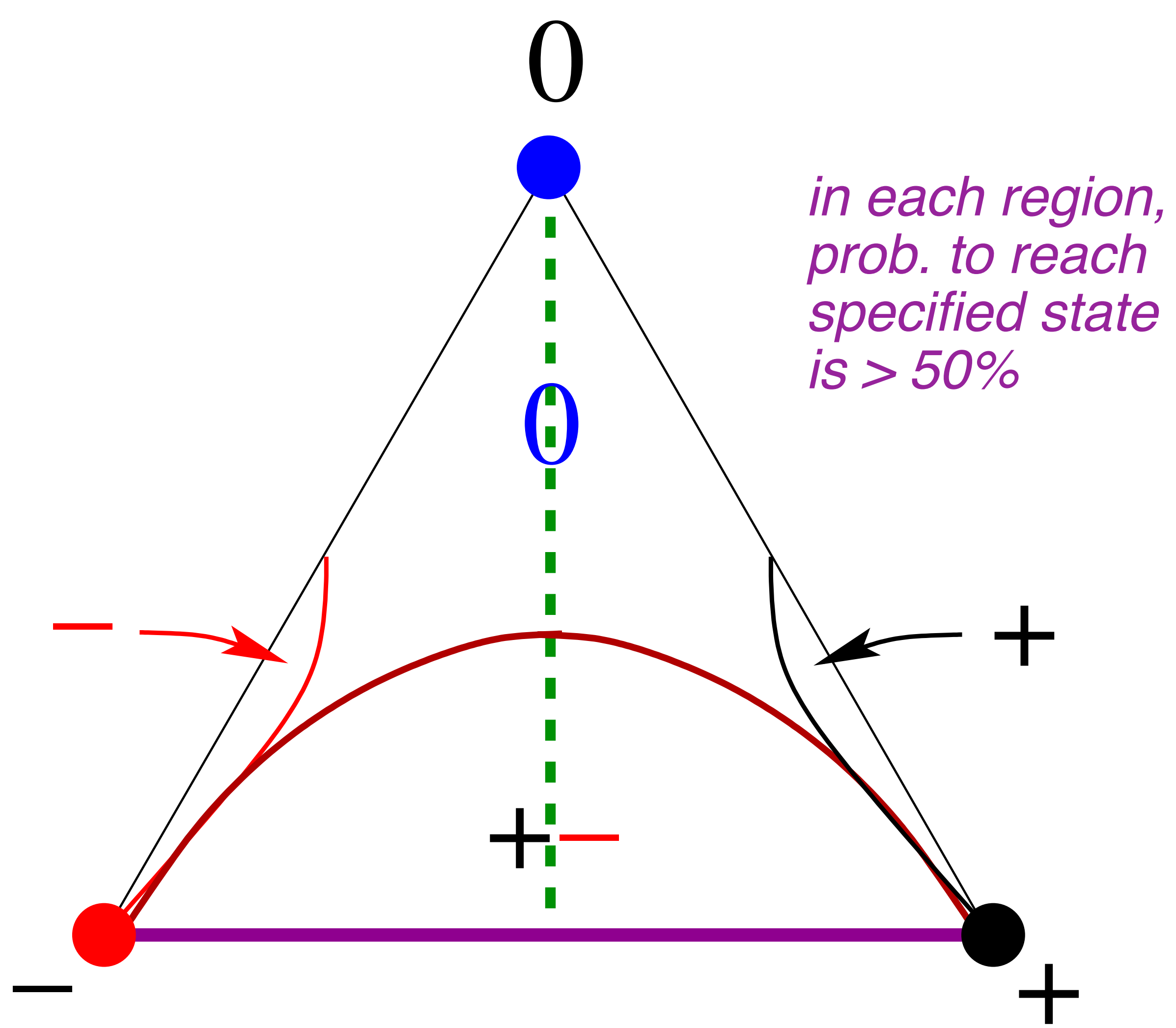
$$F(\rho_-,\rho_+) = \sum_{n\ \mathrm{odd}} \frac{2(2n+1)}{n(n+1)} \sqrt{\rho_-\rho_+}\,(\rho_- + \rho_+)^n\, P_n^1\!\left(\frac{\rho_- - \rho_+}{\rho_- + \rho_+}\right)$$

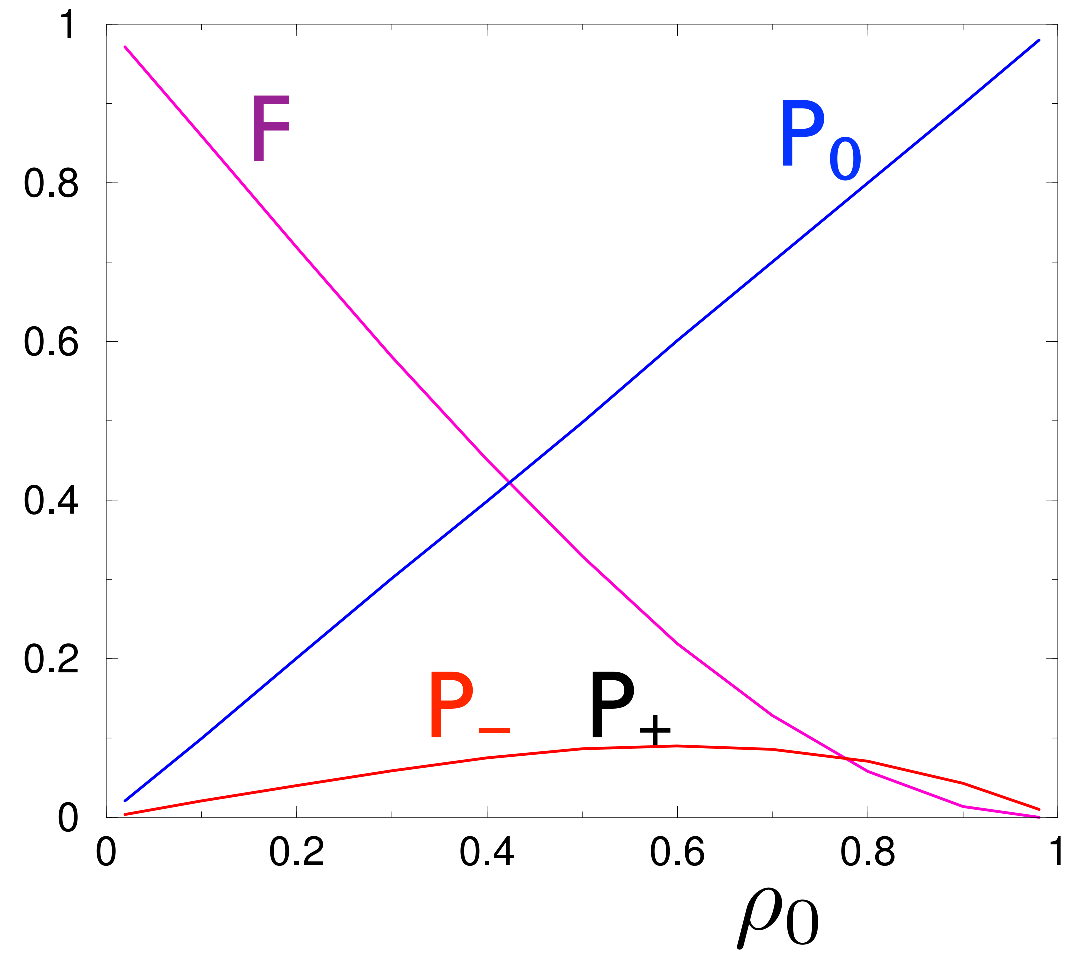
$$F(\rho_0) = 1 - \frac{1-(1-\rho_0)^2}{\sqrt{1+(1-\rho_0)^2}}$$

symmetric limit



Phase Diagram & Final State Probabilities

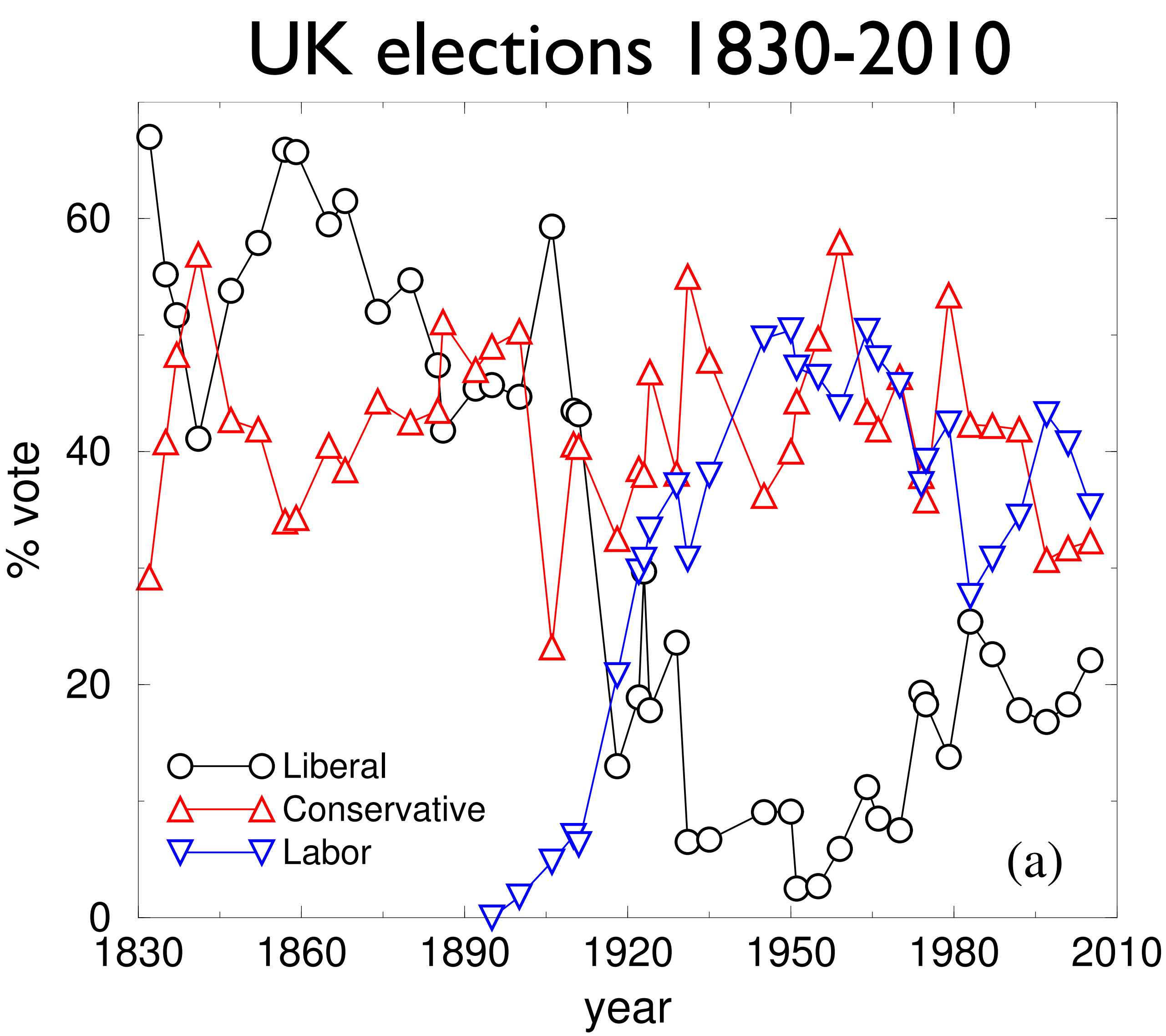


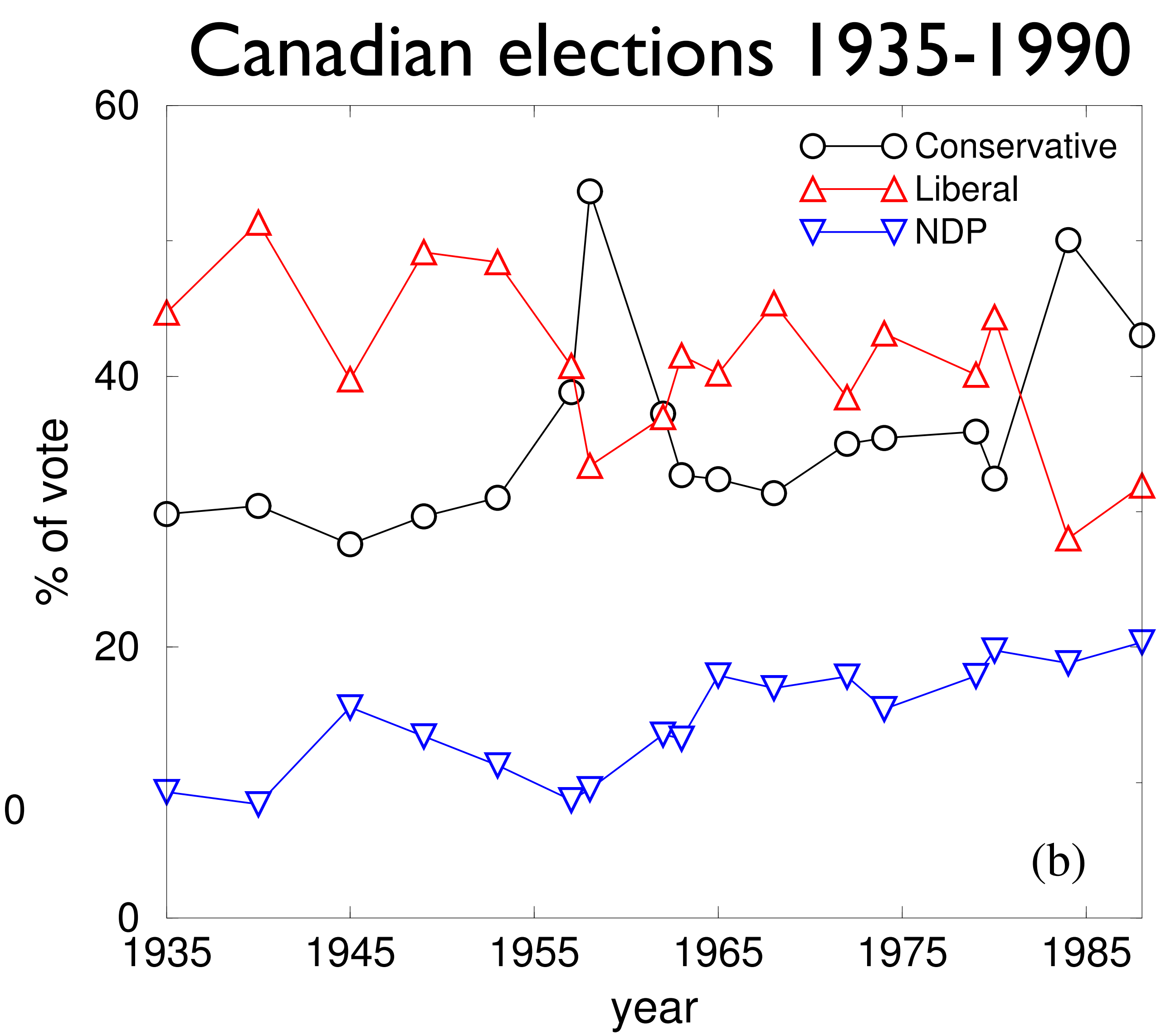


moral: extremism promotes deadlock

Strategic Voting

¿vote *for* first choice? ¿vote *against* last choice?





Strategic Voter Model

evolution of the densities a, b, c:

$$\dot{a} =$$

$$\dot{b} =$$

$$\dot{c} =$$

Strategic Voter Model

evolution of the densities a, b, c:

temperature

$$\dot{a} = T(b + c - 2a)$$

$$\dot{b} = T(c + a - 2b)$$

$$\dot{c} = T(a + b - 2c)$$

Strategic Voter Model

evolution of the densities a, b, c:

temperature / strategic voting

$$\dot{a} = T(b + c - 2a) + r_{AC}\, ac + r_{AB}\, ab$$

$$\dot{b} = T(c + a - 2b) + r_{BA}\, ba + r_{BC}\, bc$$

$$\dot{c} = T(a + b - 2c) + r_{CA}\, ca + r_{CB}\, cb$$

Strategic Voter Model

evolution of the densities a, b, c:

temperature — strategic voting

$$\dot{a} = T(b + c - 2a) + r_{AC}\, ac + r_{AB}\, ab$$

$$\dot{b} = T(c + a - 2b) + r_{BA}\, ba + r_{BC}\, bc$$

$$\dot{c} = T(a + b - 2c) + r_{CA}\, ca + r_{CB}\, cb$$

strategic voting rates:

$$r_{AB} = -r_{BA} = \begin{cases} +r & B \text{ minority} \\ 0 & C \text{ minority} \\ -r & A \text{ minority} \end{cases}$$

two natural choices

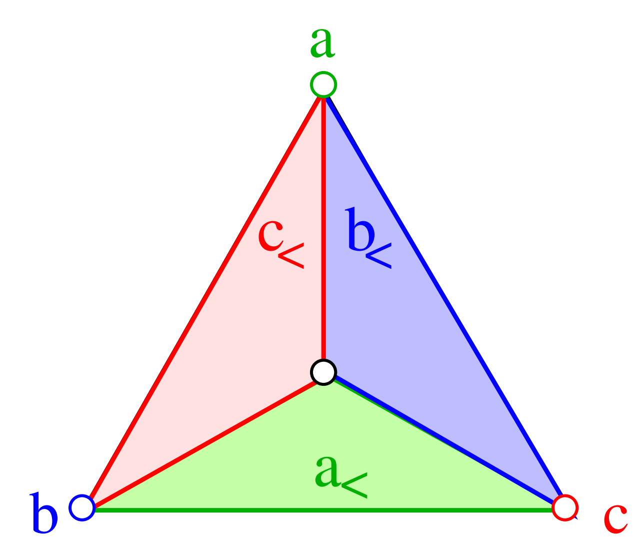
$r = \text{const.}$ too drastic

$r = r_0[(a+b)/2 - c]$ better

$$\dot{a} = T(1 - 3a) + rac$$

→ $$\dot{b} = T(1 - 3b) + rbc \qquad \text{in } c_< \text{ sector}$$

$$\dot{c} = T(1 - 3c) - rc(1 - c)$$



Strategic Voter Model

$$\dot{c} = (1-3c)T - \frac{r_0}{2}c(1-c)(1-3c)$$
$$\equiv -\frac{3r_0}{2}(c-c_-)(c-c_+)(c-c_3)$$

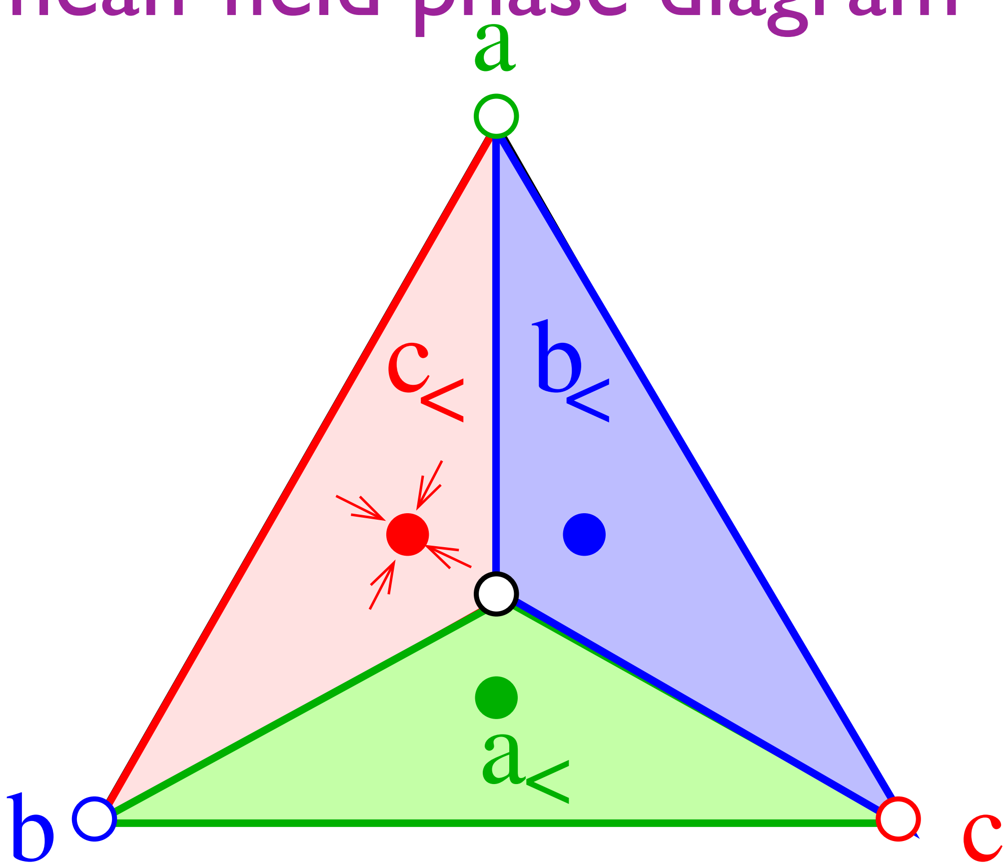
$$c_3 = \tfrac{1}{3}$$
$$c_\pm = \tfrac{1}{2}\left(1 \pm \sqrt{1-8x_0}\right)$$
$$x_0 \equiv \tfrac{T}{r_0}$$

$$\left[\frac{c(t)-c_3}{c(0)-c_3}\right]^{\alpha_3}\left[\frac{c(t)-c_+}{c(0)-c_+}\right]^{\alpha_+}\left[\frac{c(t)-c_-}{c(0)-c_-}\right]^{\alpha_-} = e^{-3(c_+-c_-)r_0 t/2}$$

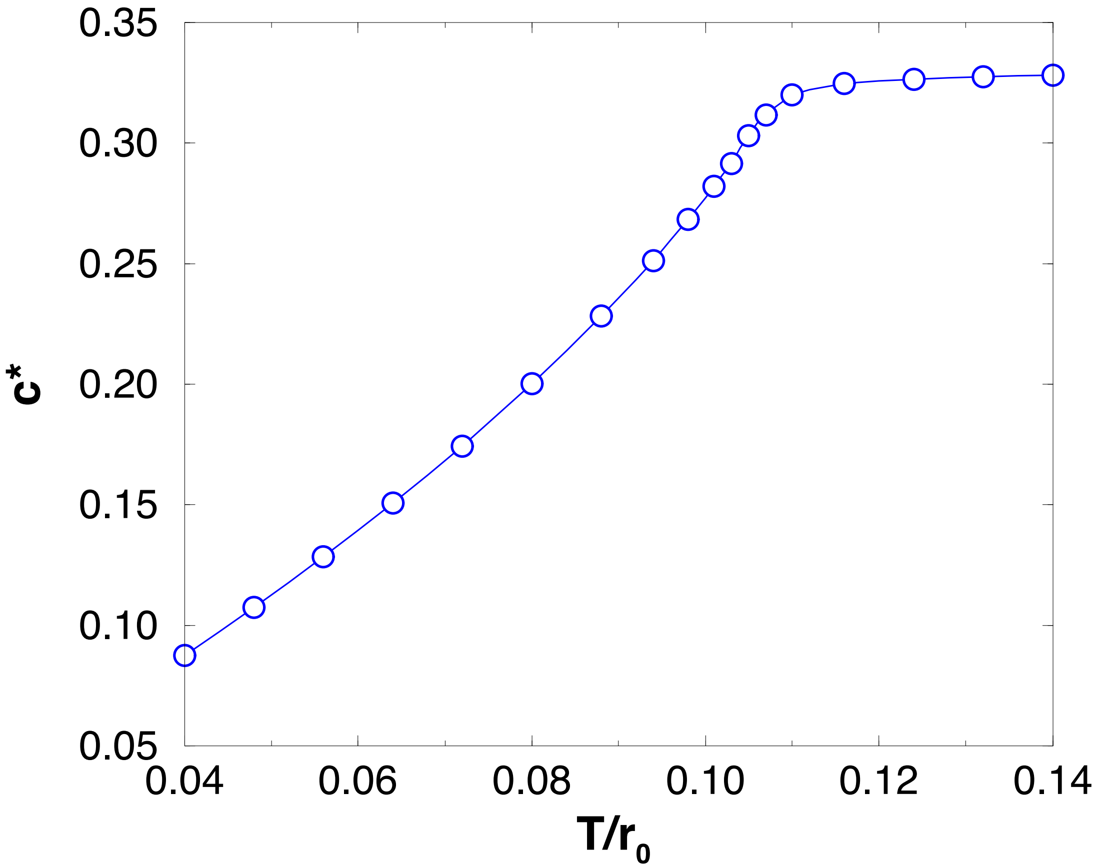
$$\alpha_\pm = \frac{1}{c_3-c_\pm} \qquad \alpha_3 = \alpha_- - \alpha_+ = \frac{c_+-c_-}{(c_3-c_+)(c_3-c_-)}$$

Strategic Voter Model

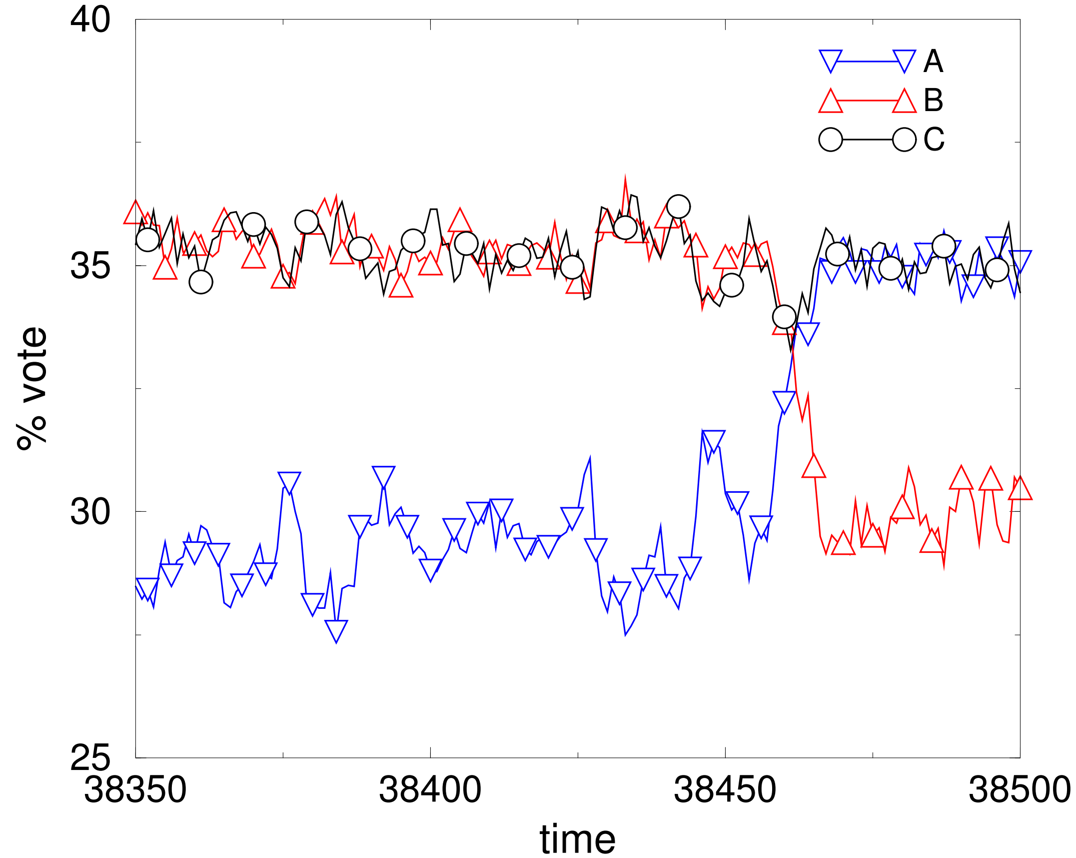
mean-field phase diagram

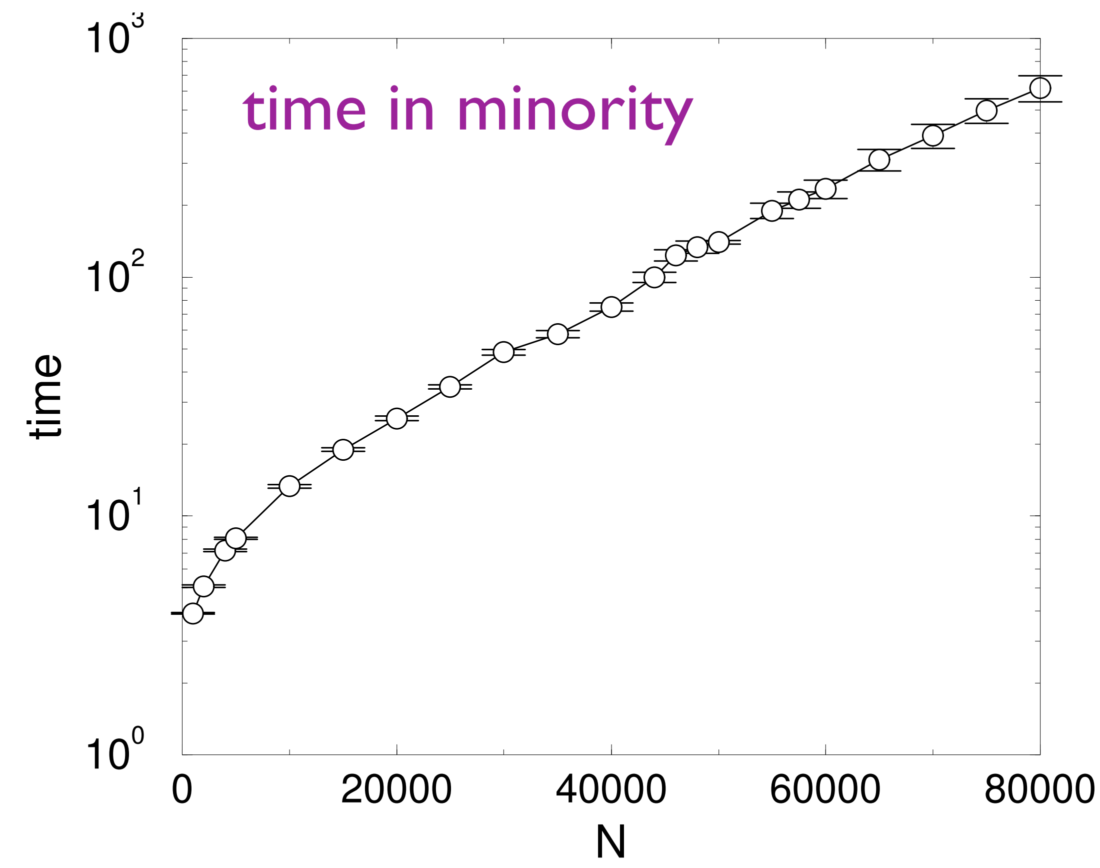


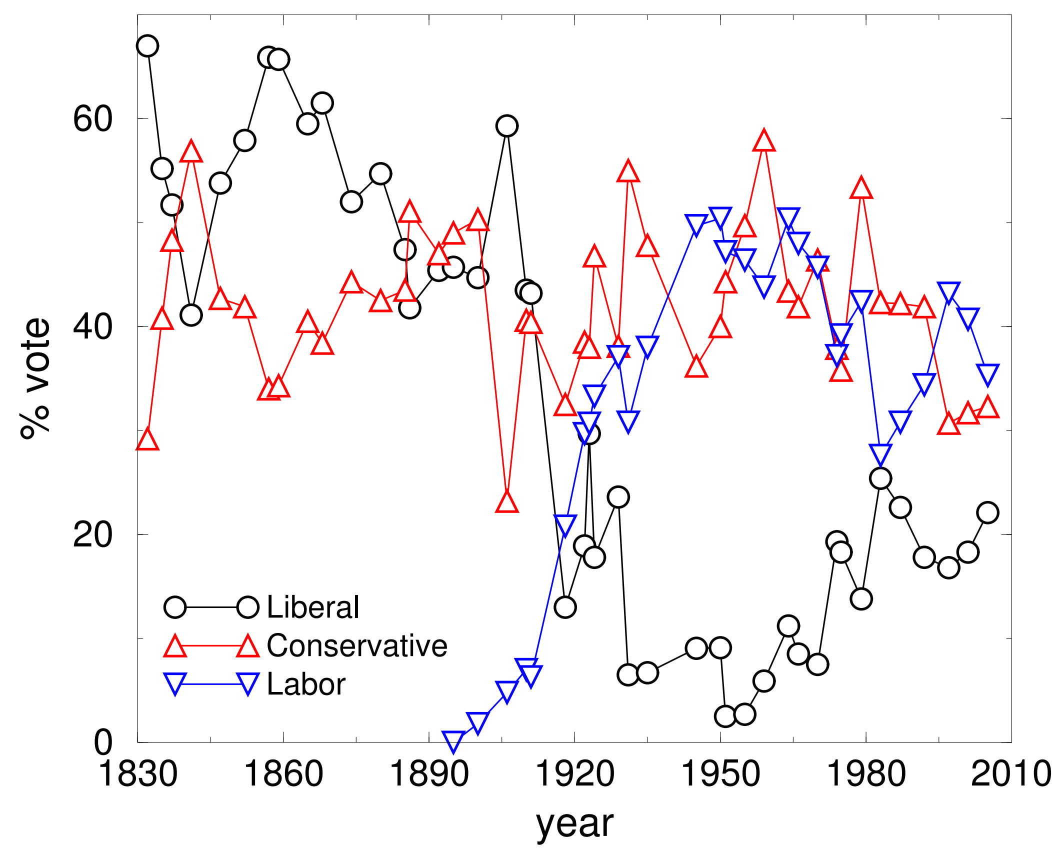
location of fixed point



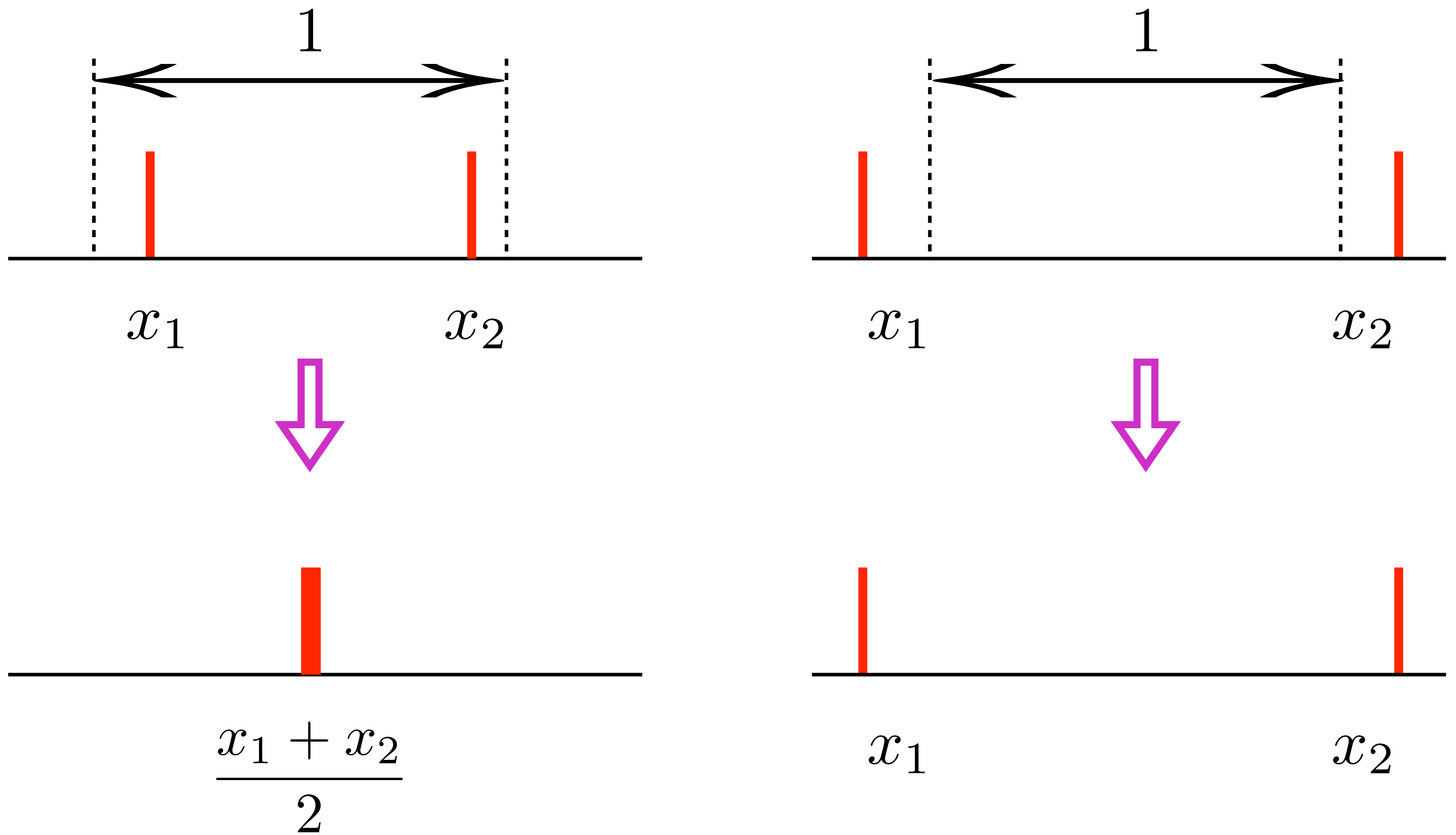
Simulations of Strategic Voter Model







Bounded Compromise Model



If $|x_2 - x_1| < 1$ compromise

If $|x_2 - x_1| > 1$ no interaction

The Opinion Distribution

P(x,t) = probability that agent has opinion x at time t

Fundamental parameter: Δ the diversity (initial opinion range)

Δ<1: consensus
Δ>1: fragmentation

$$w \sim e^{-\Delta t/2}$$

$$\frac{\partial P(x,t)}{\partial t} = \iint_{|x_1-x_2|<1} dx_1 dx_2\, P(x_1,t)P(x_2,t) \times \left[\delta\left(x - \tfrac{1}{2}(x_1+x_2)\right) - \delta(x-x_1)\right]$$

gain by averaging opinions

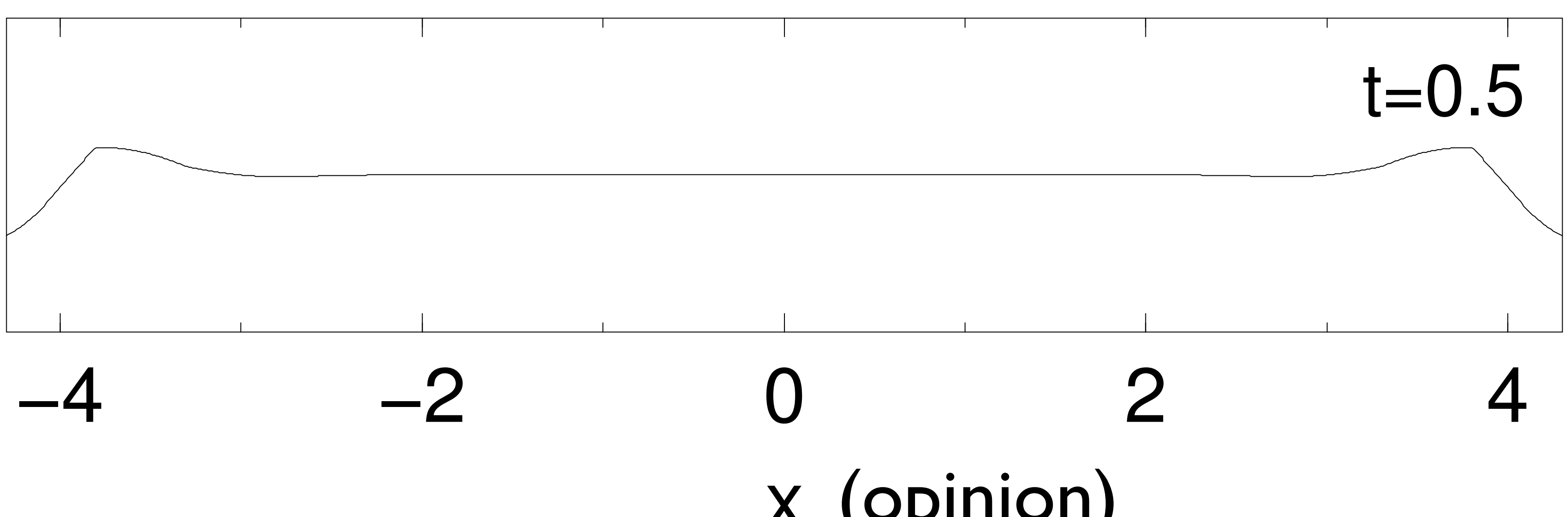
loss by interaction

same as Maxwell model for inelastic collisions
& inelastic collapse phenomena

Ben-Naim and Krapivsky (2000)
Baldassarri, Marconi, Puglisi (2001)
Ben-Naim, Krapivsky, SR (2003)

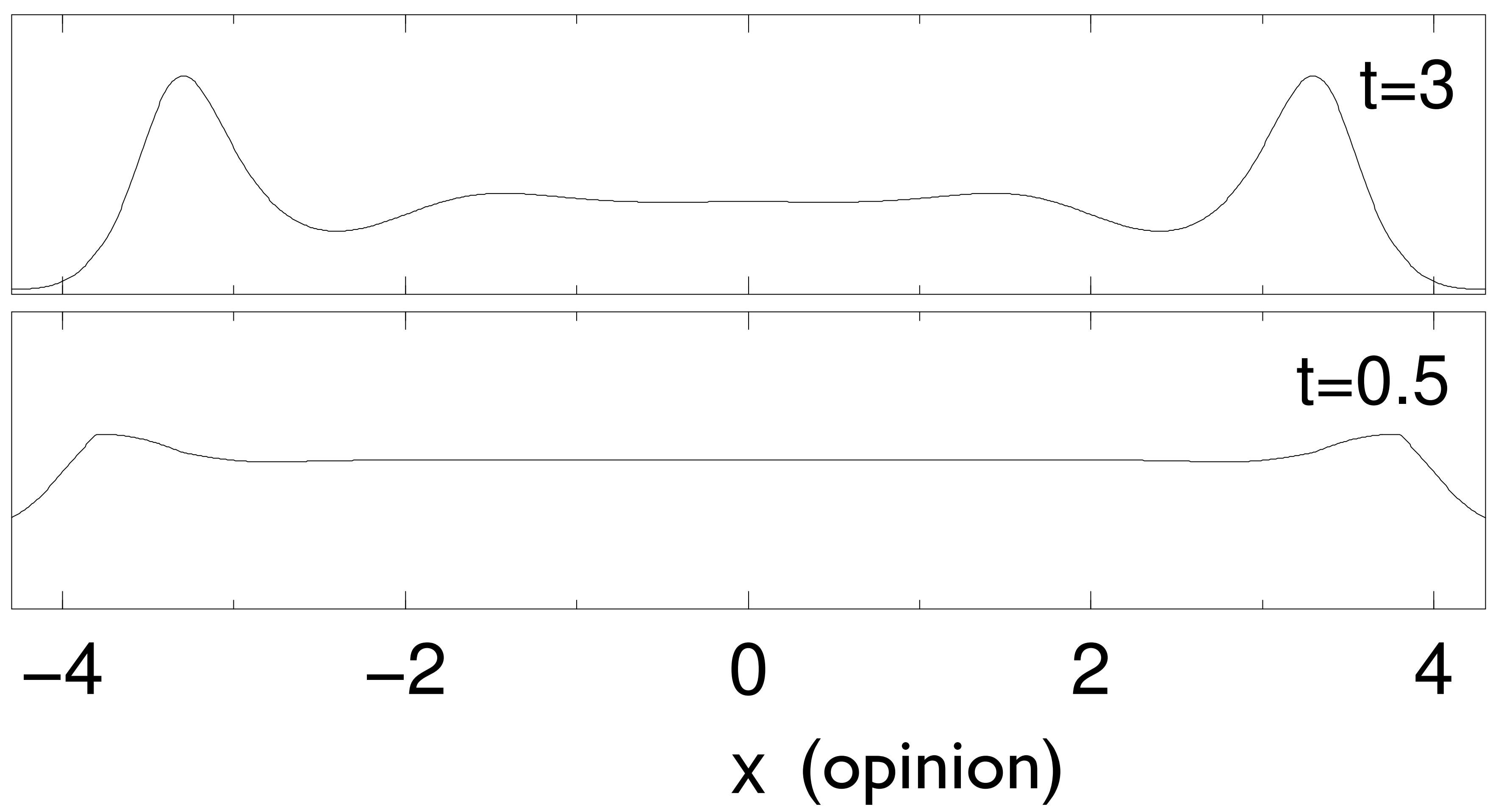
Early time evolution (for Δ=4.2)

integrate master equation rather than simulate!



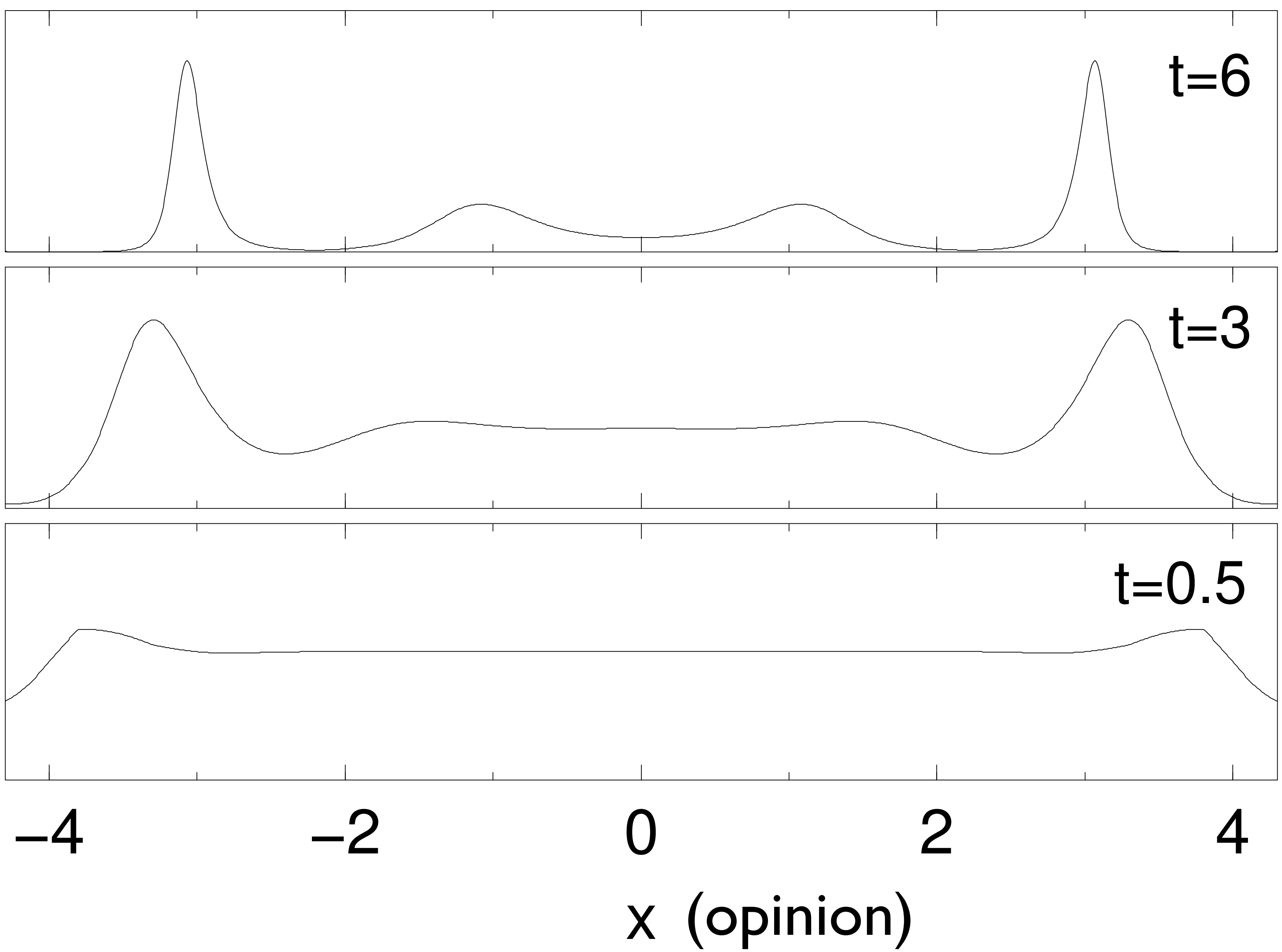
Early time evolution (for Δ=4.2)

integrate master equation rather than simulate!



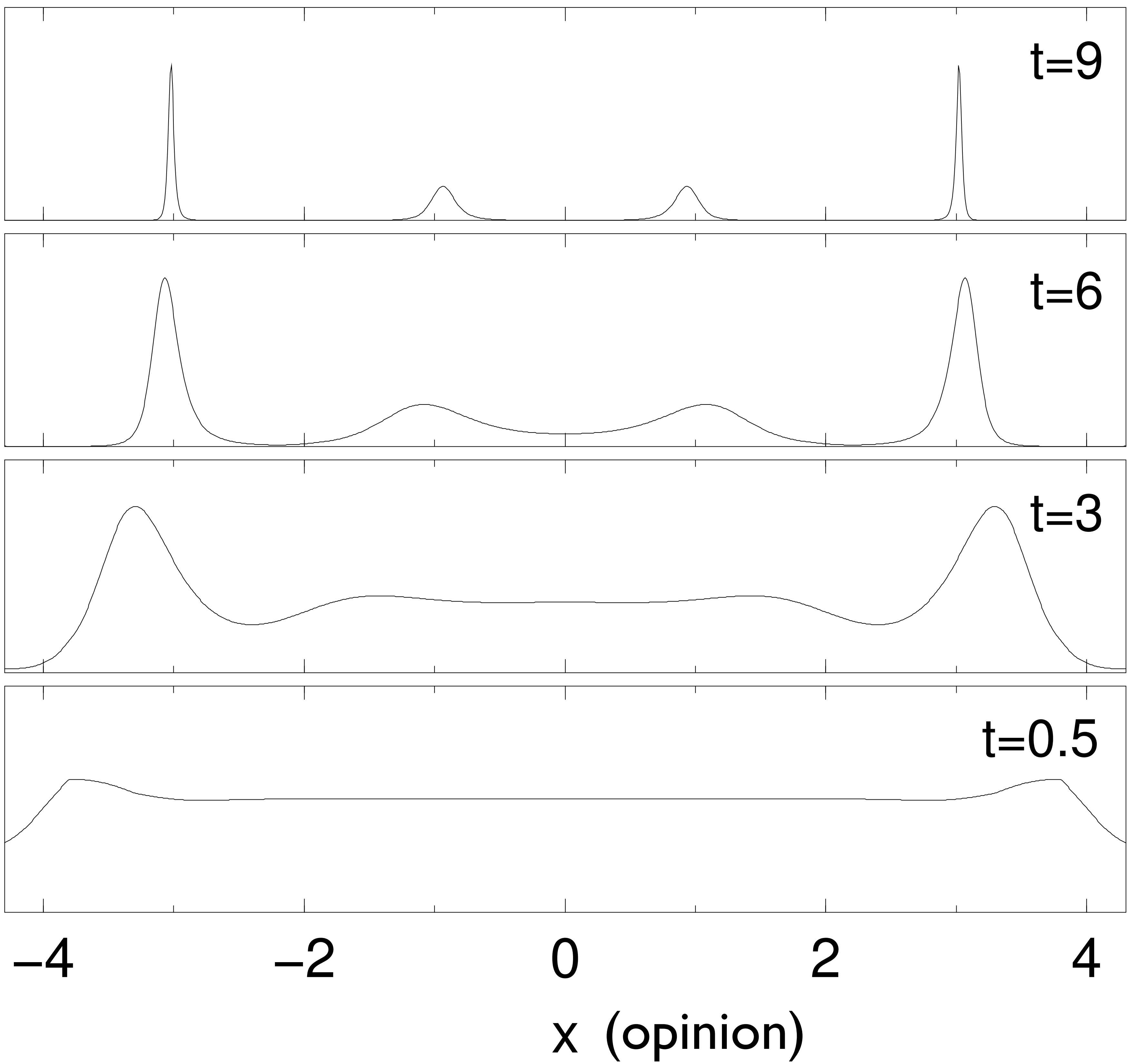
Early time evolution (for Δ=4.2)

integrate master equation rather than simulate!

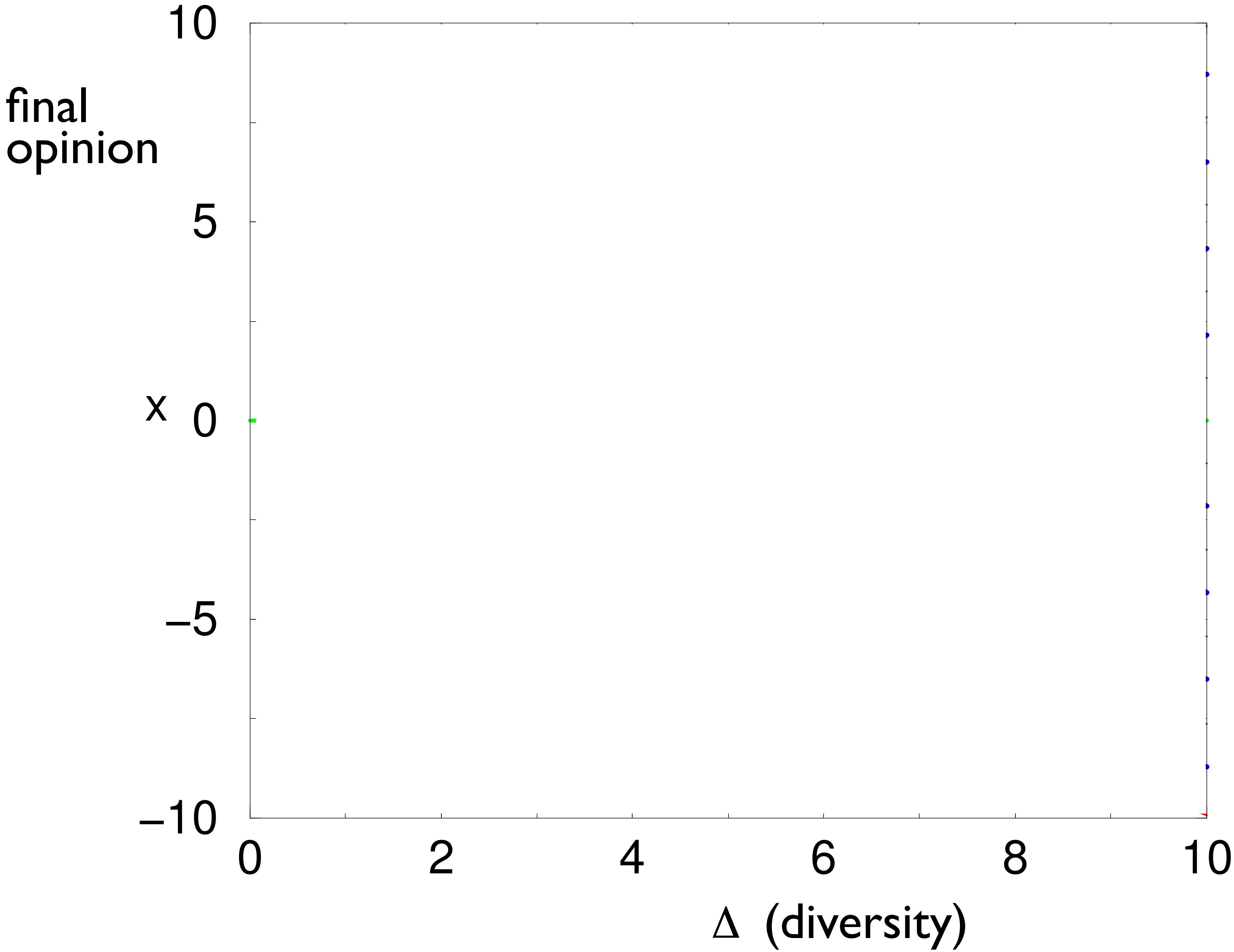


Early time evolution (for Δ=4.2)

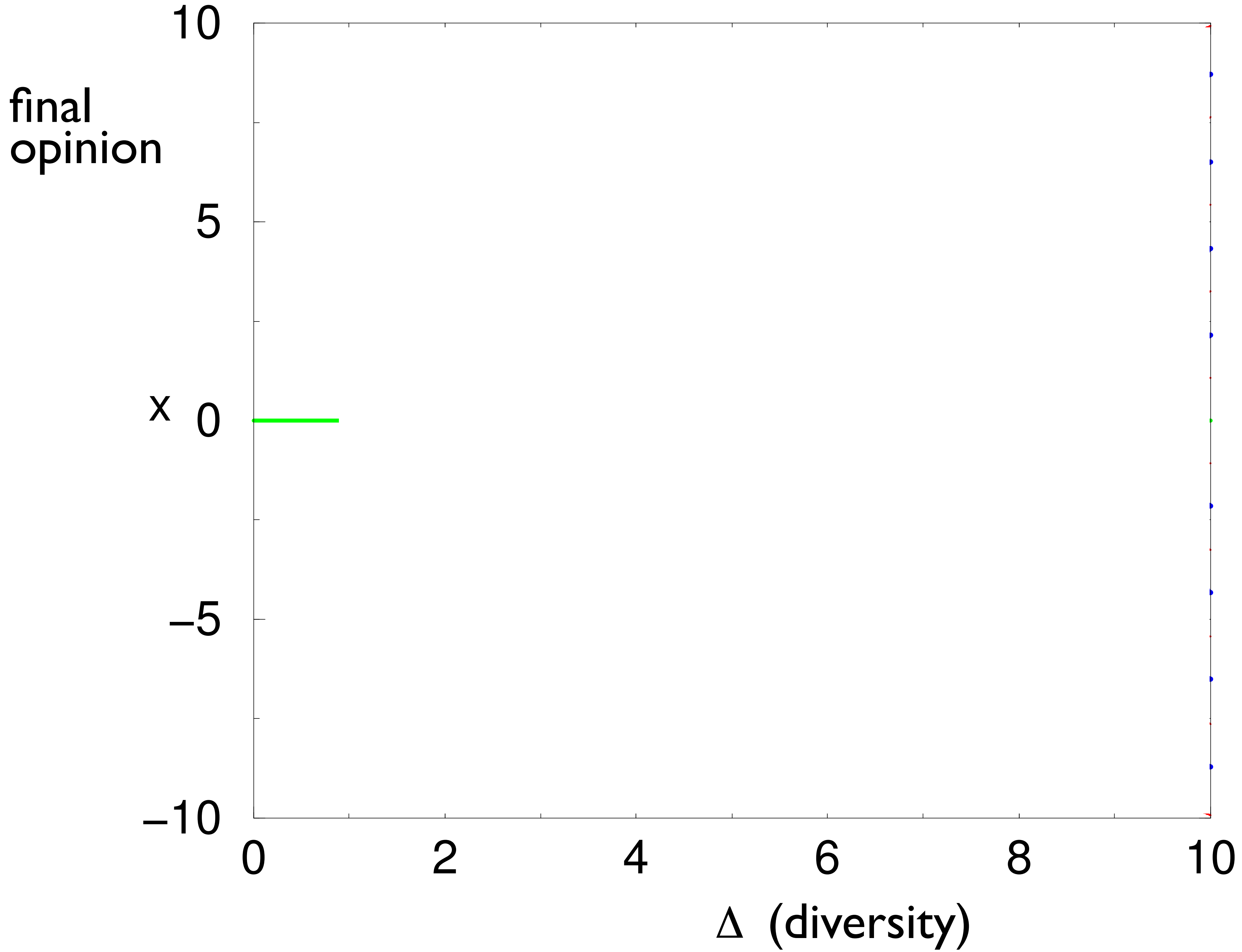
integrate master equation rather than simulate!



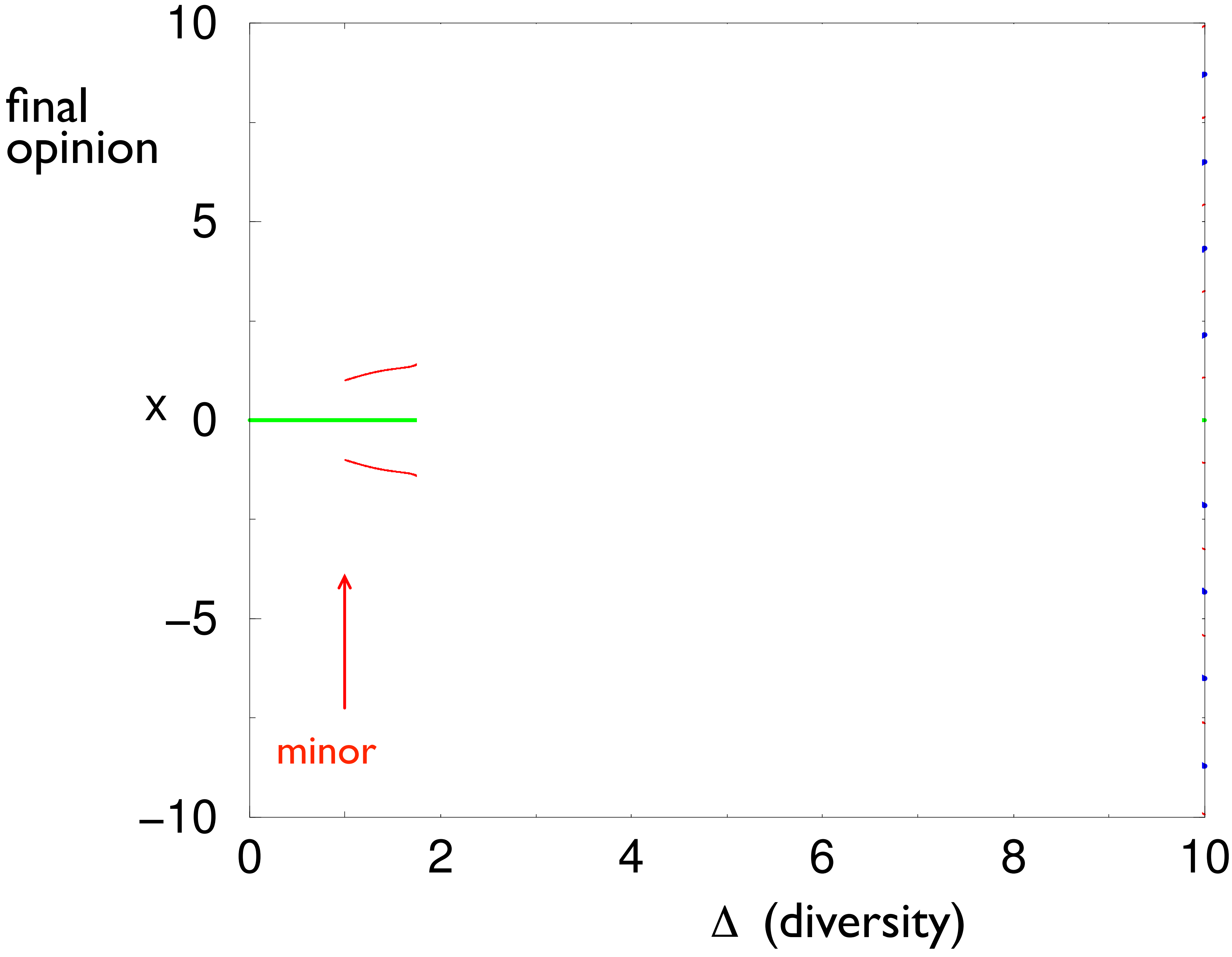
Fragmentation Sequence



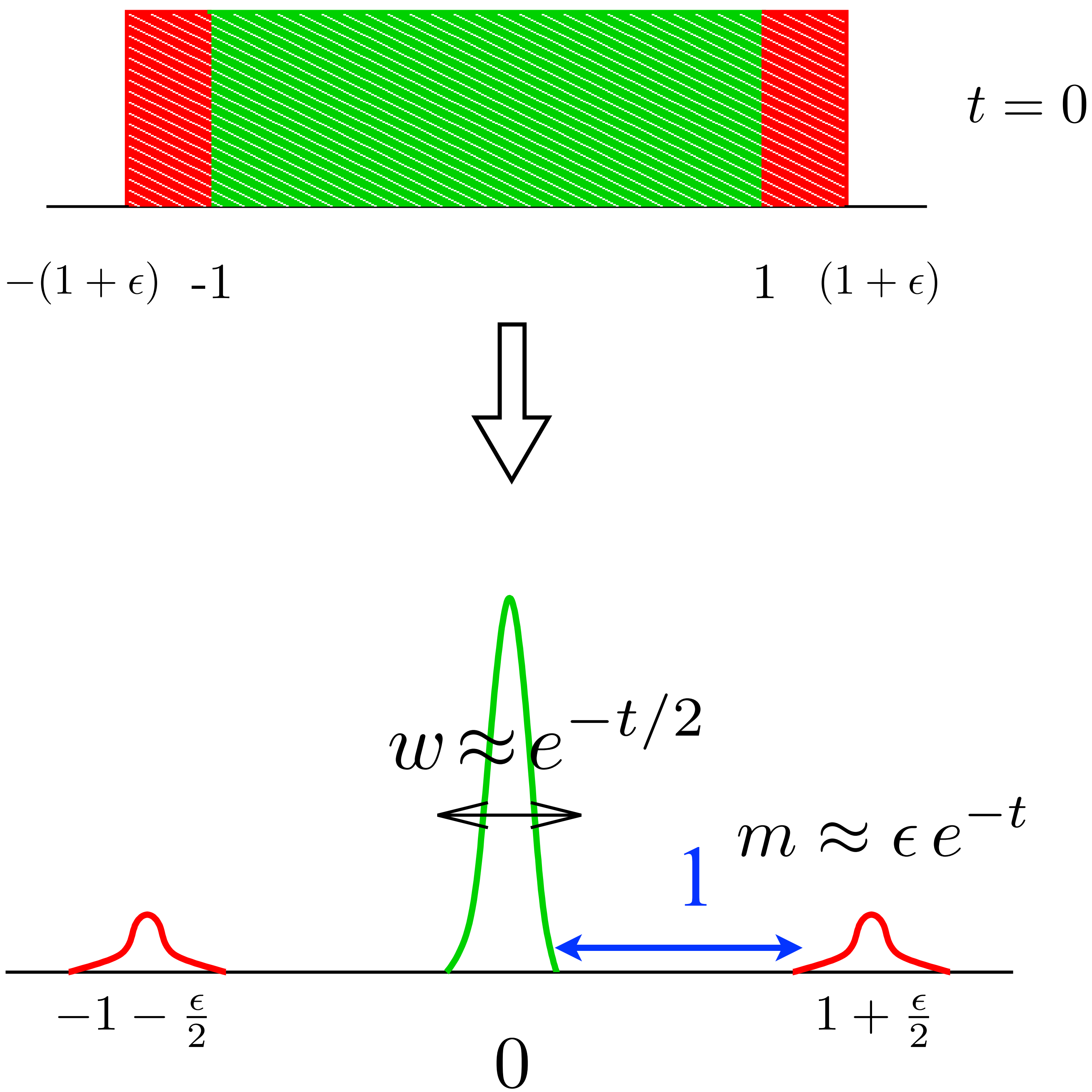
Fragmentation Sequence



Fragmentation Sequence



Birth of Extremists

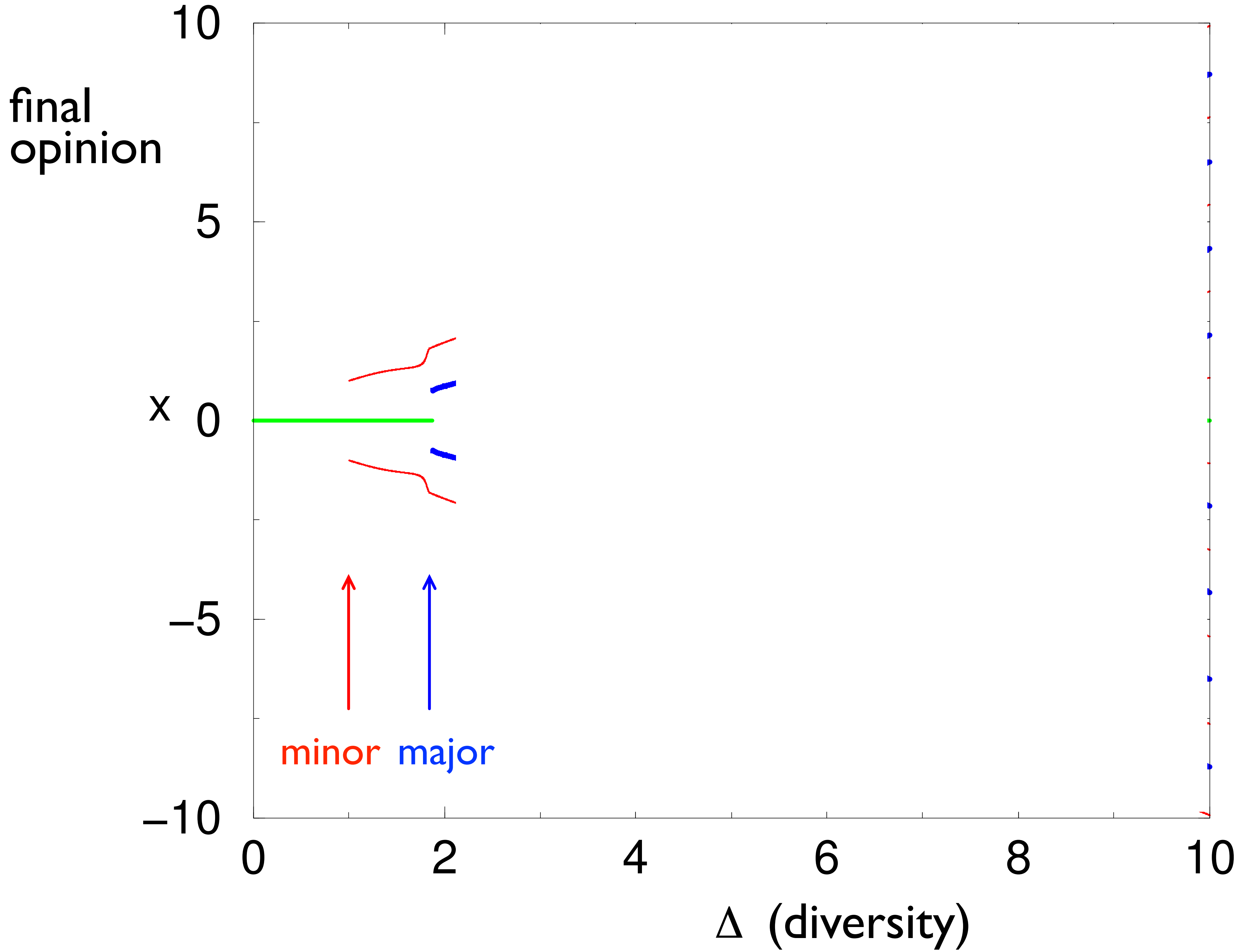


separation:

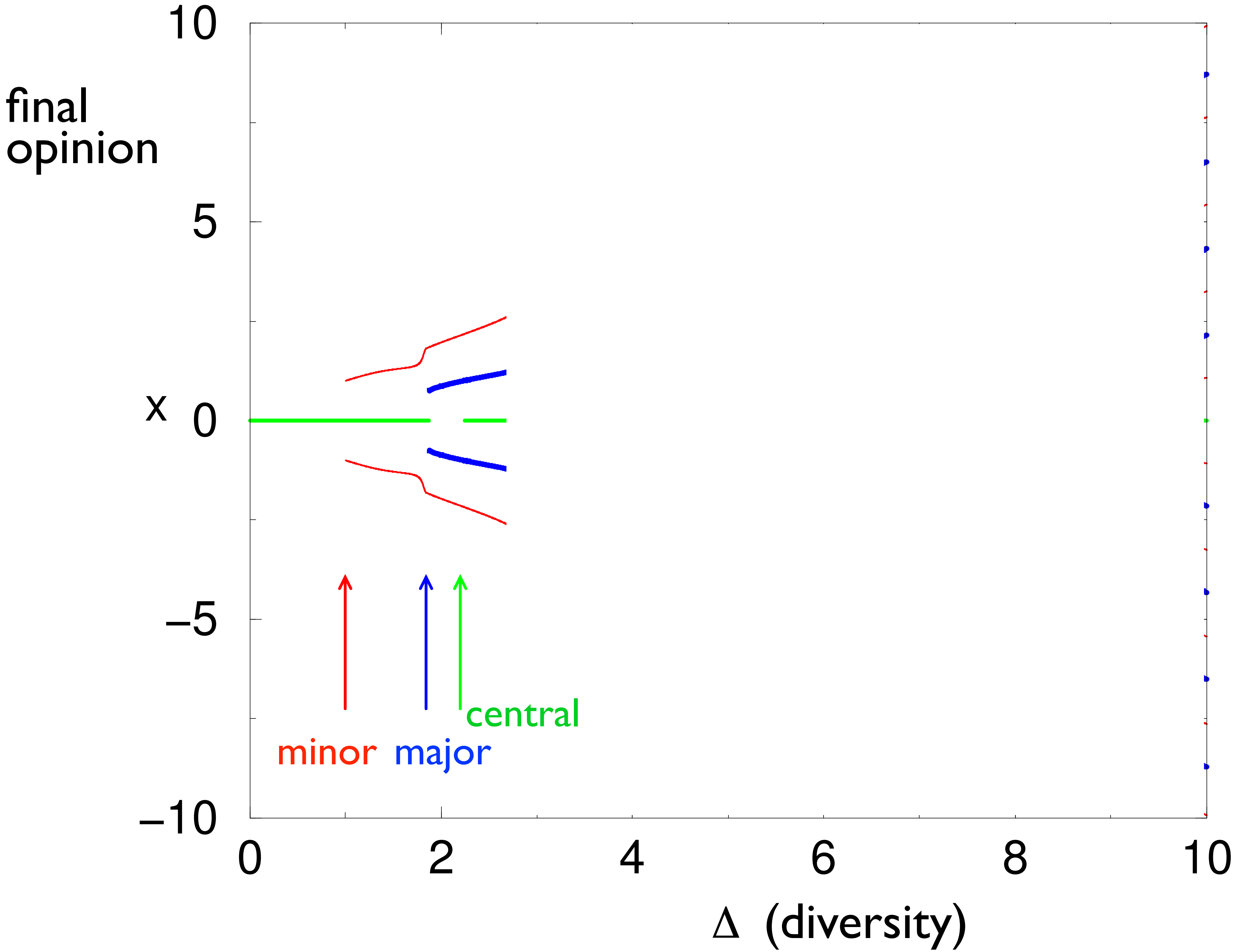
$w = \epsilon = e^{-t_{\rm sep}/2}$

$\rightarrow m(t_{\rm sep}) \propto \epsilon^3$

Fragmentation Sequence



Fragmentation Sequence



A Possible Realization

1993 Canadian Federal Election

year	PQ	NDP	L	PC	SC	R/CA
1979		26	114	136	6	
1980		32	147	103		
1984		30	40	211		
1988		43	83	169		
1993	54	9	177	2		52

The Dynamics of Persuasion

Sid Redner, Santa Fe Institute (physics.bu.edu/~redner)

CIRM, Luminy France, January 5-9, 2015

T. Antal, E. Ben-Naim, P. Chen, P. L. Krapivsky, M. Mobilia, V. Sood, F. Vazquez, D. Volovik

+ support from NSF

Modeling Consensus:

- introduction to the voter model
- voter model on complex networks
- voting with some confidence
- majority rule

lecture 1

Modeling Discord & Diversity:

- 3-state voter models
- strategic voting
- bounded compromise
- dynamics of social balance
- Axelrod model

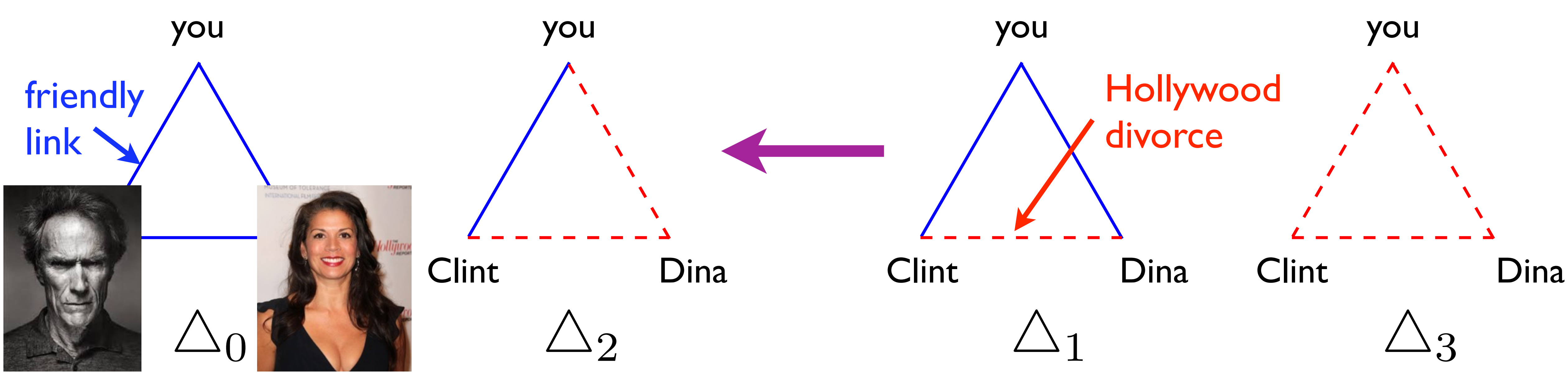
lecture 2

lecture 3

Dynamics of Social Balance

Investigate dynamical rules that promote evolution to a *balanced* state

Socially Balanced States



unfrustrated/balanced

frustrated/imbalanced

Social Balance

a friend of my friend
an enemy of my enemy } *is my friend;*

a friend of my enemy
an enemy of my friend } *is my enemy.*

Static properties of signed graphs:

Balanced states on complete graph must either be

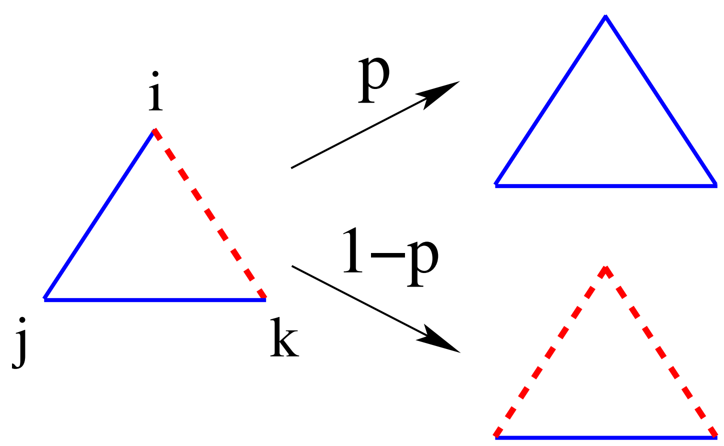
- *utopia:* only friendly links
- *bipolar:* two mutually antagonistic cliques

Cartwright & Harary (1956)

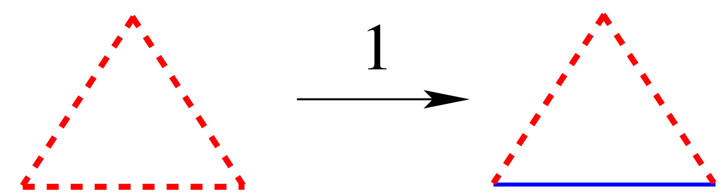
Two Natural Evolution Rules

Local Triad Dynamics:

reduce imbalance *in one triad* by single update



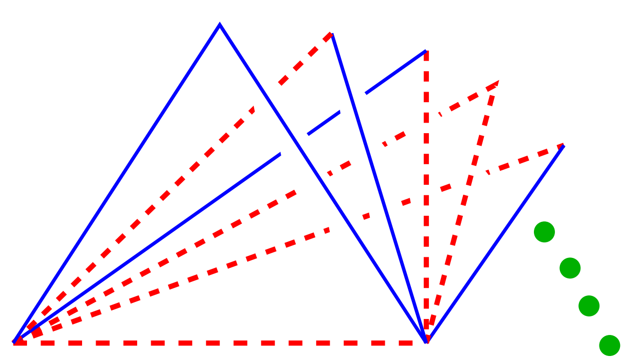
p: amity parameter



→ Balance transition as a function of p

Global Triad Dynamics:

reduce *global imbalance* by single update



→ Outcome unknown

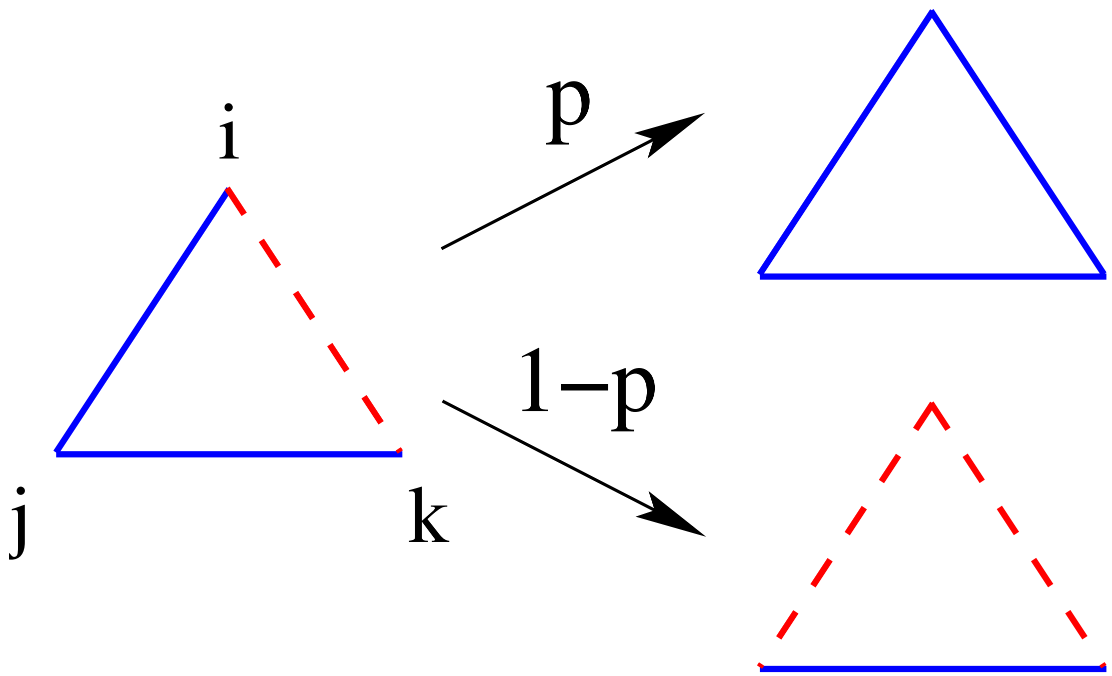
Tantalizing connections to spin glasses & jamming phenomena

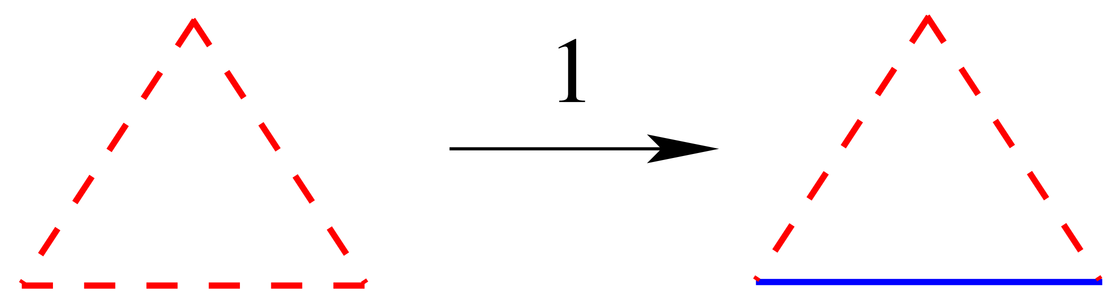
Local Triad Dynamics on Arbitrary Networks

(social graces of the clueless)

1. Pick a random imbalanced (frustrated) triad
2. Reverse a single link so that the triad becomes balanced

   probability p: unfriendly → friendly;   probability 1-p: friendly → unfriendly





Fundamental parameter p:

- p=1/3: flip a random link in the triad equiprobably
- p>1/3: predisposition toward tranquility
- p<1/3: predisposition toward hostility

The Evolving State

rate equation for the density of friendly links ρ:

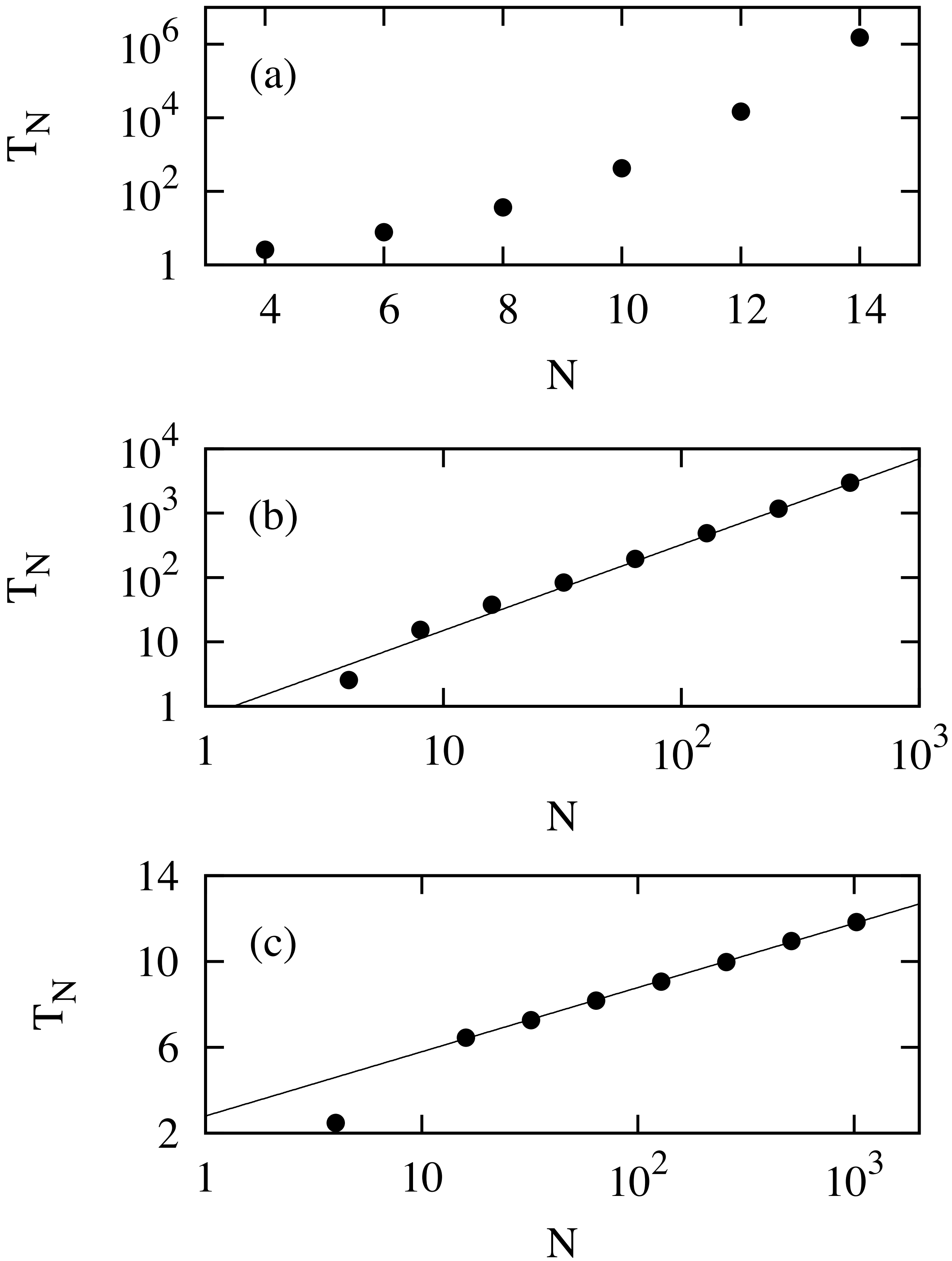
$- \to +$ in $\triangle_1$ $\quad$ $+ \to -$ in $\triangle_1$ $\quad$ $- \to +$ in $\triangle_3$

$$\frac{d\rho}{dt} = 3\rho^2(1-\rho)[p-(1-p)] + (1-\rho)^3$$
$$= 3(2p-1)\rho^2(1-\rho) + (1-\rho)^3$$

$$\rho(t) \sim \begin{cases} \rho_\infty + Ae^{-Ct} & p < 1/2; \\ 1 - \dfrac{1-\rho_0}{\sqrt{1+2(1-\rho_0)^2 t}} & p = 1/2; \\ 1 - e^{-3(2p-1)t} & p > 1/2. \end{cases}$$

- $p < 1/2$: rapid approach to frustrated steady state
- $p = 1/2$: slow relaxation to utopia
- $p > 1/2$: rapid attainment of utopia

Simulations for a Finite Society



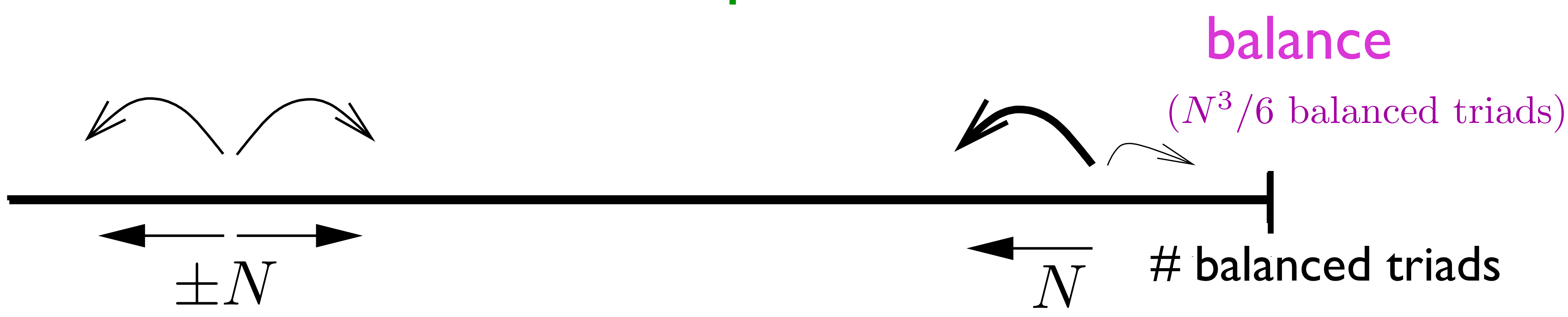
$$p < \frac{1}{2}, \qquad T_N \sim e^{N^2}$$

$$p = \frac{1}{2}, \qquad T_N \sim N^{4/3}$$

$$p > \frac{1}{2}, \qquad T_N \sim \frac{\ln N}{2p - 1}$$

Fate of a Finite Society

p<1/2: effective random walk picture



$D \propto N^2$ $\qquad$ $v \propto N$

$$\rightarrow T_N \sim e^{v\mathcal{L}_N/D} \sim e^{N^2}$$

p>1/2: inversion of the rate equation

$$u \sim e^{-3(2p-1)t} \approx N^{-2} \quad \rightarrow \quad T_N \sim \frac{\ln N}{2p-1}$$

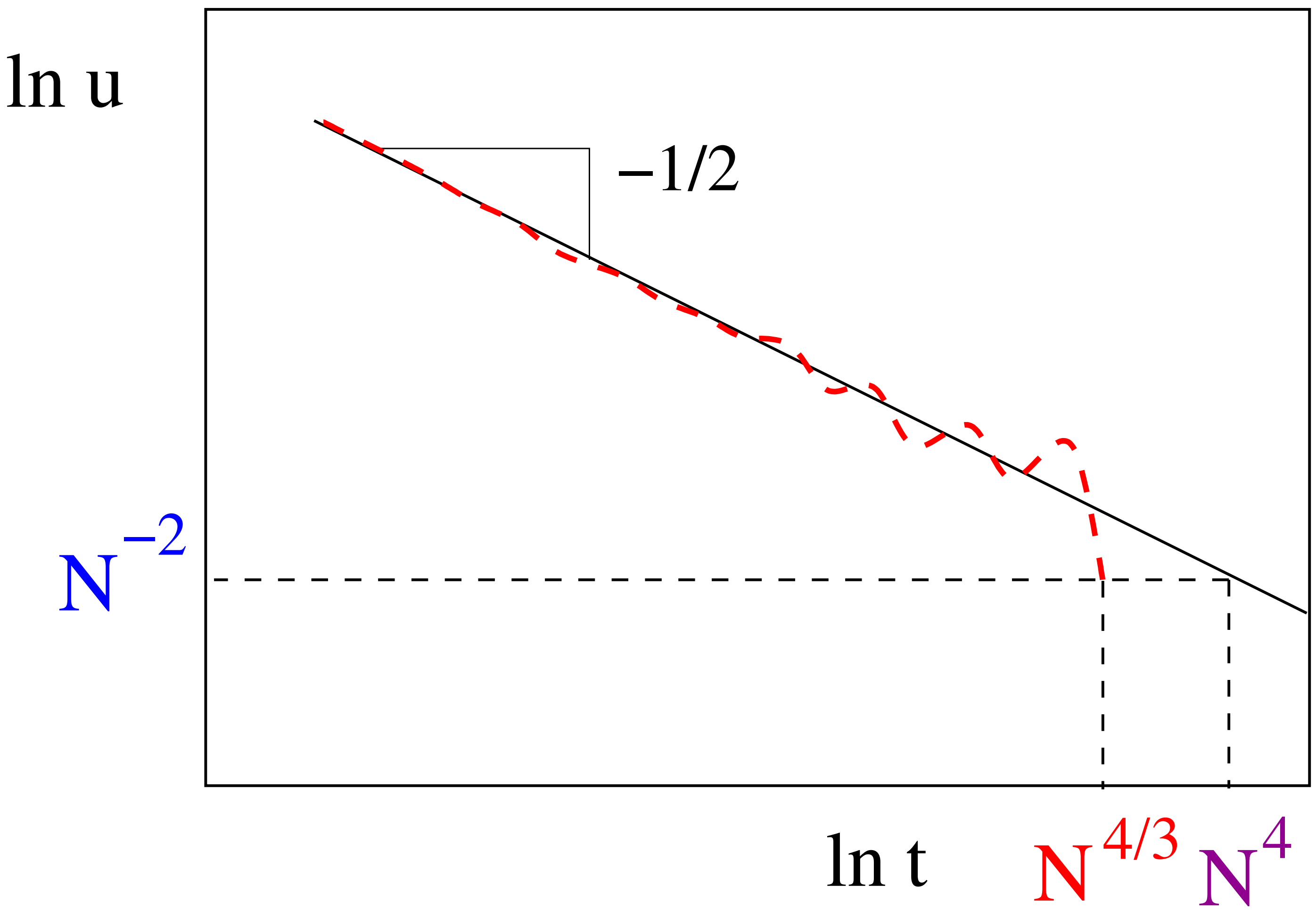
u=1-ρ, the unfriendly link density

p=1/2

naive rate equation estimate:

$$u \equiv 1 - \rho \propto t^{-1/2} \approx N^{-2} \quad \rightarrow \; T_N \sim N^4$$

incorporating fluctuations as balance is approached:



$$\begin{aligned} U &= Lu + \sqrt{L}\,\eta \\ &\sim \frac{L}{\sqrt{t}} + \sqrt{L}\,t^{1/4} \end{aligned}$$

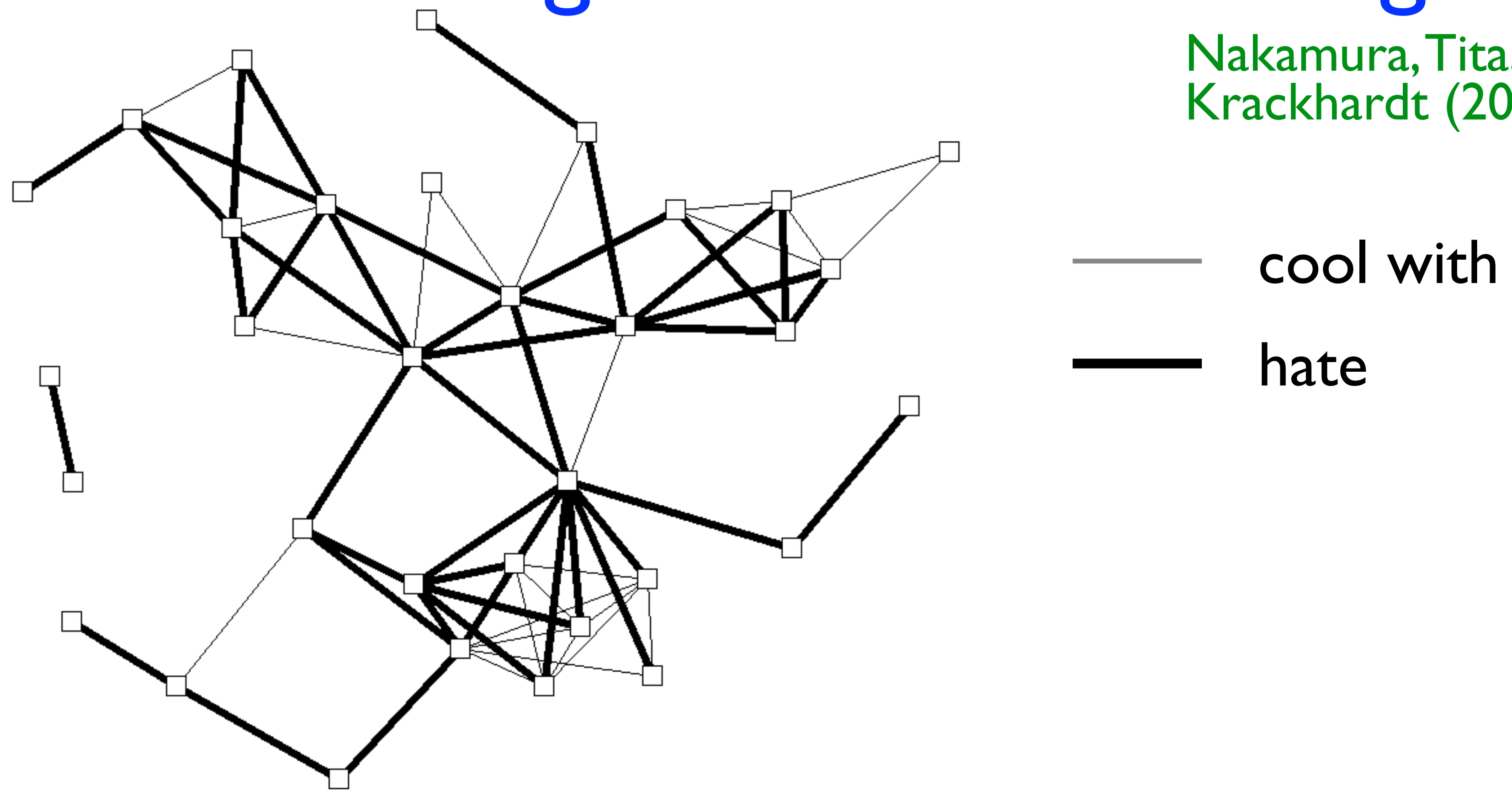
equating the 2 terms in U:

$$T_N \sim L^{2/3} \sim N^{4/3}$$

Possible Application I: Long Beach Street Gangs

Nakamura, Tita, & Krackhardt (2007)

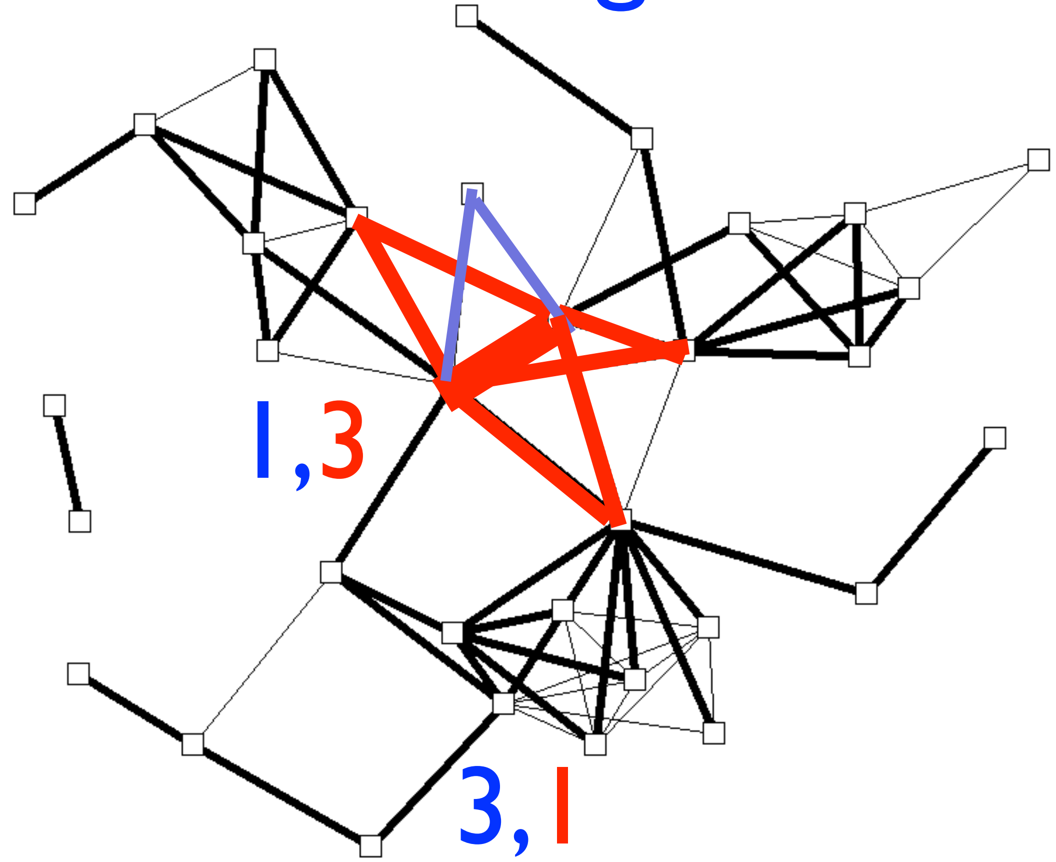
gang relations



Possible Application I: Long Beach Street Gangs

Nakamura, Tita, & Krackhardt (2007)

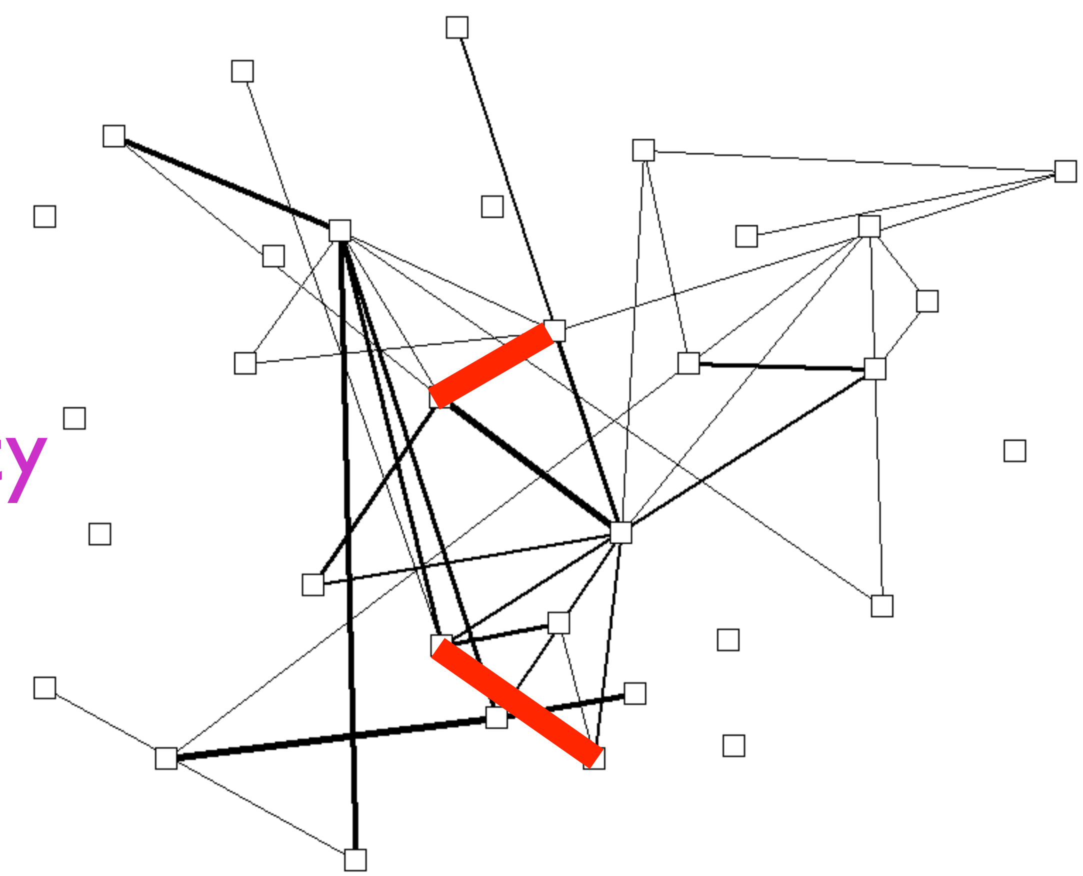
gang relations



cool with

hate

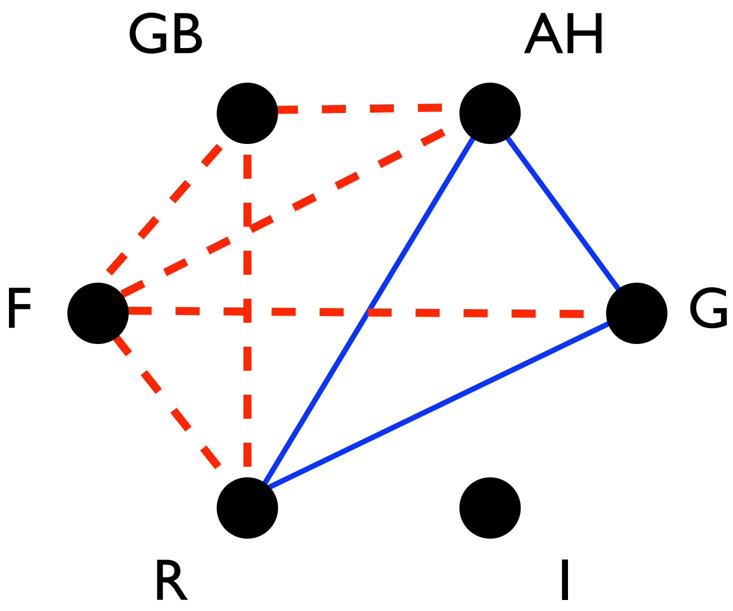
violence frequency



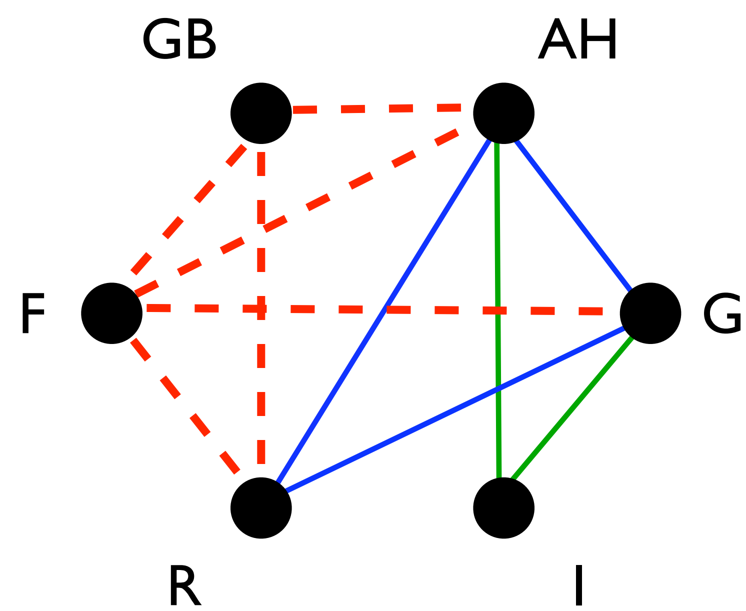
low incidence

high incidence

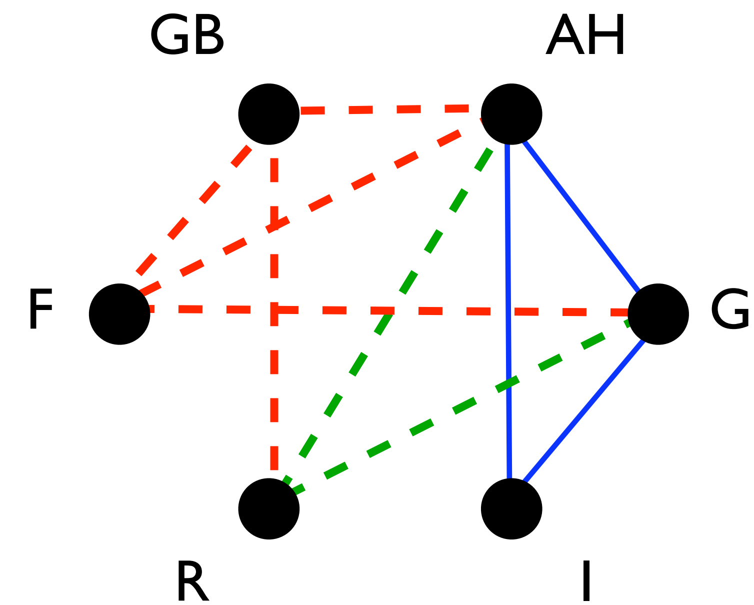
Possible Application II: A Historical Lesson



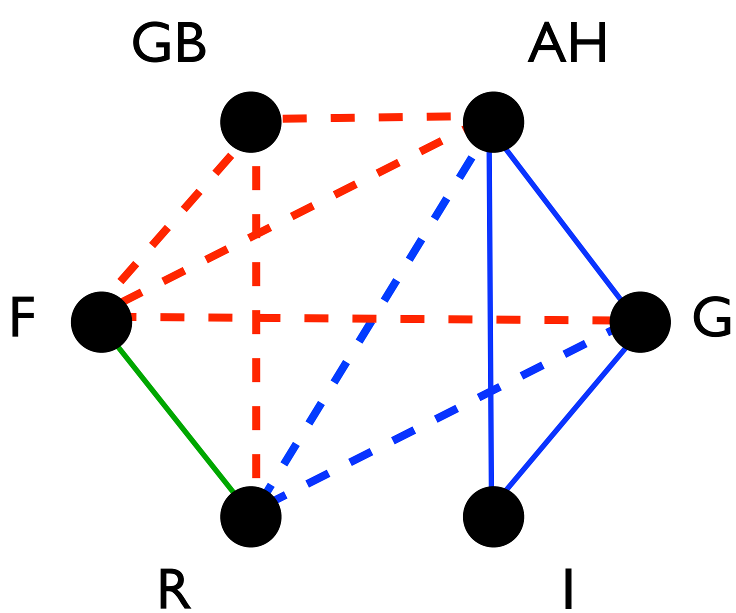
3 Emperor's League 1872-81



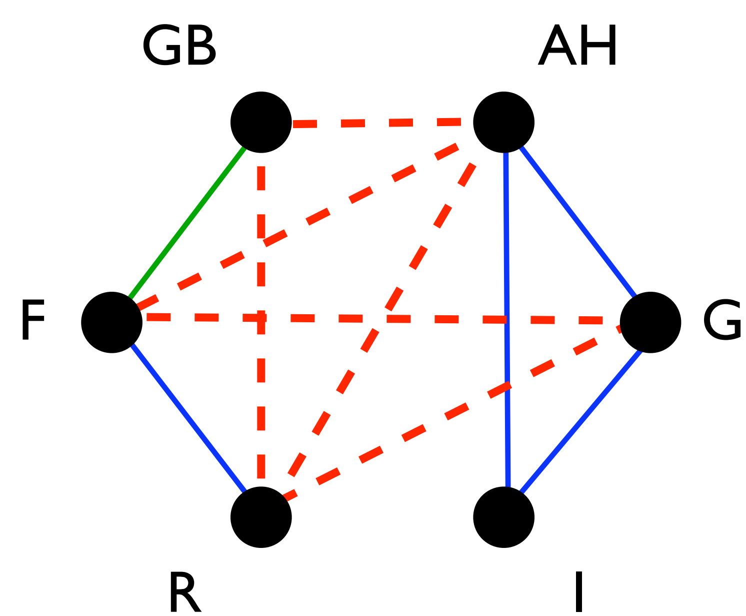
Triple Alliance 1882



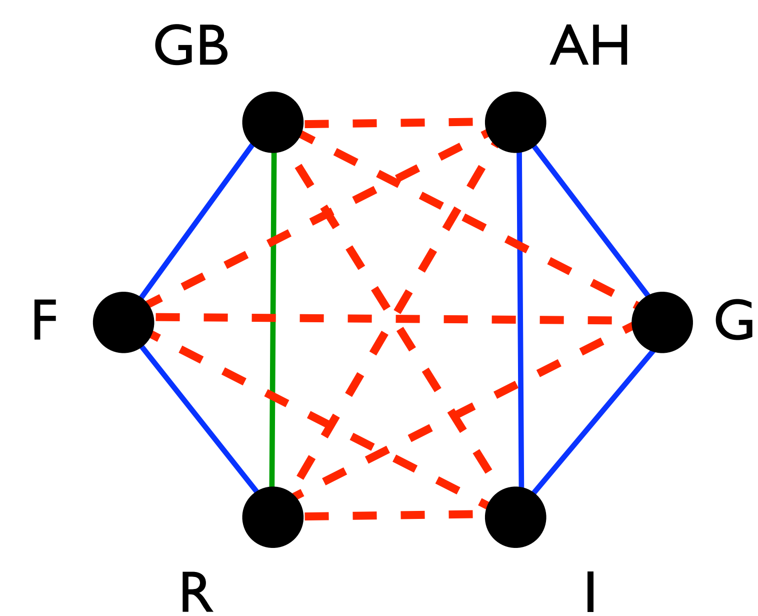
German-Russian Lapse 1890



French-Russian Alliance 1891-94



Entente Cordiale 1904



British-Russian Alliance 1907

Axelrod Model

Axelrod (1997)

You:

car	cuisine	recreation	politics	abode
BMW	**meat**	**skiing**	leftist	suburb house
SUV	vegetarian	hiking	rightist	**city apartment**
Ford	vegan	fishing	anarchist	homeless
Trabant	fast weekly	hunting	**apathetic**	pied-à-terre
bicycle	dieting	running	fascist	city house

← F features →

↑ q traits ↓

Me:

car	cuisine	recreation	politics	abode
BMW	meat	skiing	leftist	suburb house
SUV	**vegetarian**	hiking	rightist	city apartment
Ford	vegan	fishing	anarchist	homeless
Trabant	fast weekly	hunting	**apathetic**	pied-à-terre
bicycle	dieting	**running**	fascist	**city house**

Axelrod Model

Axelrod (1997)

You:

car	cuisine	recreation	politics	abode
BMW	*meat*	*skiing*	*leftist*	*suburb house*
SUV	*vegetarian*	*hiking*	*rightist*	*city apartment*
Ford	*vegan*	*fishing*	*anarchist*	*homeless*
Trabant	*fast weekly*	*hunting*	*apathetic*	*pied-à-terre*
bicycle	*dieting*	*running*	*fascist*	*city house*

q traits

← F features →

Me:

car	cuisine	recreation	politics	abode
BMW	*meat*	*skiing*	*leftist*	*suburb house*
SUV	*vegetarian*	*hiking*	*rightist*	*city apartment*
Ford	*vegan*	*fishing*	*anarchist*	*homeless*
Trabant	*fast weekly*	*hunting*	*apathetic*	*pied-à-terre*
bicycle	*dieting*	*running*	*fascist*	*city house*

Axelrod Model

Axelrod (1997)

You:

car	cuisine	recreation	politics	abode
BMW	meat	*skiing*	leftist	suburb house
SUV	*vegetarian*	hiking	rightist	*city apartment*
Ford	vegan	fishing	anarchist	homeless
Trabant	fast weekly	hunting	*apathetic*	pied-à-terre
bicycle	dieting	running	fascist	city house

q traits

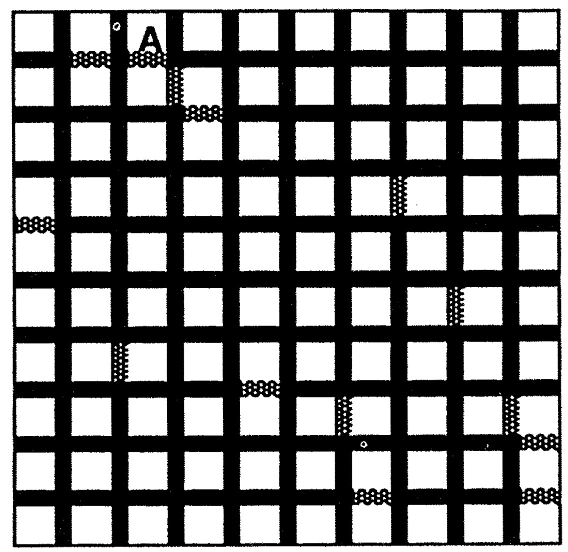
F features

Me:

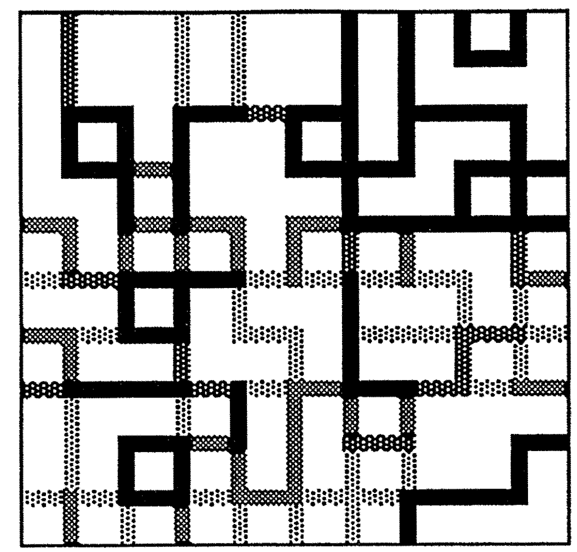
car	cuisine	recreation	politics	abode
BMW	meat	skiing	leftist	suburb house
SUV	*vegetarian*	hiking	rightist	city apartment
Ford	vegan	fishing	anarchist	homeless
Trabant	fast weekly	hunting	*apathetic*	pied-à-terre
bicycle	dieting	*running*	fascist	*city house*

Axelrod's Simulation

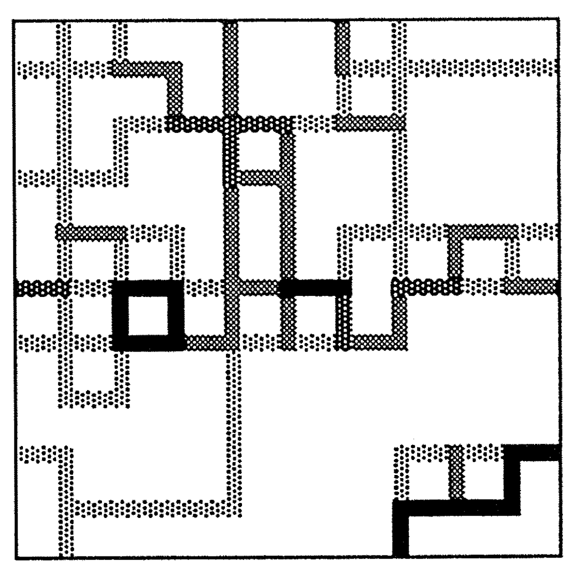
F=5, q=10, 10x10 lattice



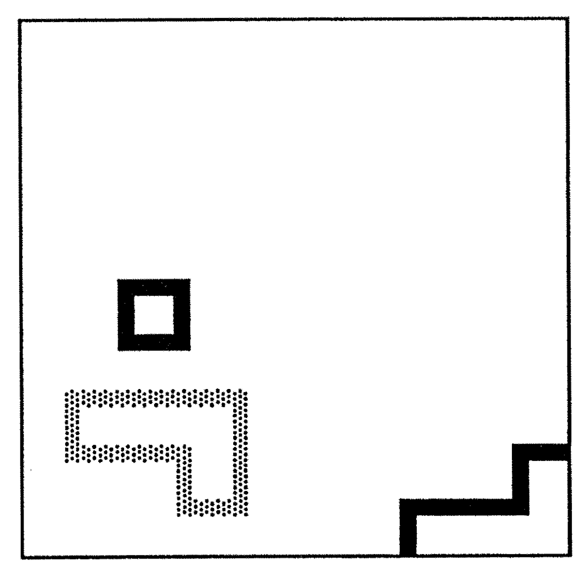
t=0



t=200



t=400



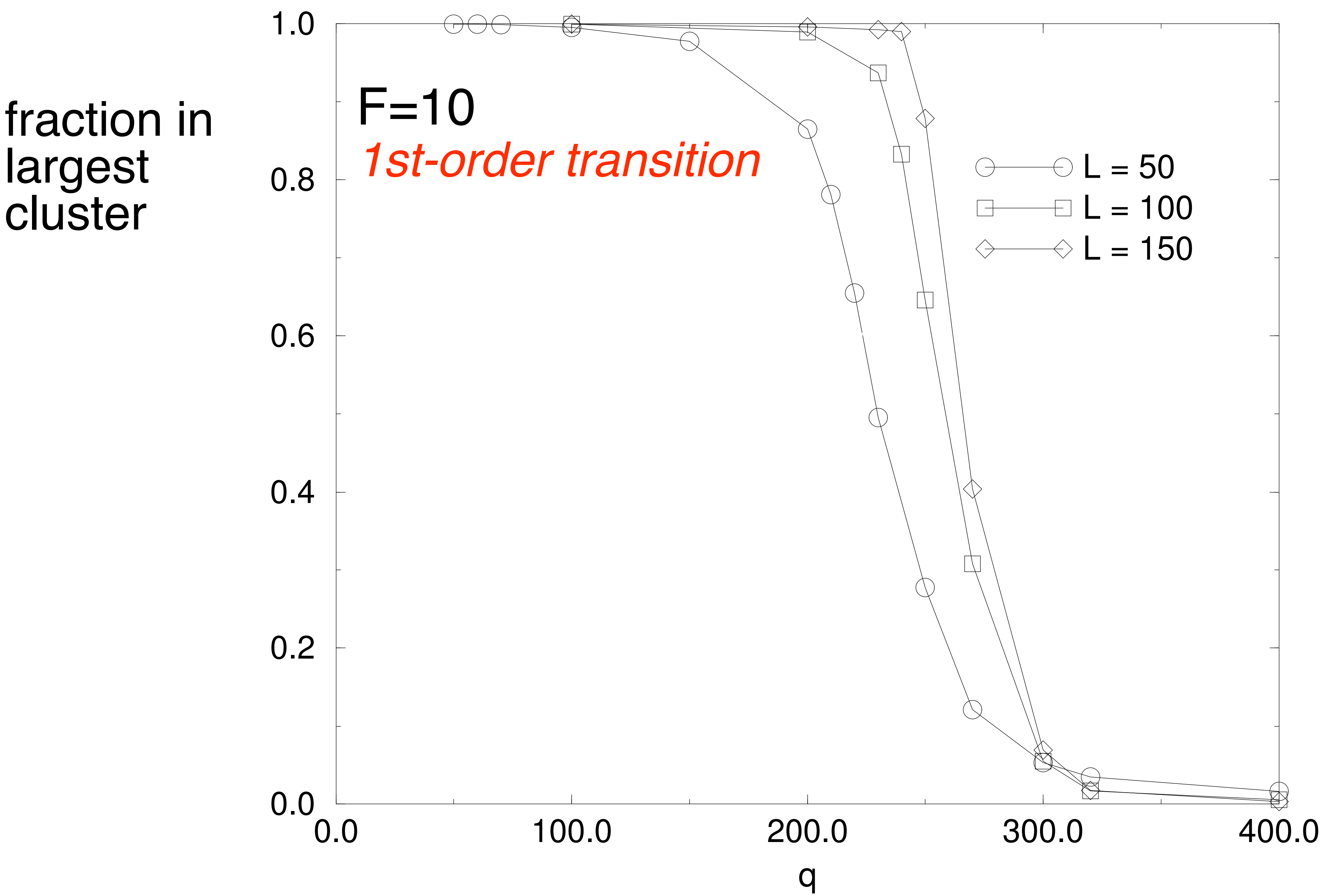
t=800

Axelrod Model

Castellano, Marsili & Vespignani (2000)

simulations on 150 x 150 square lattice

small (F,q): consensus
large (F,q): fragmentation

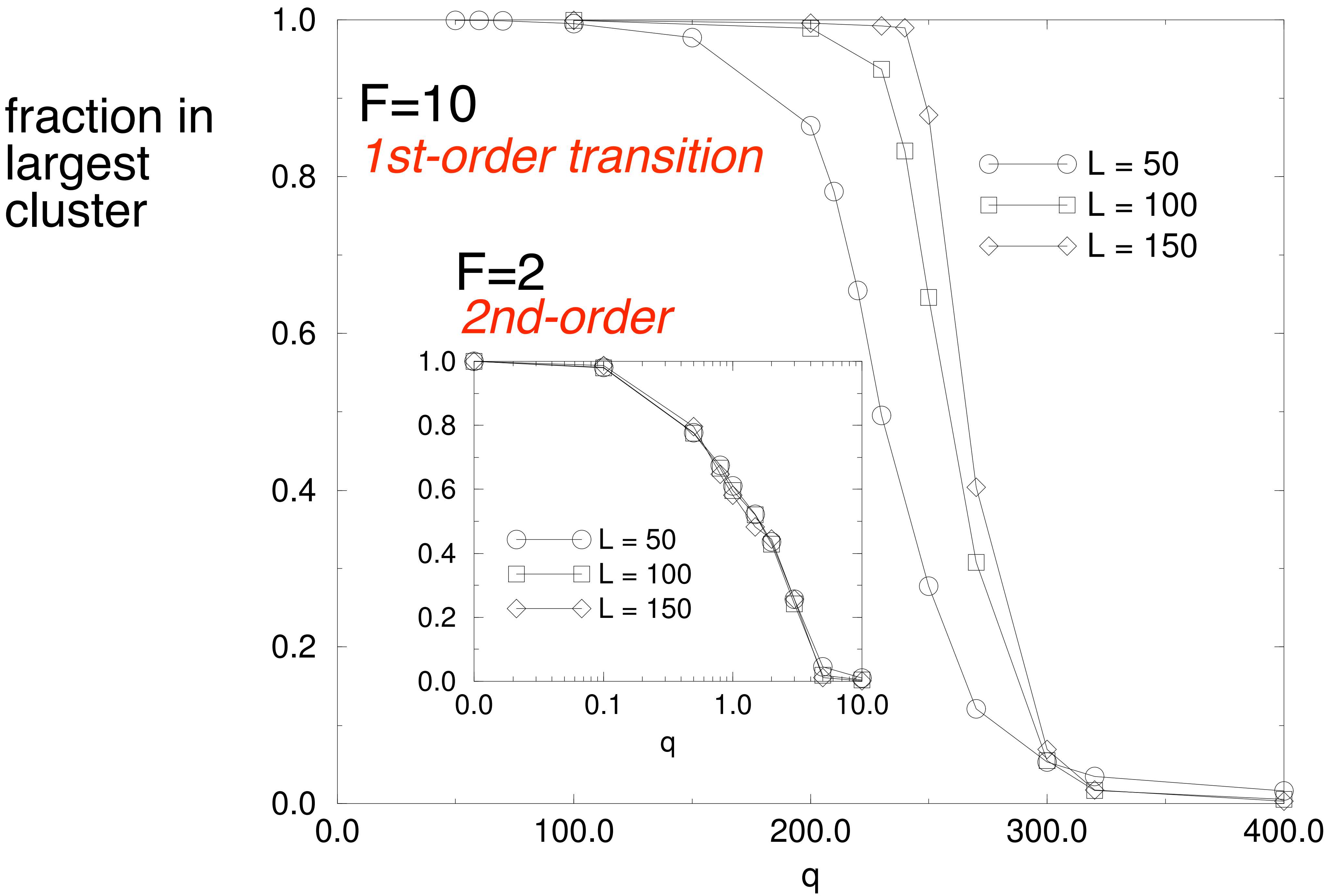


Axelrod Model

Castellano, Marsili & Vespignani (2000)

simulations on 150 x 150 square lattice

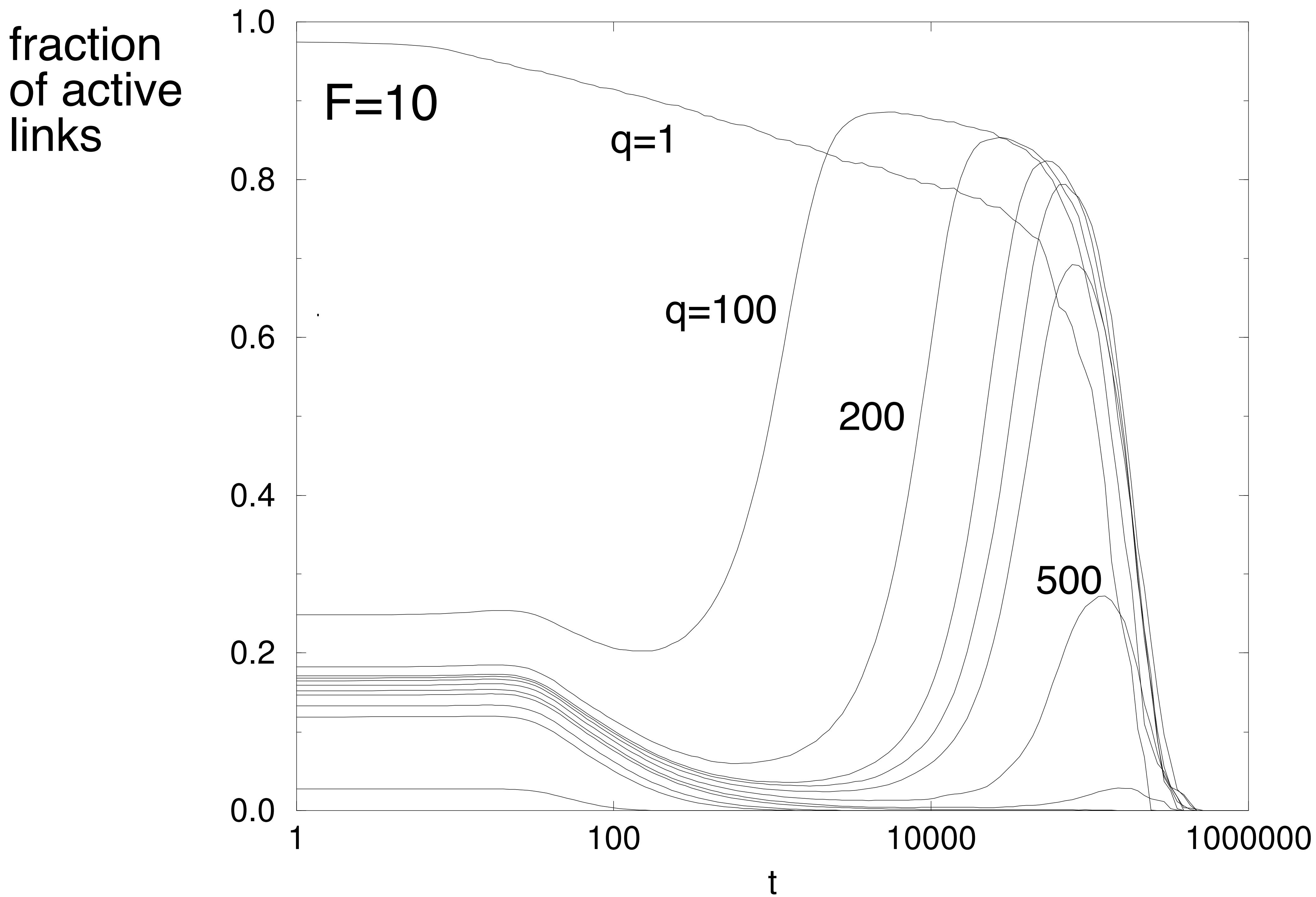
small (F,q): consensus
large (F,q): fragmentation



Axelrod Model

Castellano, Marsili & Vespignani (2000)

simulations on 150 x 150 square lattice



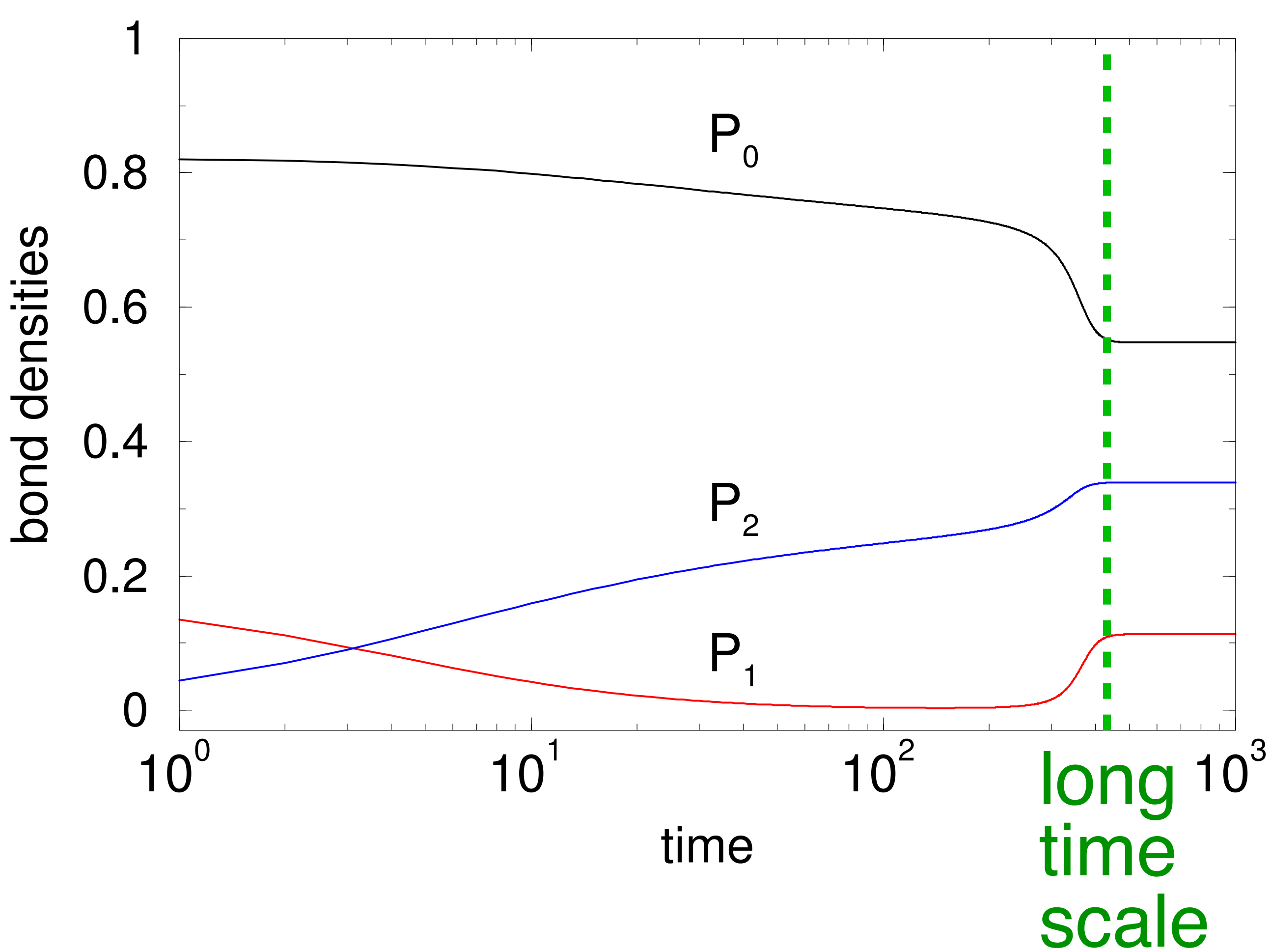
!

Axelrod Model with F=2
on 4-regular graph

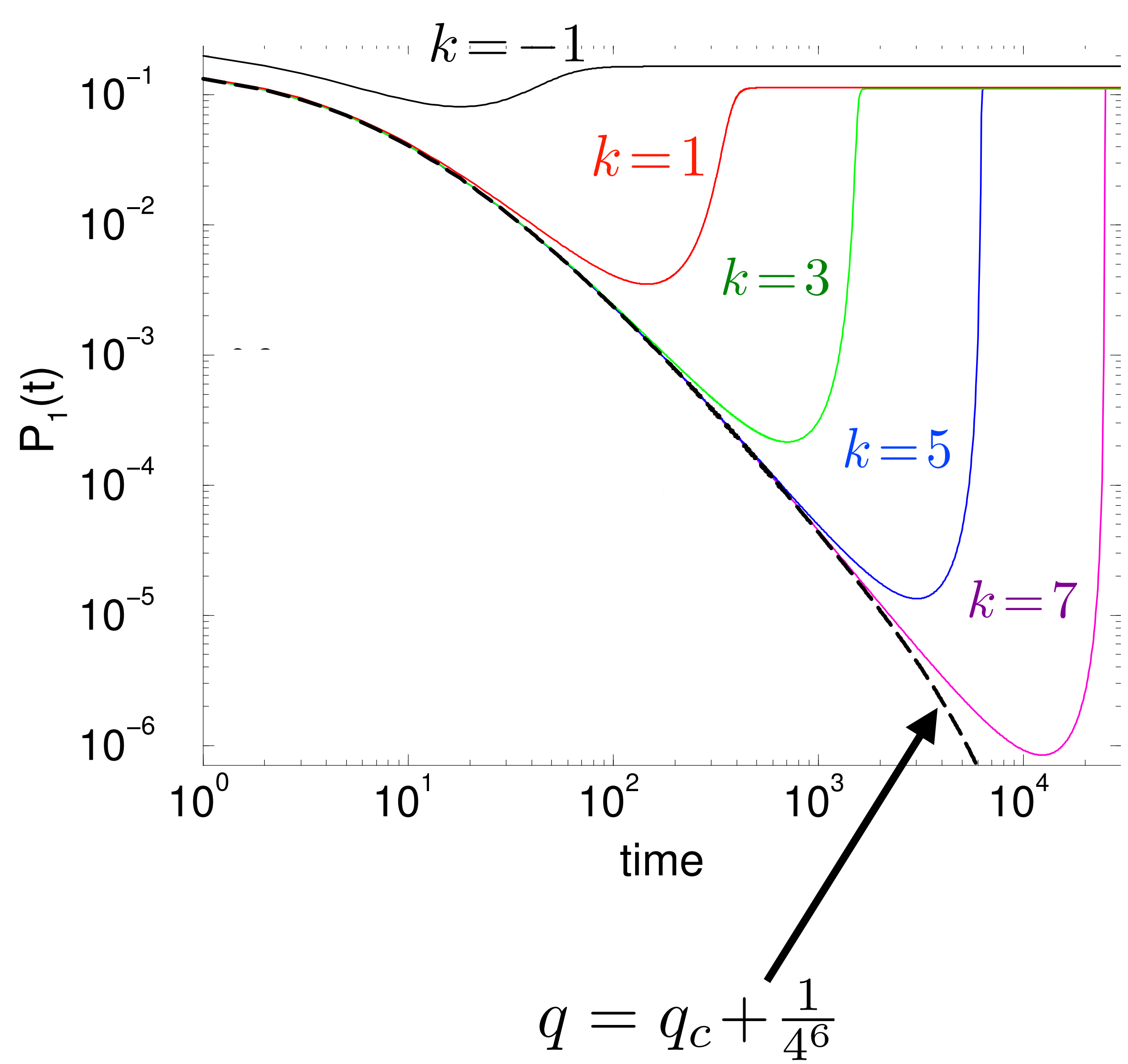
$P_m \equiv$ fraction of links with m common features

$m = 0, 2$ inactive; $m = 1$ active

$q = q_c - \frac{1}{4}$



$q = q_c - \frac{1}{4^k}$



Master Equations for Bond Densities

$$\dot{P}_0 = \frac{z-1}{z} P_1 \Big[-\lambda P_0 + \tfrac{1}{2} P_1 \Big]$$

$$\dot{P}_1 = -\frac{P_1}{z} + \frac{z-1}{z} P_1 \Big[\lambda P_0 - \tfrac{1}{2}(1+\lambda) P_1 + P_2 \Big]$$

direct processes

$$\dot{P}_2 = \frac{P_1}{z} + \frac{z-1}{z} P_1 \left[\tfrac{1}{2} \lambda P_1 - P_2 \right]$$

direct process for $\dot{P}_1$: choose random link

$$\Delta N_1 = -\tfrac{1}{2} P_1 \qquad \rightarrow \frac{\Delta P_1}{\Delta t} = -\frac{\tfrac{1}{2} P_1 / L}{1/N} = -\frac{P_1}{z}$$

Master Equations for Bond Densities

$$\dot{P}_0 = \frac{z-1}{z} P_1 \left[-\lambda P_0 + \tfrac{1}{2} P_1 \right]$$

indirect processes

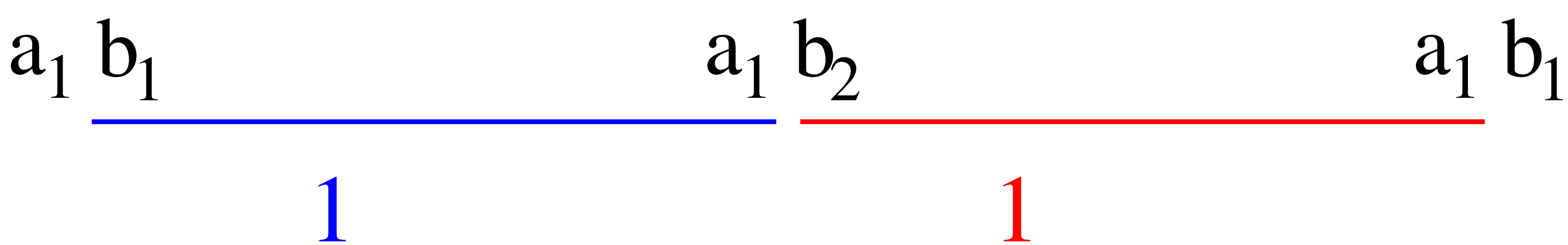
$$\dot{P}_1 = -\frac{P_1}{z} + \frac{z-1}{z} P_1 \left[\lambda P_0 - \tfrac{1}{2}(1+\lambda) P_1 + P_2 \right]$$

$$\dot{P}_2 = \frac{P_1}{z} + \frac{z-1}{z} P_1 \left[\tfrac{1}{2}\lambda P_1 - P_2 \right]$$

direct process for $\dot{P}_1$: choose random link

$$\Delta N_1 = -\tfrac{1}{2} P_1 \qquad \rightarrow \frac{\Delta P_1}{\Delta t} = -\frac{\frac{1}{2} P_1 / L}{1/N} = -\frac{P_1}{z}$$

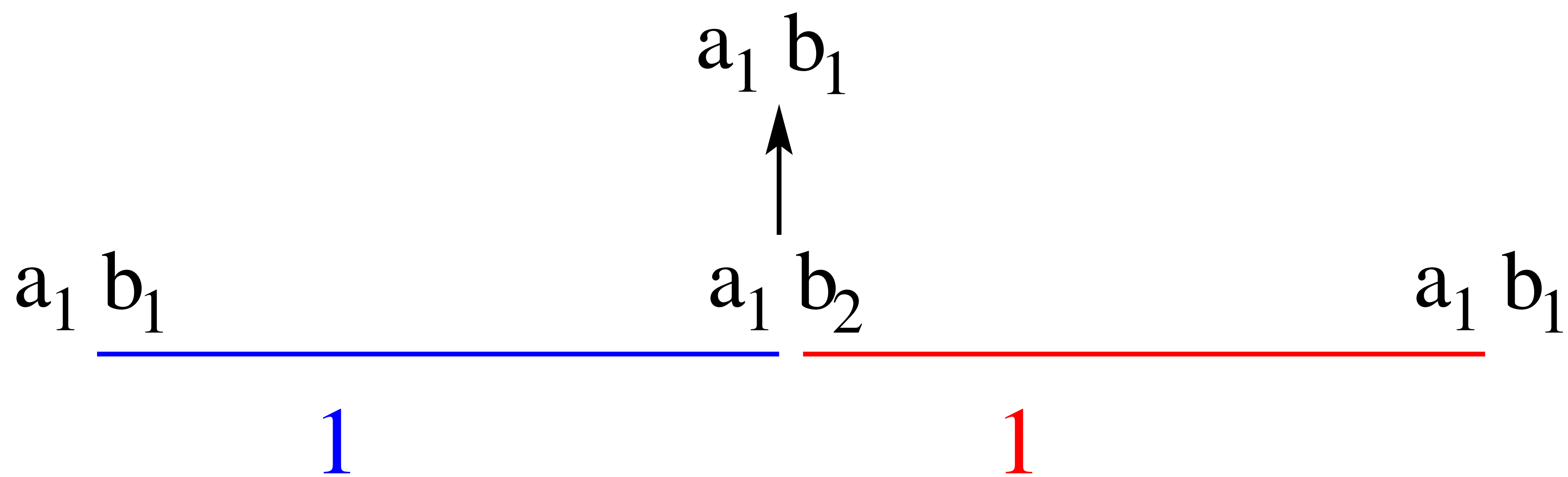
indirect processes for $\dot{P}_2$: 1 → 2



direct process for $\dot{P}_1$: choose random link

$$\Delta N_1 = -\tfrac{1}{2} P_1 \qquad \rightarrow \frac{\Delta P_1}{\Delta t} = -\frac{\frac{1}{2} P_1 / L}{1/N} = -\frac{P_1}{z}$$

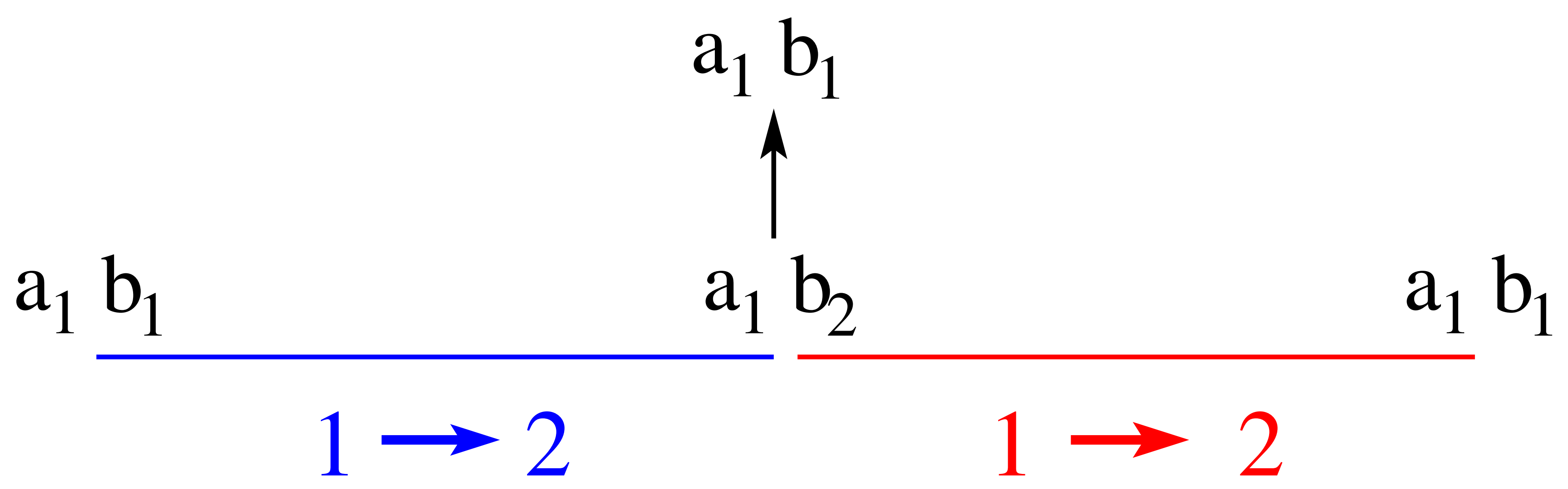
indirect processes for $\dot{P}_2$: 1 → 2



direct process for $\dot{P}_1$:   choose random link

$$\Delta N_1 = -\frac{1}{2} P_1 \qquad \to \frac{\Delta P_1}{\Delta t} = -\frac{\frac{1}{2} P_1 / L}{1/N} = -\frac{P_1}{z}$$

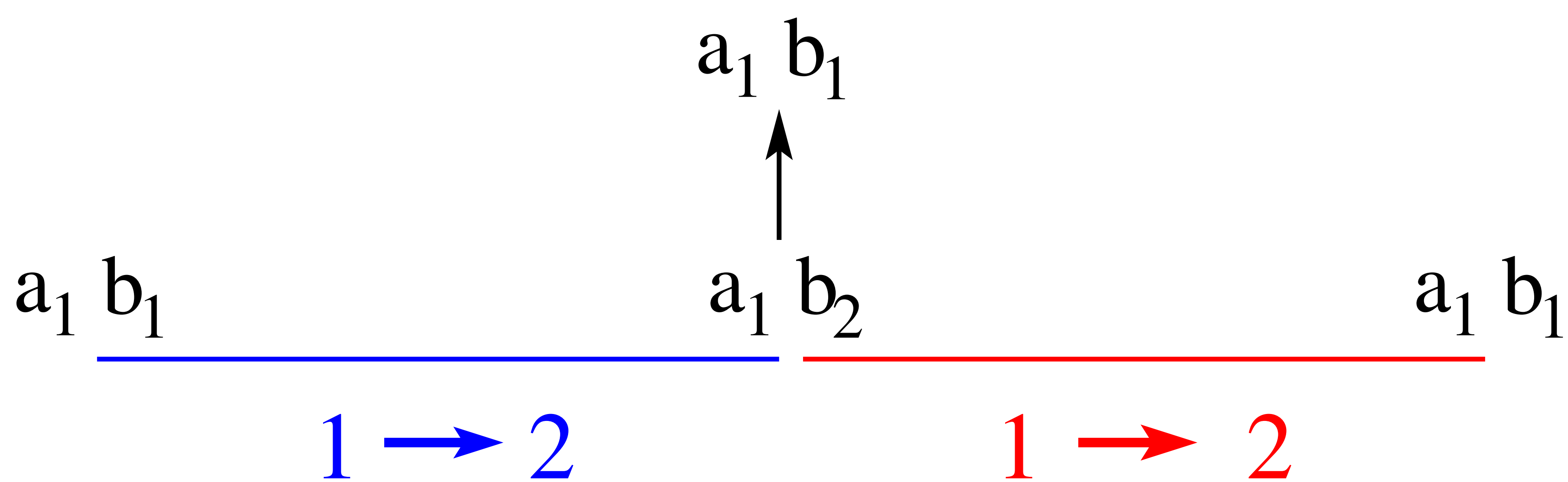
indirect processes for $\dot{P}_2$:   1 → 2



direct process for $\dot{P}_1$: choose random link

$$\Delta N_1 = -\tfrac{1}{2} P_1 \qquad \rightarrow \frac{\Delta P_1}{\Delta t} = -\frac{\frac{1}{2} P_1 / L}{1/N} = -\frac{P_1}{z}$$

indirect processes for $\dot{P}_2$: 1 → 2



$$\text{rate} \;\; \tfrac{1}{2} P_1 \lambda \underset{\text{mean field}}{\longrightarrow} \tfrac{1}{2} P_1 (q-1)^{-1}$$

Master Equations for Bond Densities

$$d\tau \equiv \tfrac{z-1}{z} P_1 \, dt$$
$$x \equiv P_0$$
$$y \equiv P_1$$
$$P_2 = 1 - P_0 - P_1$$

$$\dot{P}_0 = \frac{z-1}{z} P_1 \Big[-\lambda P_0 + \tfrac{1}{2} P_1 \Big]$$

$$\dot{P}_1 = -\frac{P_1}{z} + \frac{z-1}{z} P_1 \Big[\lambda P_0 - \tfrac{1}{2}(1+\lambda) P_1 + P_2 \Big]$$

$$\dot{P}_2 = \quad \frac{P_1}{z} + \frac{z-1}{z} P_1 \left[\tfrac{1}{2} \lambda P_1 - P_2 \right]$$

Master Equations for Bond Densities

$$x' = -\lambda x + \tfrac{1}{2} y$$

$$y' = \left(1 - \tfrac{1}{\eta}\right) + (\lambda - 1)x - \left(\tfrac{3+\lambda}{2}\right) y$$

$$d\tau \equiv \tfrac{z-1}{z} P_1 \, dt$$

$$x \equiv P_0$$

$$y \equiv P_1$$

$$P_2 = 1 - P_0 - P_1$$

Master Equations for Bond Densities

$$x' = -\lambda x + \tfrac{1}{2} y$$

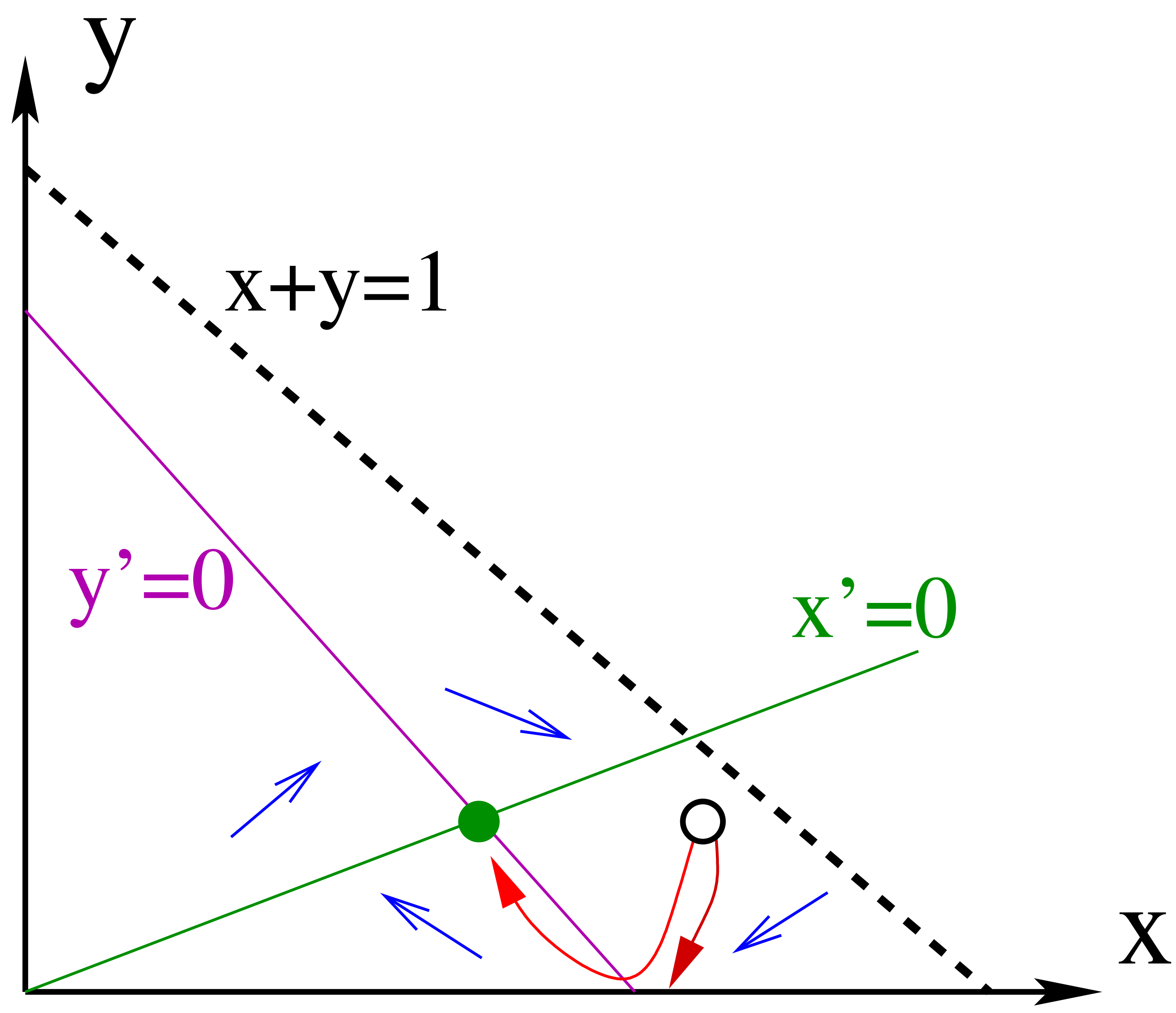
$$y' = \left(1 - \tfrac{1}{\eta}\right) + (\lambda - 1)x - \left(\tfrac{3+\lambda}{2}\right) y$$

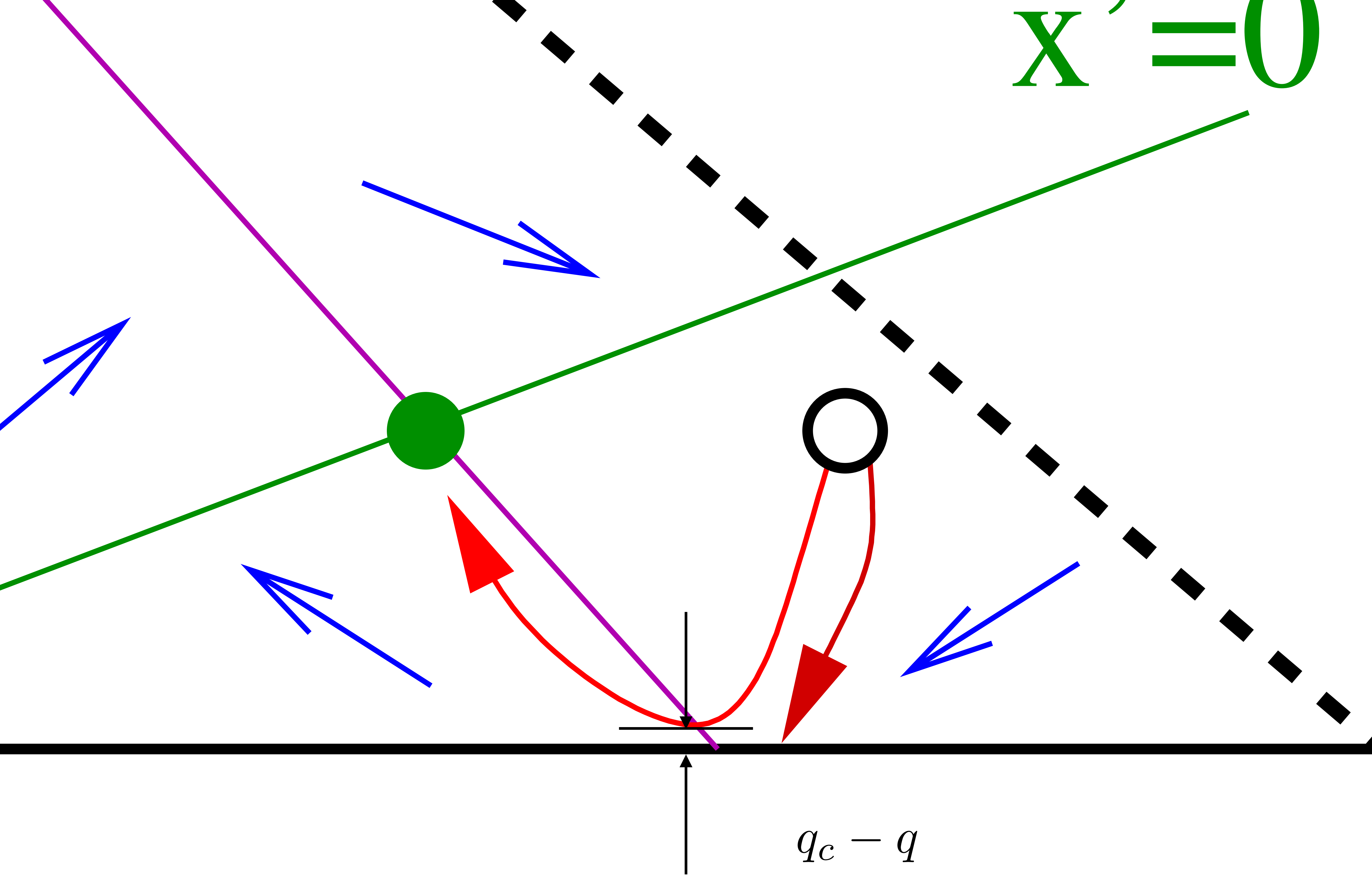
$$d\tau \equiv \tfrac{z-1}{z} P_1 \, dt$$

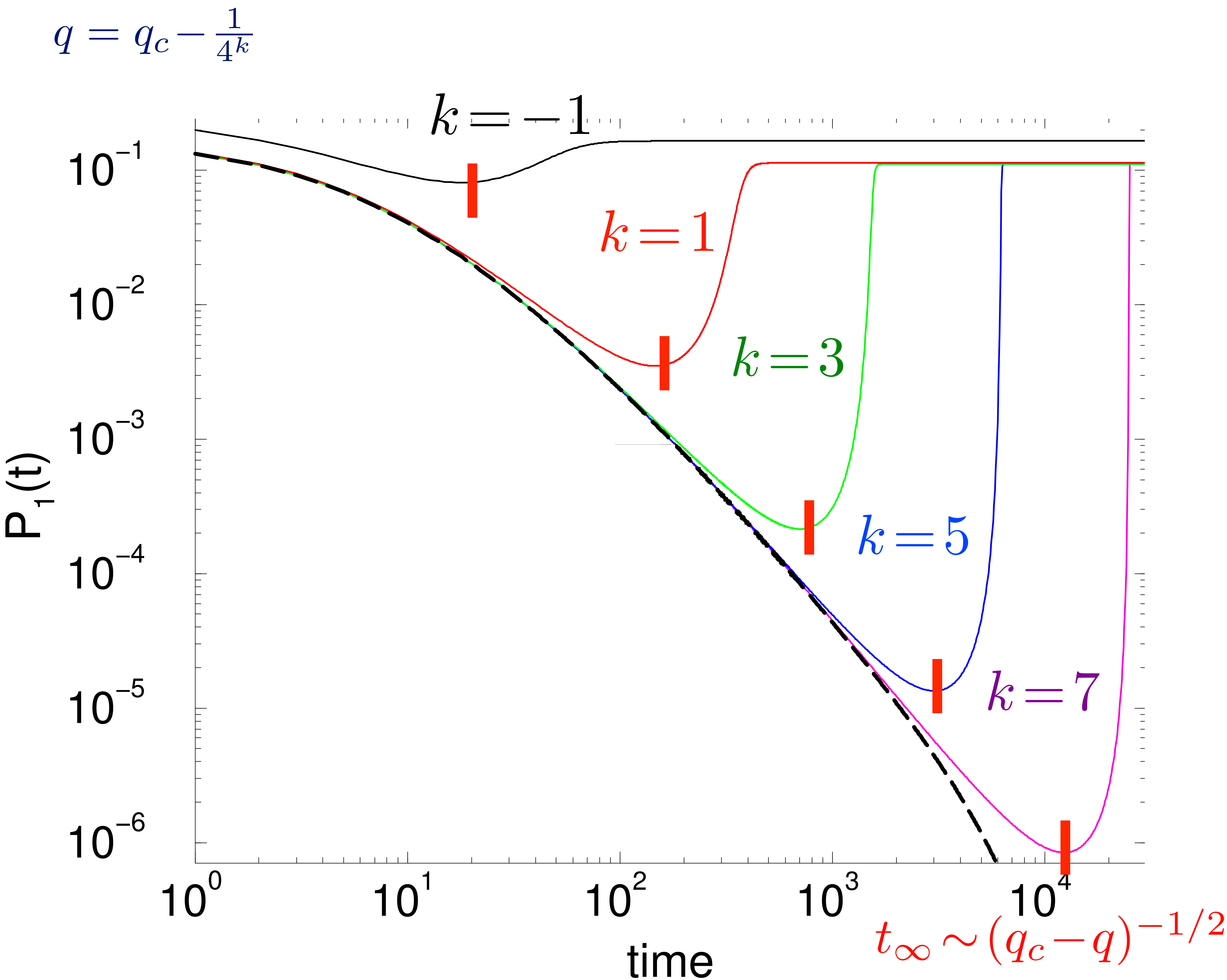
$$x \equiv P_0$$

$$y \equiv P_1$$

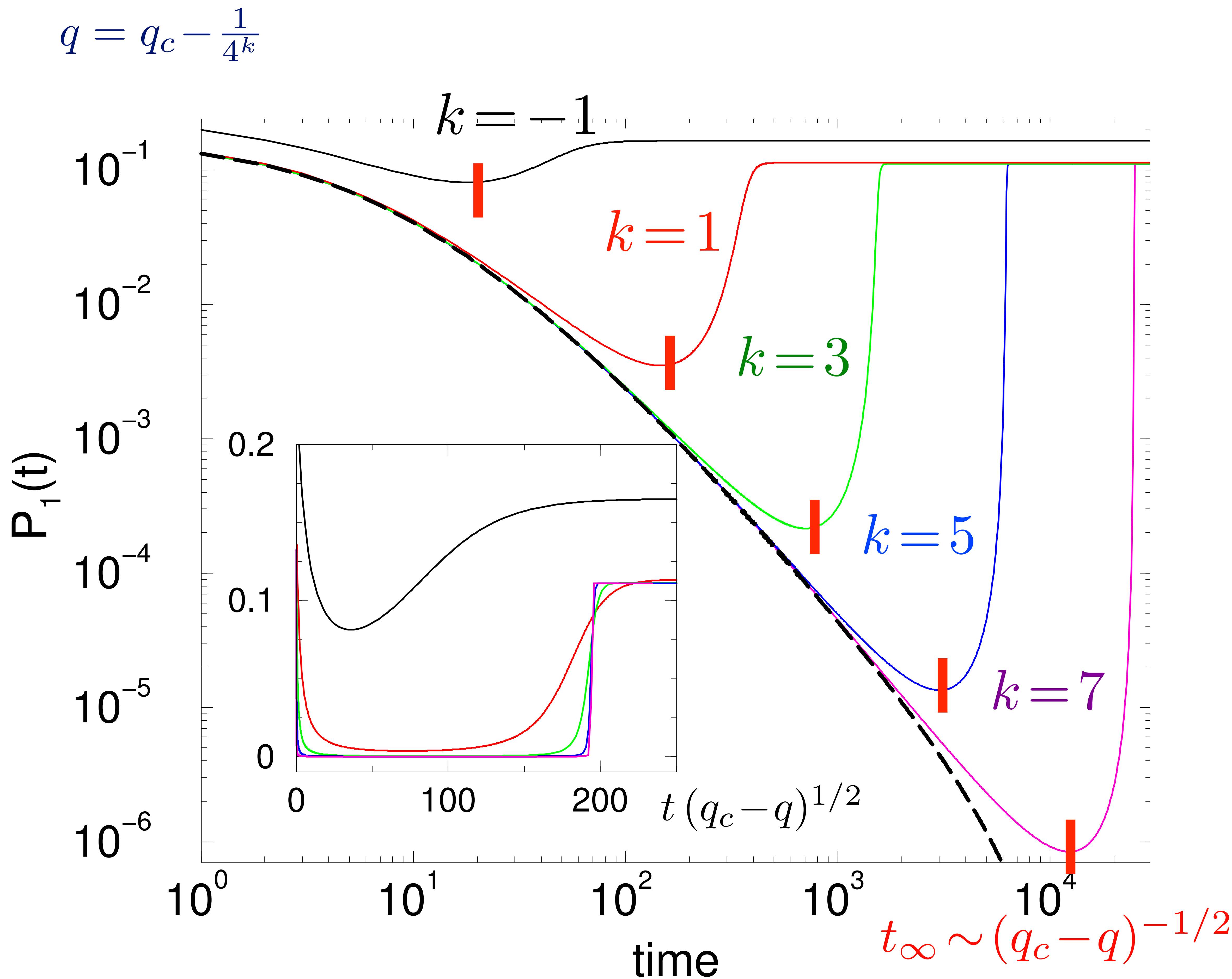
$$P_2 = 1 - P_0 - P_1$$




X'=0
$q_c - q$



$q = q_c - \frac{1}{4^k}$
$k = -1$
$k = 1$
$k = 3$
$k = 5$
$k = 7$
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
$P_1(t)$
10^0
10^1
10^2
10^3
10^4
time
$t_\infty \sim (q_c - q)^{-1/2}$



$q = q_c - \frac{1}{4^k}$
$k=-1$
$k=1$
$k=3$
$k=5$
$k=7$
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
$P_1(t)$
10^0
10^1
10^2
10^3
10^4
time
$t_\infty \sim (q_c - q)^{-1/2}$
0.2
0.1
0
100
200
$t\,(q_c - q)^{1/2}$

Axelrod Model with F=2

transition between steady state ($q<q_c$) & fragmented static state ($q>q_c$)

$q<q_c$: very slow approach to steady state with time scale $\simeq (q_c - q)^{-1/2}$

long transient in which $P_1 \simeq (q_c - q)$ before steady state is reached

Some Closing Thoughts

Voter Model well characterized, but:

- *consensus route incompletely understood on complex graphs*
- *generalizations, role of heterogeneity, role of internal beliefs,*
- *data-driven models*

Models of Diversity & Discord

- *bifurcation sequence in bounded compromise*
- *role of competing social interactions mostly unknown*
- *mathematical understanding of Axelrod model lacking*
- *data-driven models*

Notes: physics.bu.edu/~redner:

click the "slides from selected talks" link