Facilitated Asymmetric Exclusion

Complex Driven Systems - From Statistical Physics to the Life Sciences

celebrating Royce's achievements in 40 years of research

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Sid Redner (physics.bu.edu/~redner)

Alan Gabel & Paul Krapivsky (BU), arXiv:1007.3217

support from NSF DMR0906504

Basic Facts about the ASEP

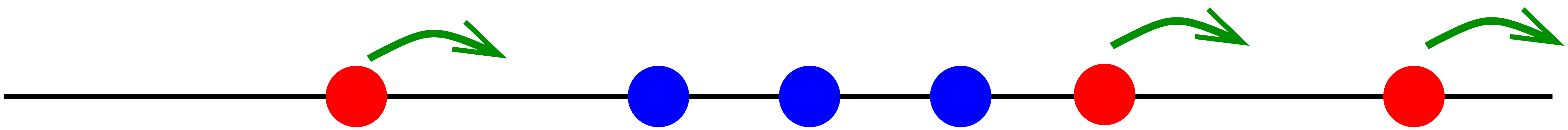
Facilitated ASEP

- steady state current
- island size distribution
- rarefaction wave discontinuity

Summary & Outlook

Asymmetric Exclusion Process (ASEP)

B. Schmittmann & R.K.P. Zia, *Phase Transitions and Critical Phenomena*, Vol.17, eds. C. Domb and J. L. Lebowitz
G. Schütz, *Phase Transitions and Critical Phenomena*, Vol. 19, eds. C. Domb and J. L. Lebowitz
B. Derrida, Phys. Repts. **301**, 65 (1998);



- particles confined to a 1d lattice
- particles move to the right only (asymmetric)
- only one particle per site (exclusion)
- hopping only if target is vacant (process)

Asymmetric Exclusion Process (ASEP)

- *simple:* only one parameter, the density ρ
- *no correlations:* steady-state current j=ρ(1-ρ)
- inhomogeneity smoothing by Burgers eqn

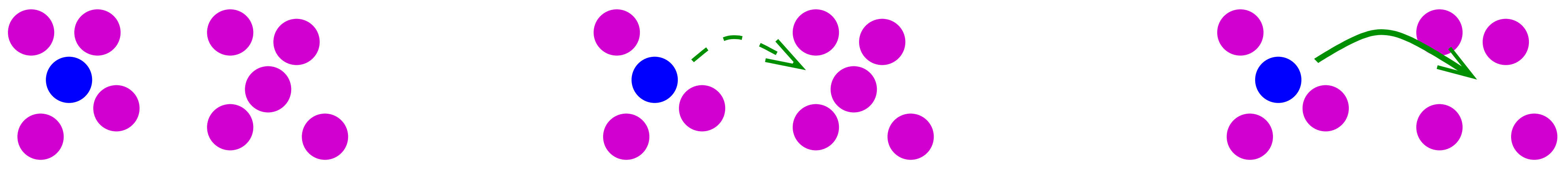
$$\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho(1-\rho)}{\partial x} = 0$$

upslope → shock wave

downstep → rarefaction wave

Why and What is Facilitation?

particle motion in glasses: higher mobility in
low-density regions



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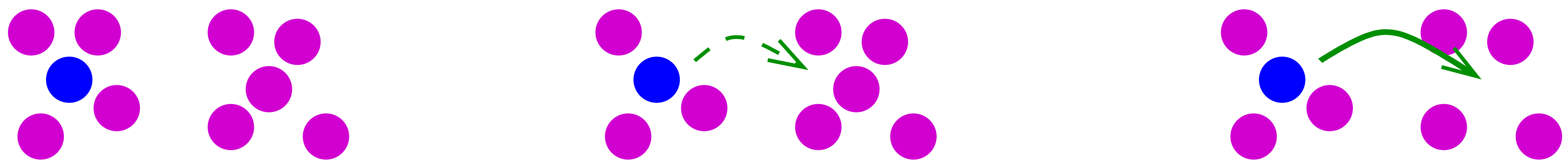

facilitated exclusion: need stimulus to move

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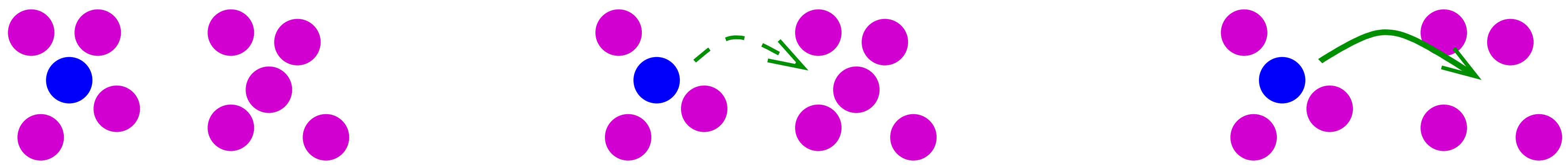

facilitated exclusion: need stimulus to move

lazy: no stimuli

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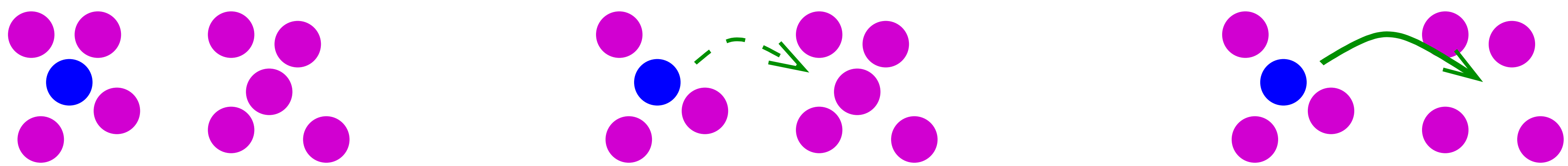


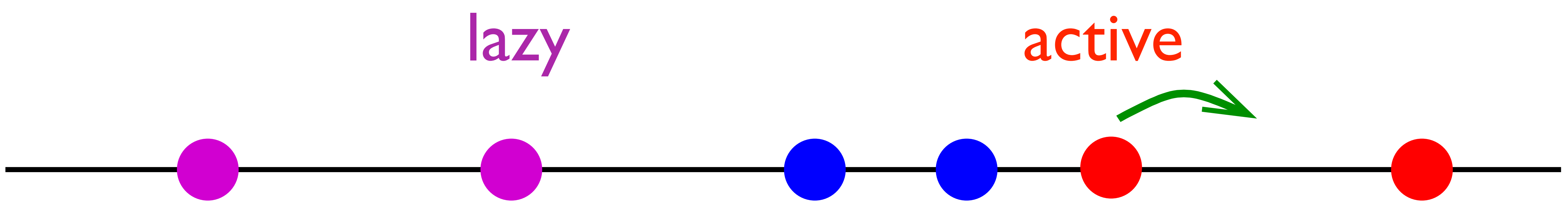
facilitated exclusion: need stimulus to move

lazy: no stimuli

mobile: kicked in the ass

Facilitated Asymmetric Exclusion Process

L. B. Shaw, R. K. P. Zia, and K. H. Lee, Phys. Rev. E **68**, 021910 (2003)
U. Basu and P. K. Mohanty, Phys. Rev. E **79**, 041143 (2009)
M. Rossi, R. Pastor-Satorras, and A. Vespignani, Phys. Rev. Lett. **85**, 1803 (2000)
M. Sellitto, Phys. Rev. Lett. **101**, 048301 (2008)



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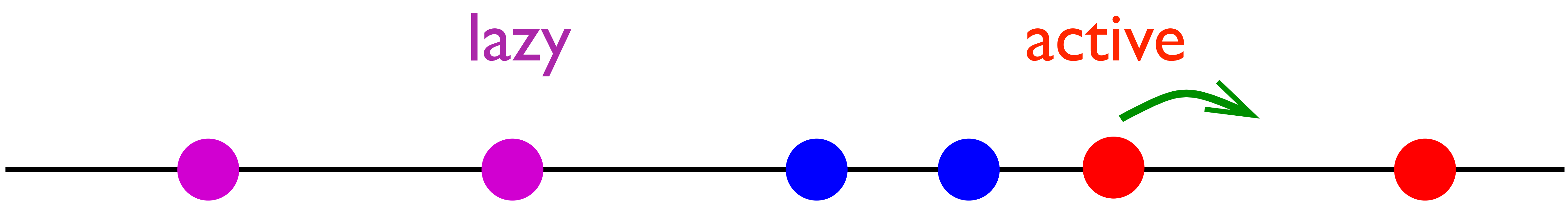

only the triplet ● ● ○ is active

Mapping to Dimer ASEP

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valid ***only*** for density >1/2

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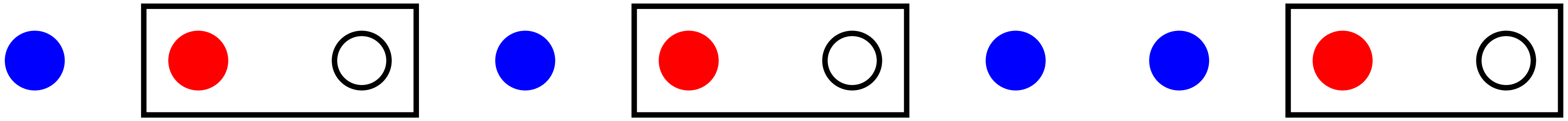

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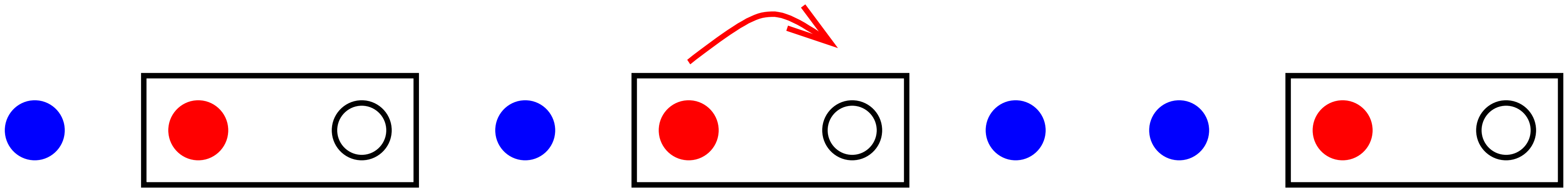

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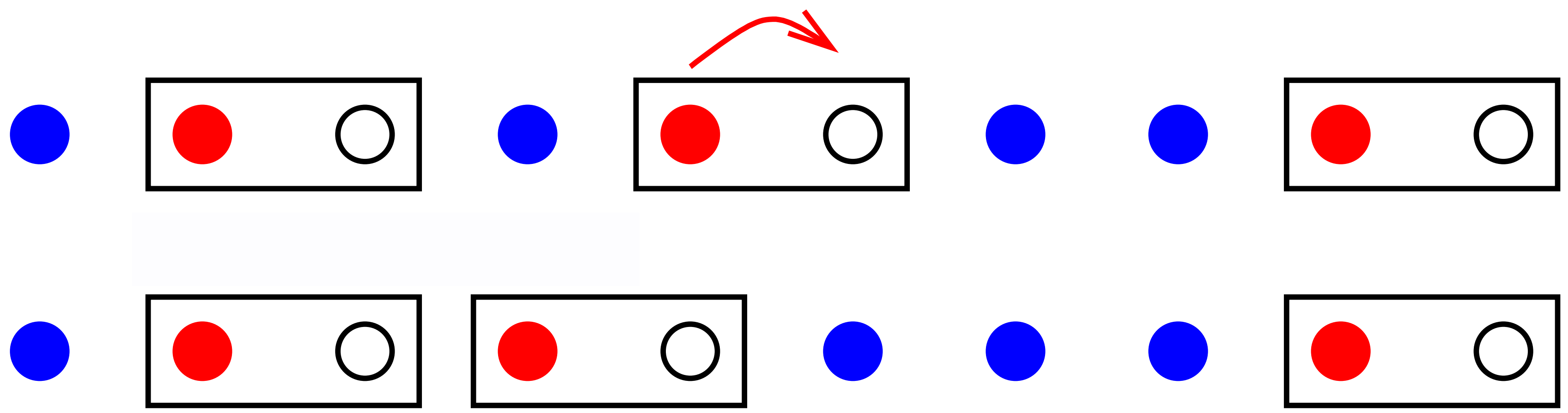

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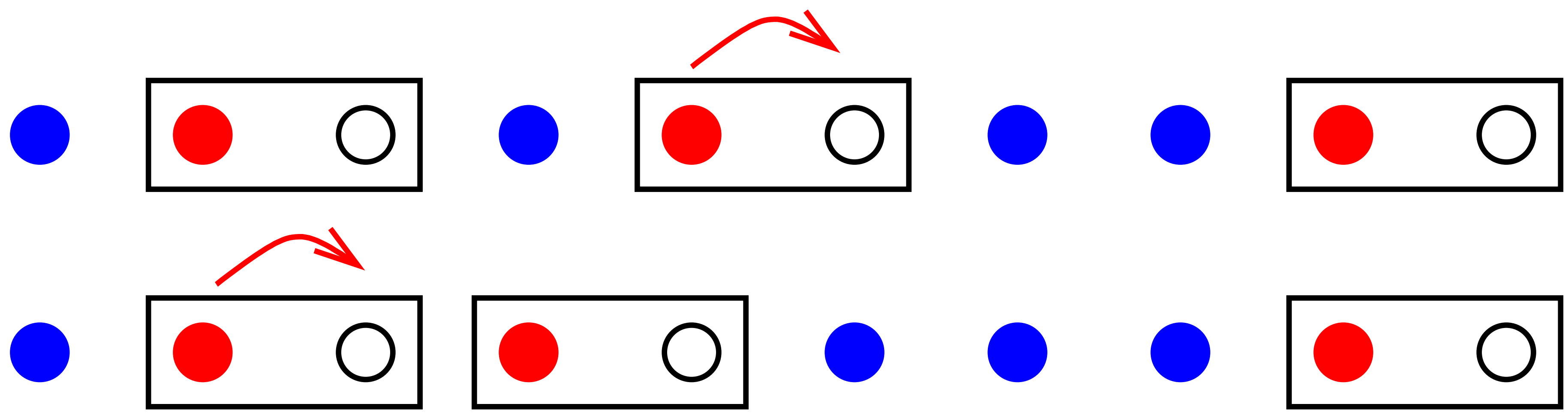

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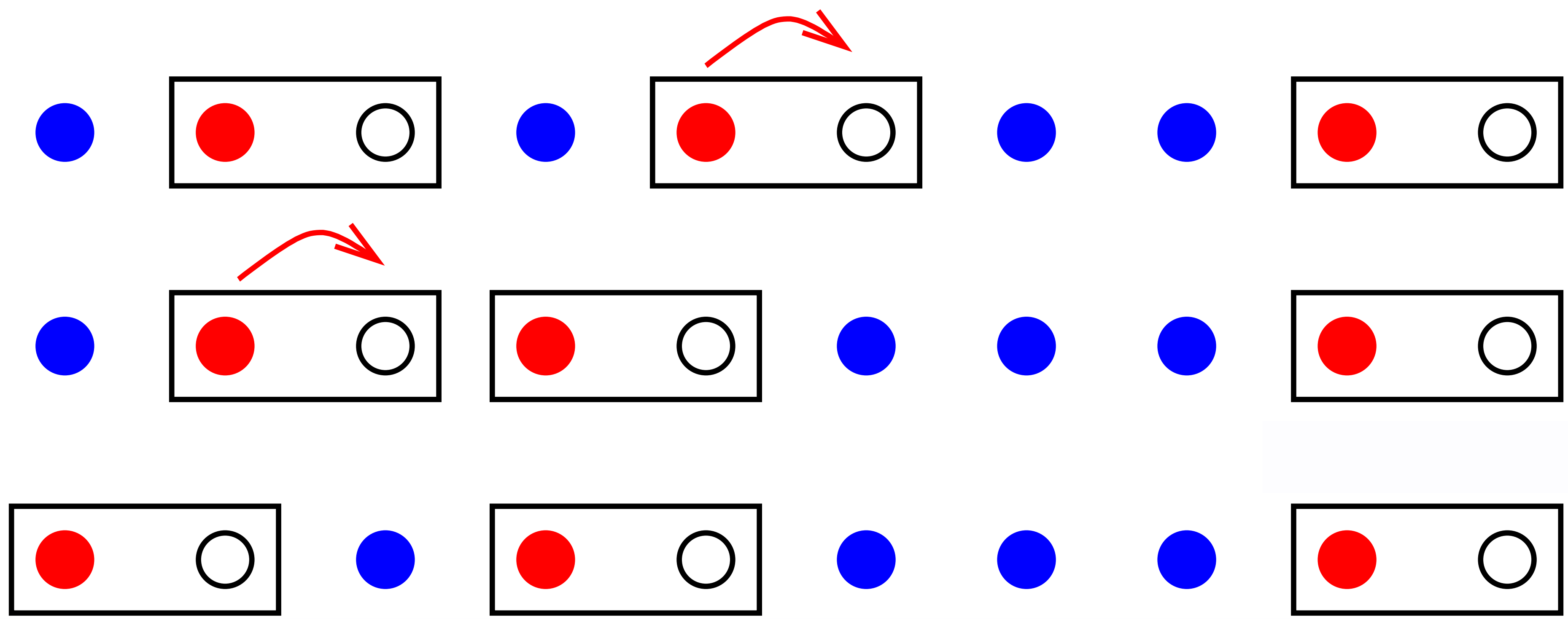

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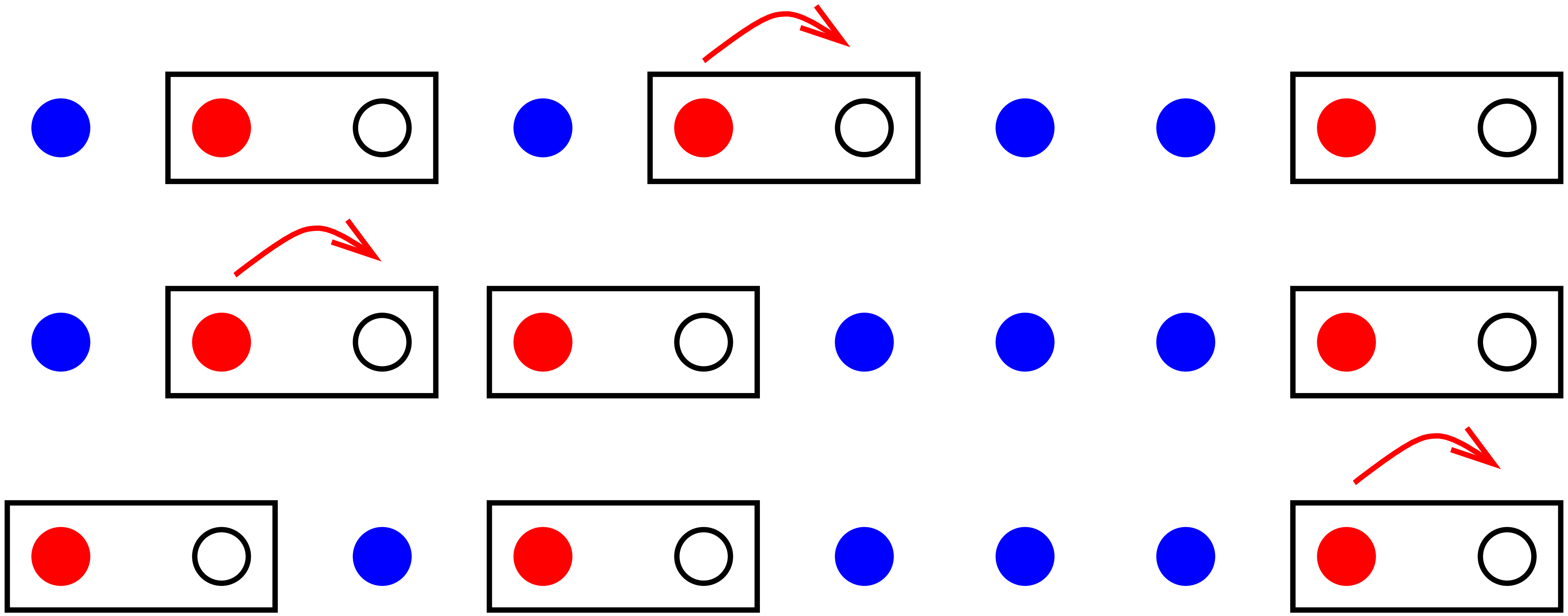

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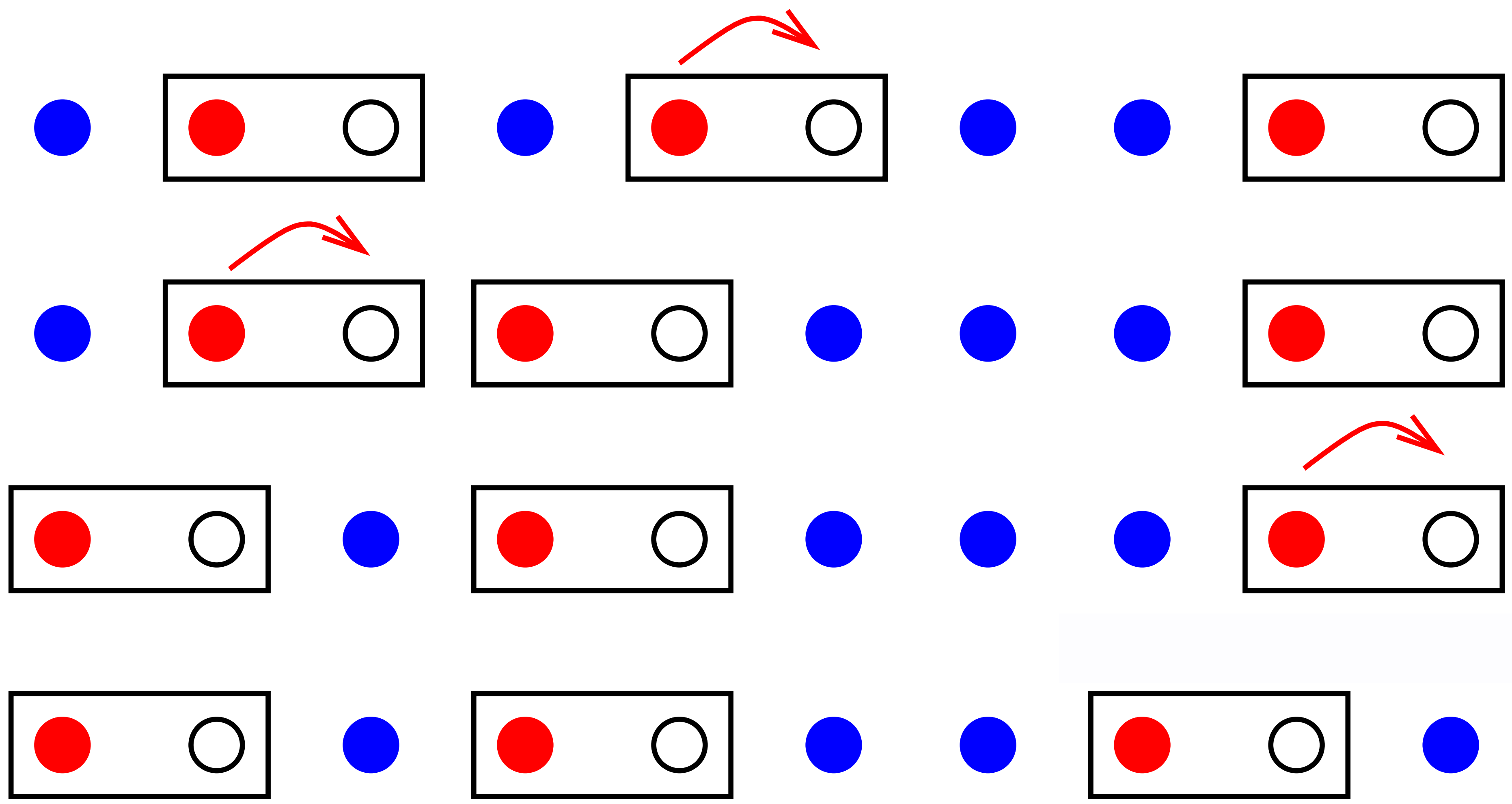

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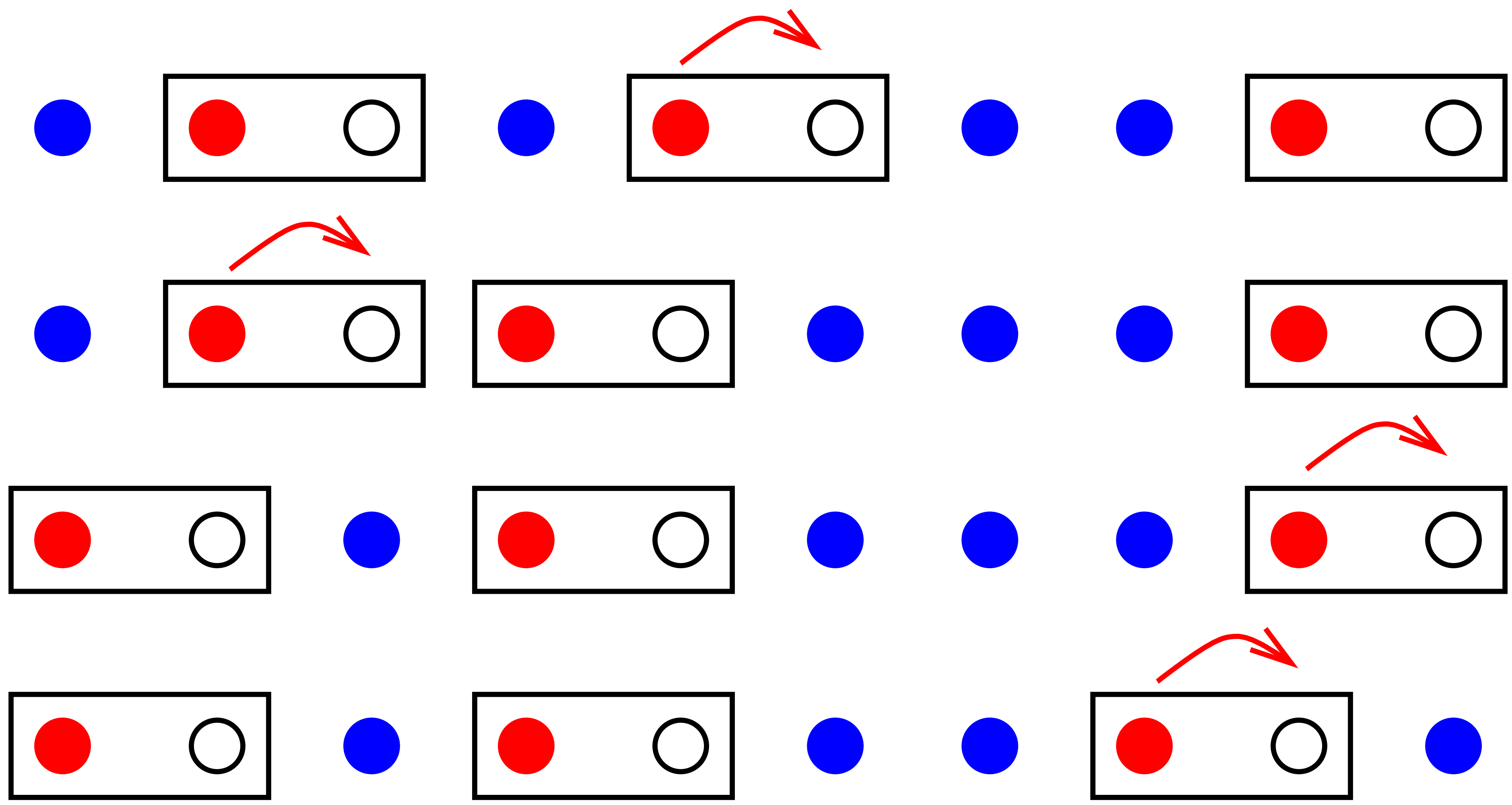

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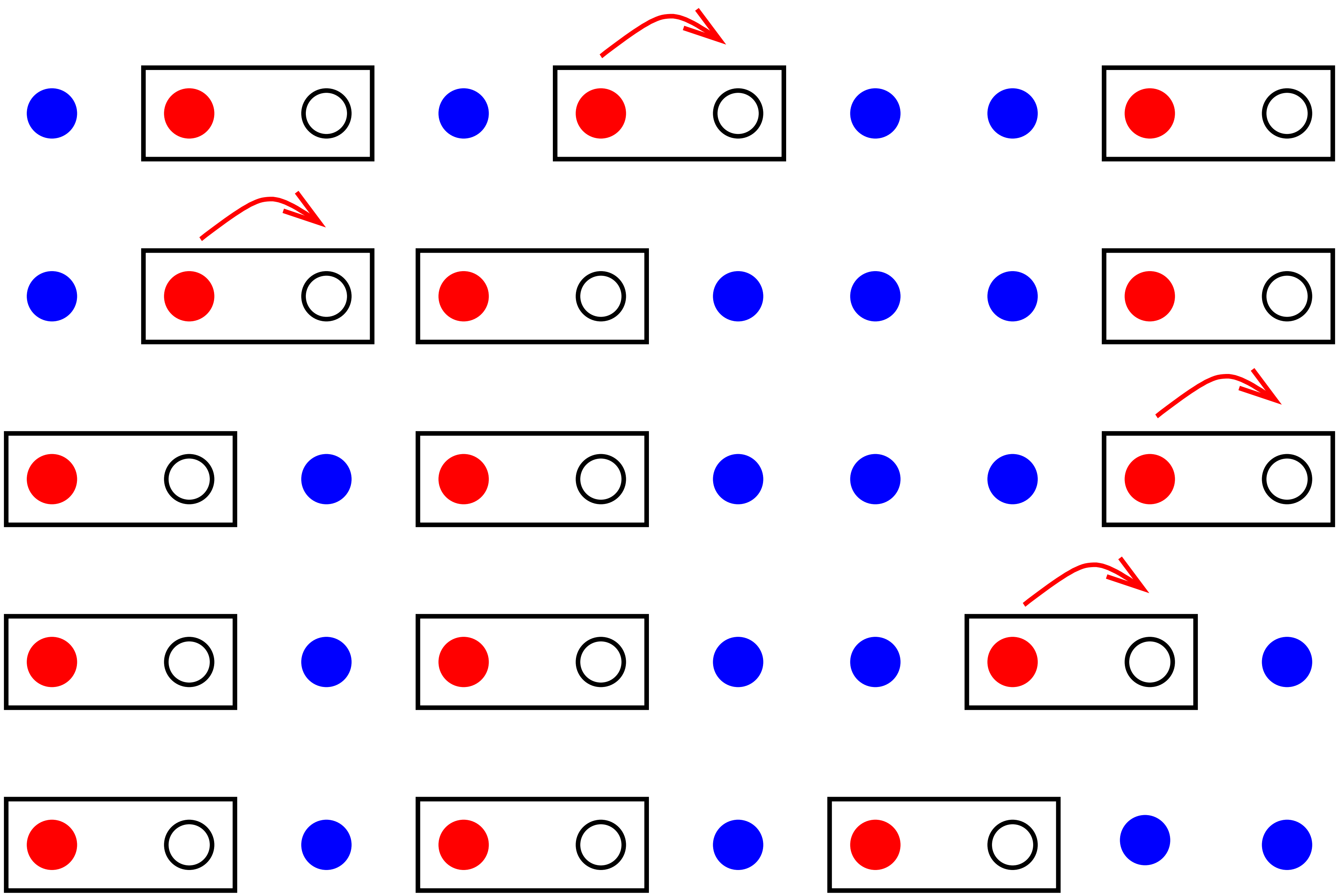
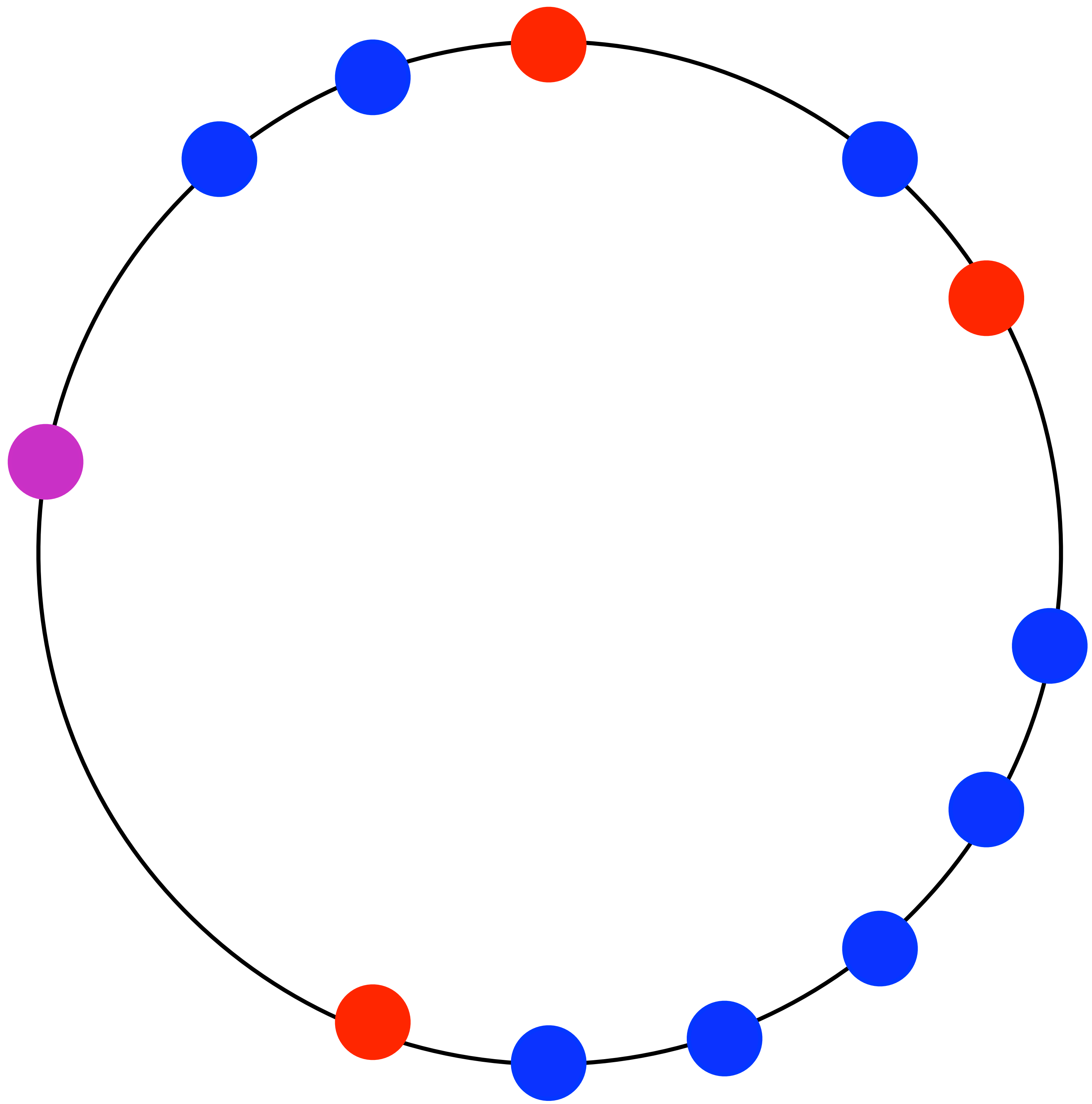
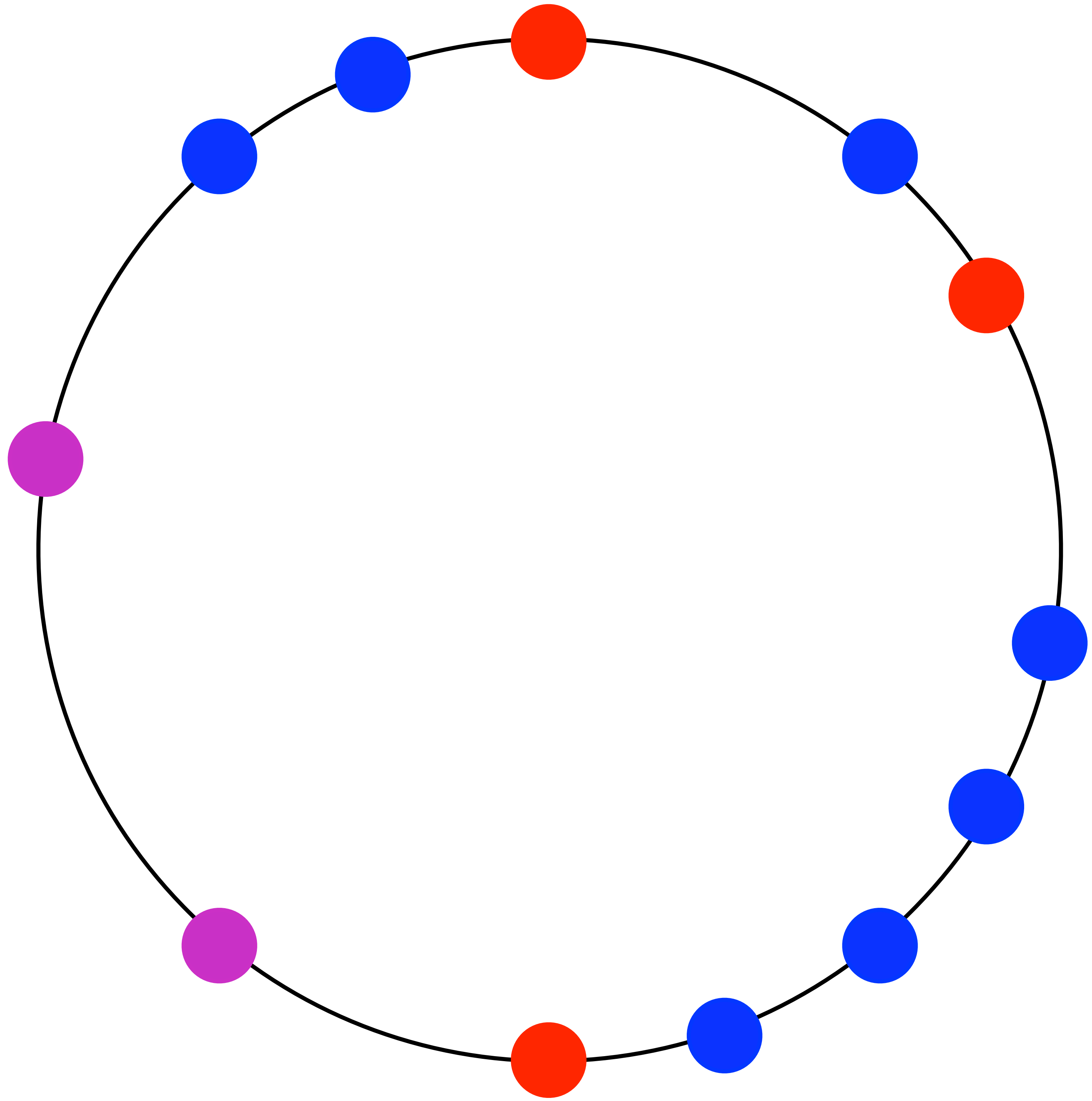


Finite Ring (density > 1/2)



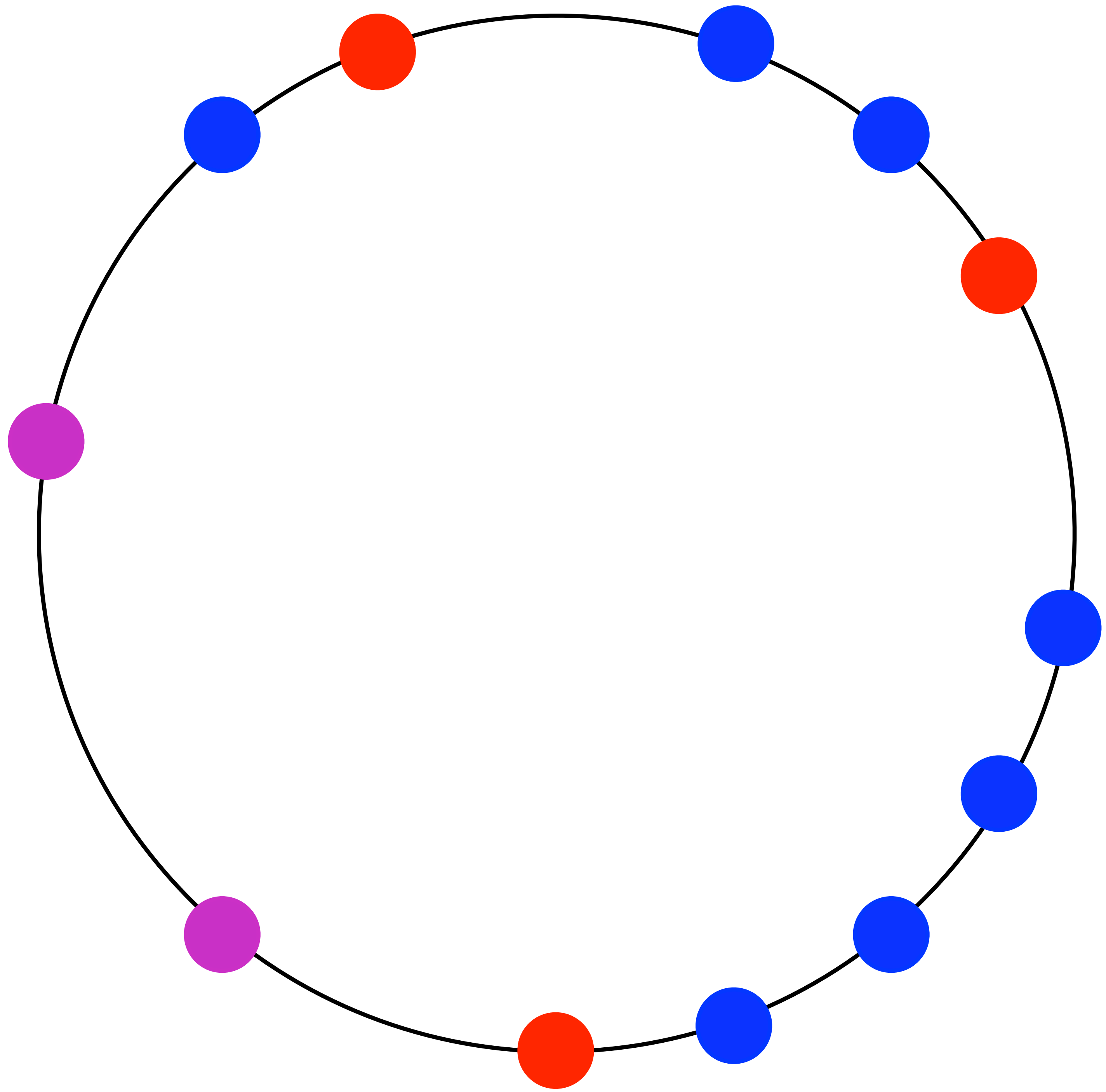
4 islands

Finite Ring (density > 1/2)



5 islands

Finite Ring (density > 1/2)



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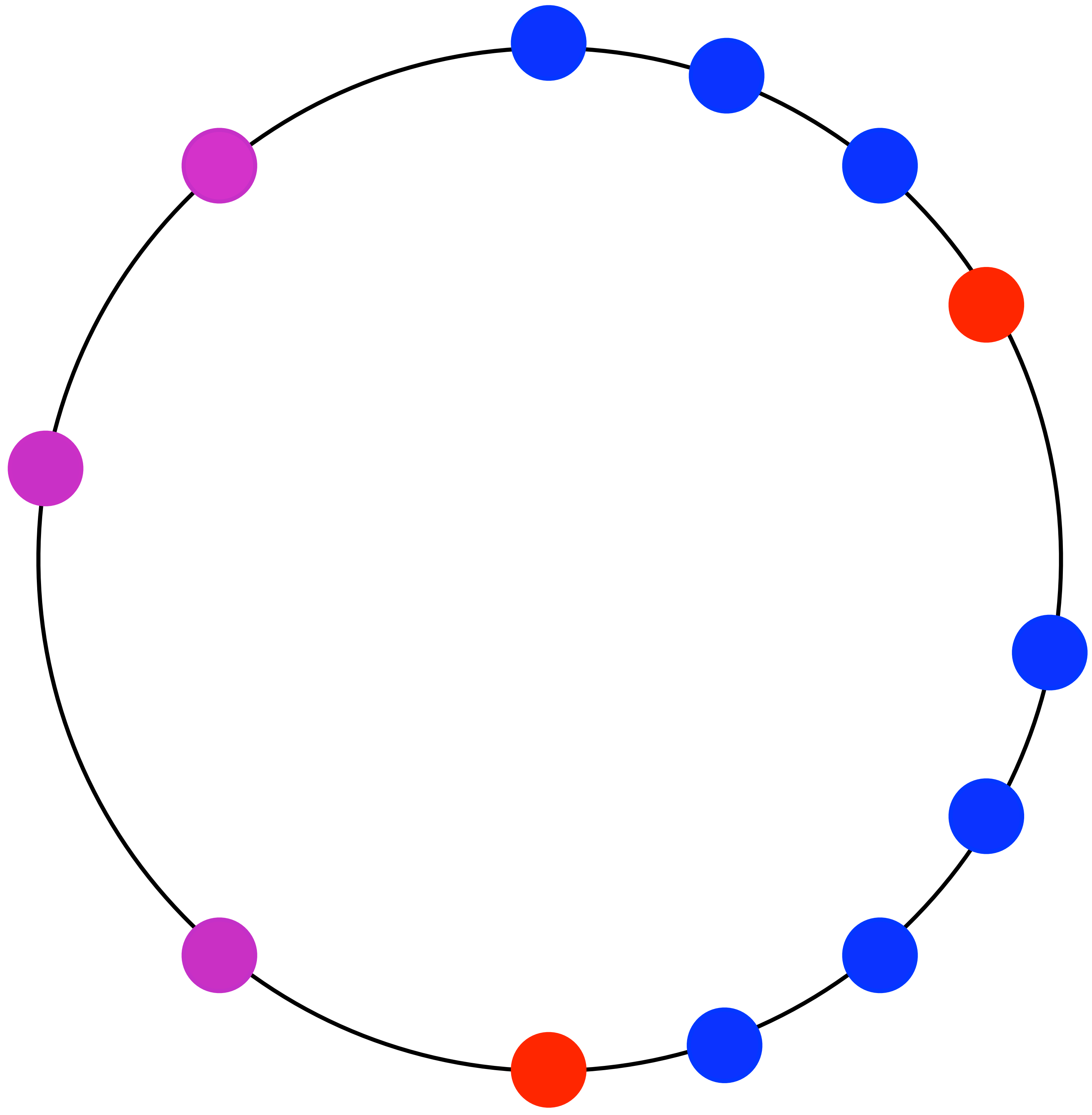

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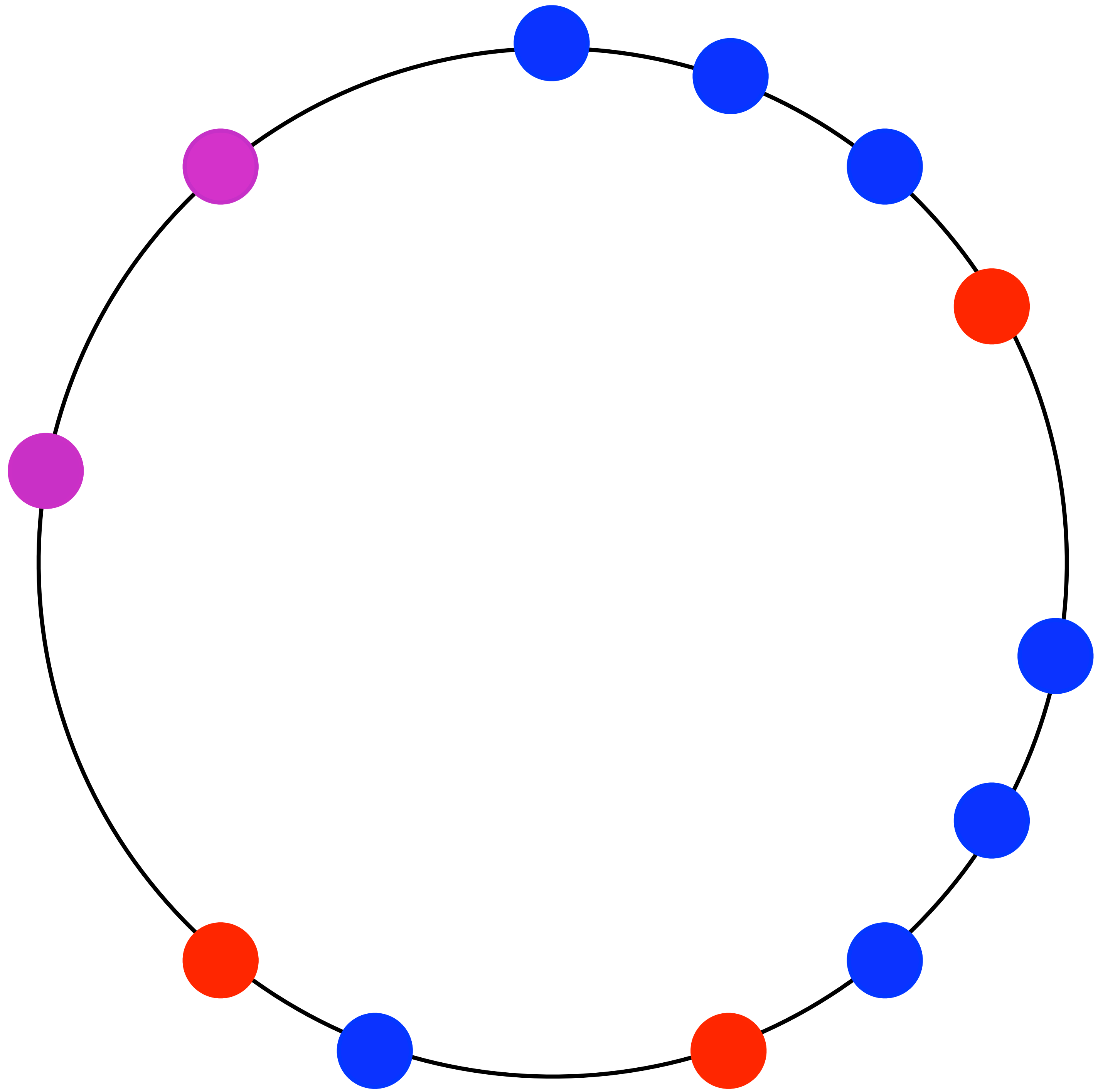

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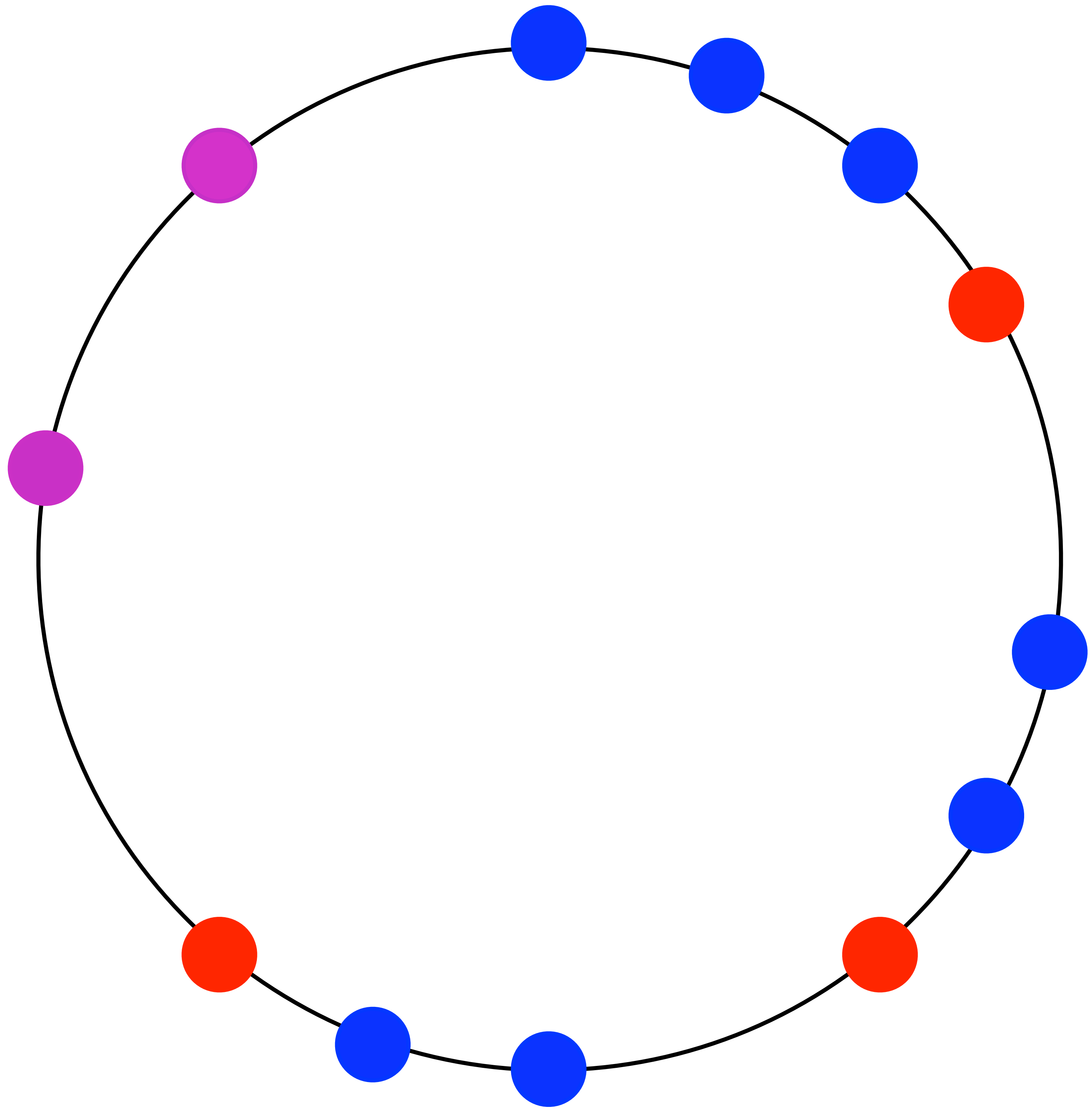

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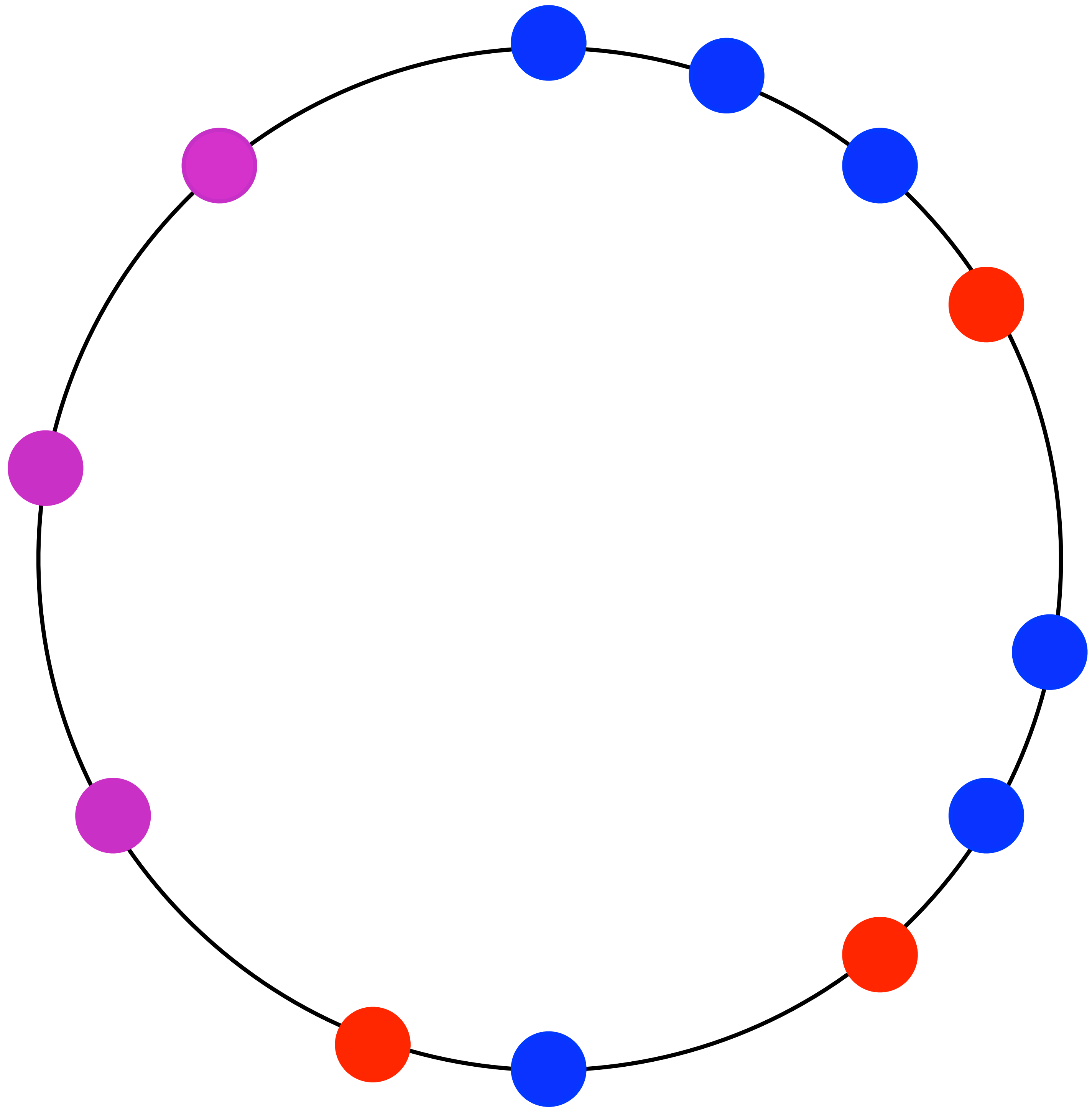
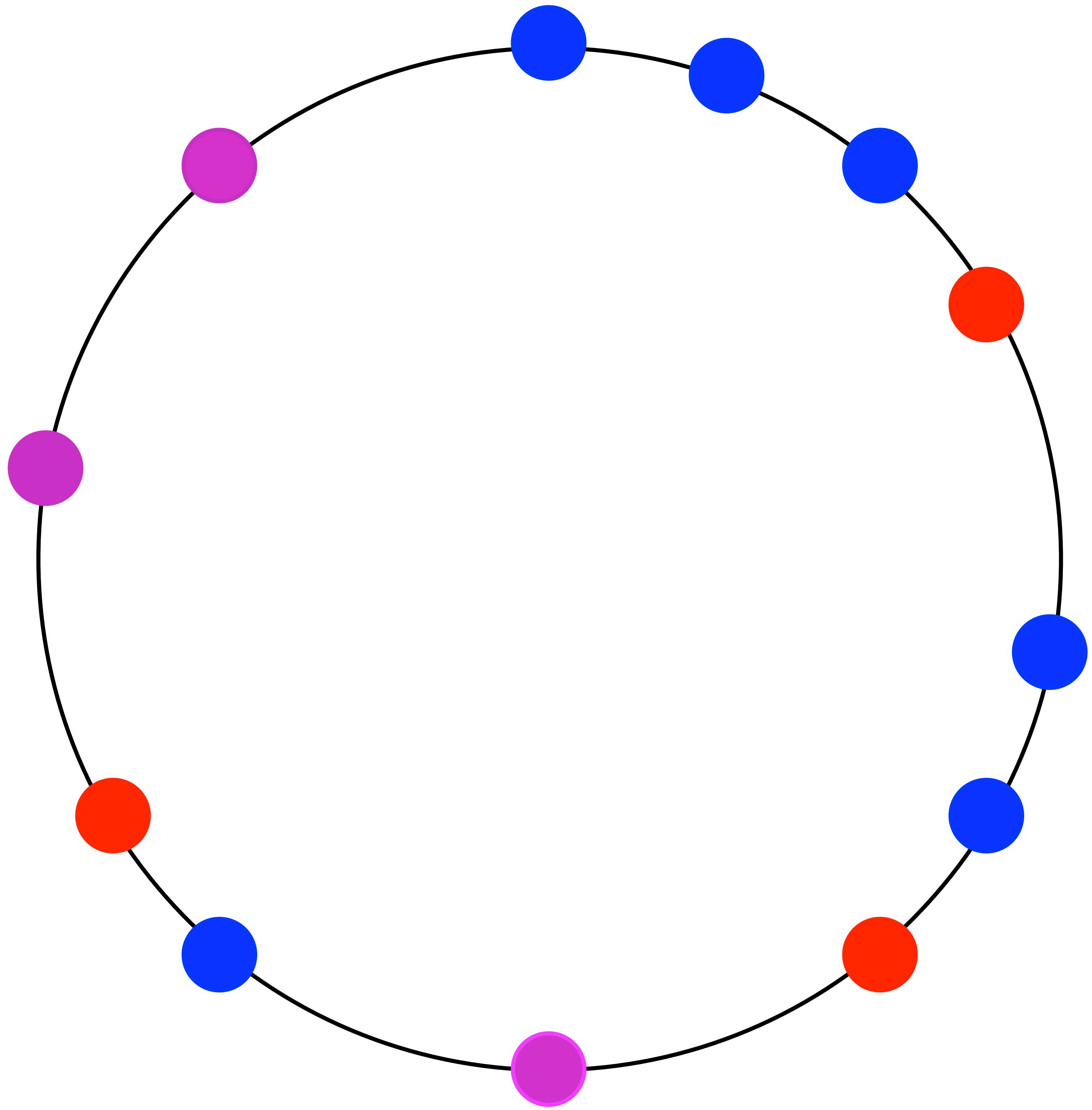


6 islands

Finite Ring (density > 1/2)



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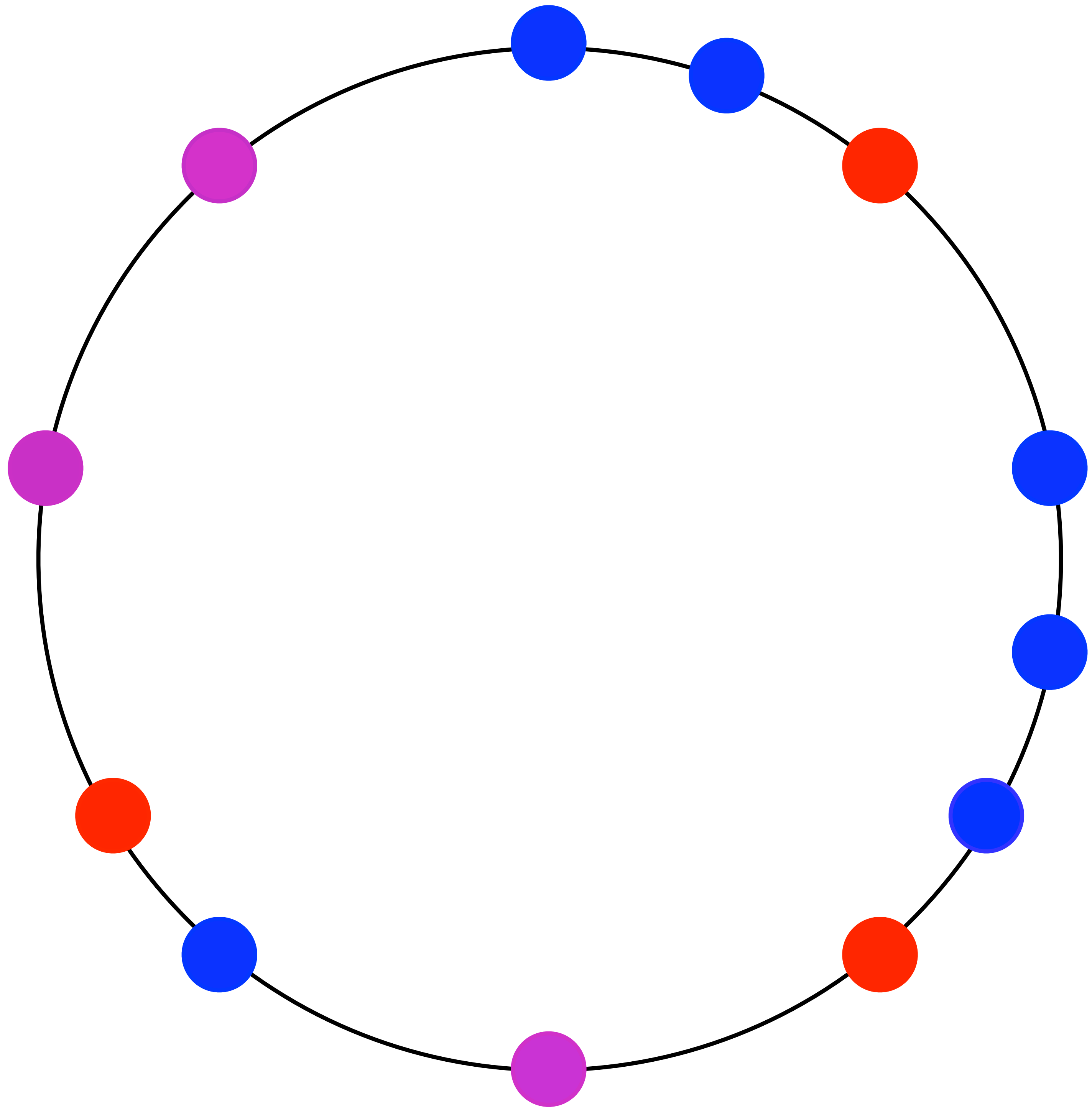

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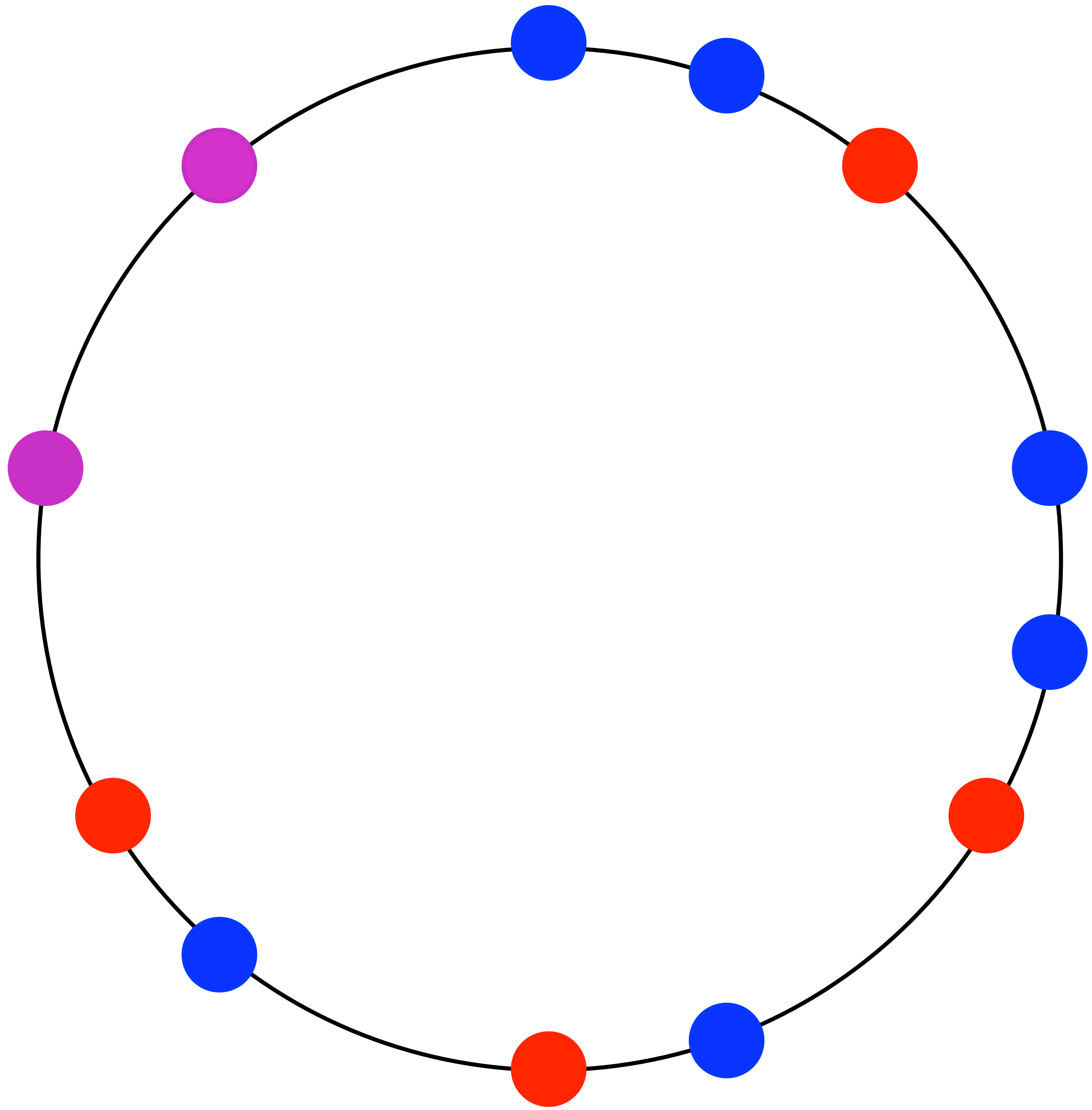

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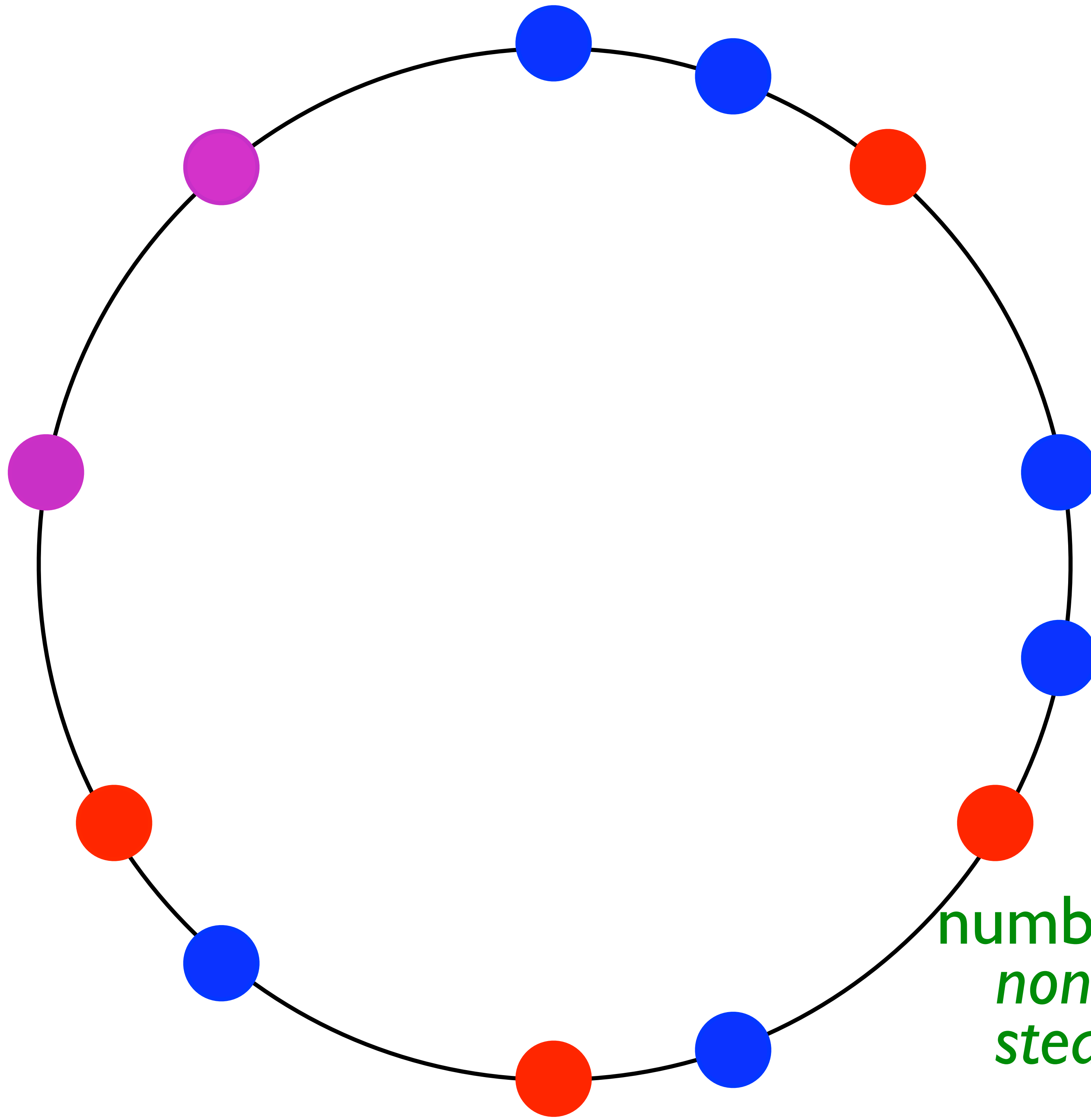

6 islands

number of islands:
non-decreasing until a steady state is achieved

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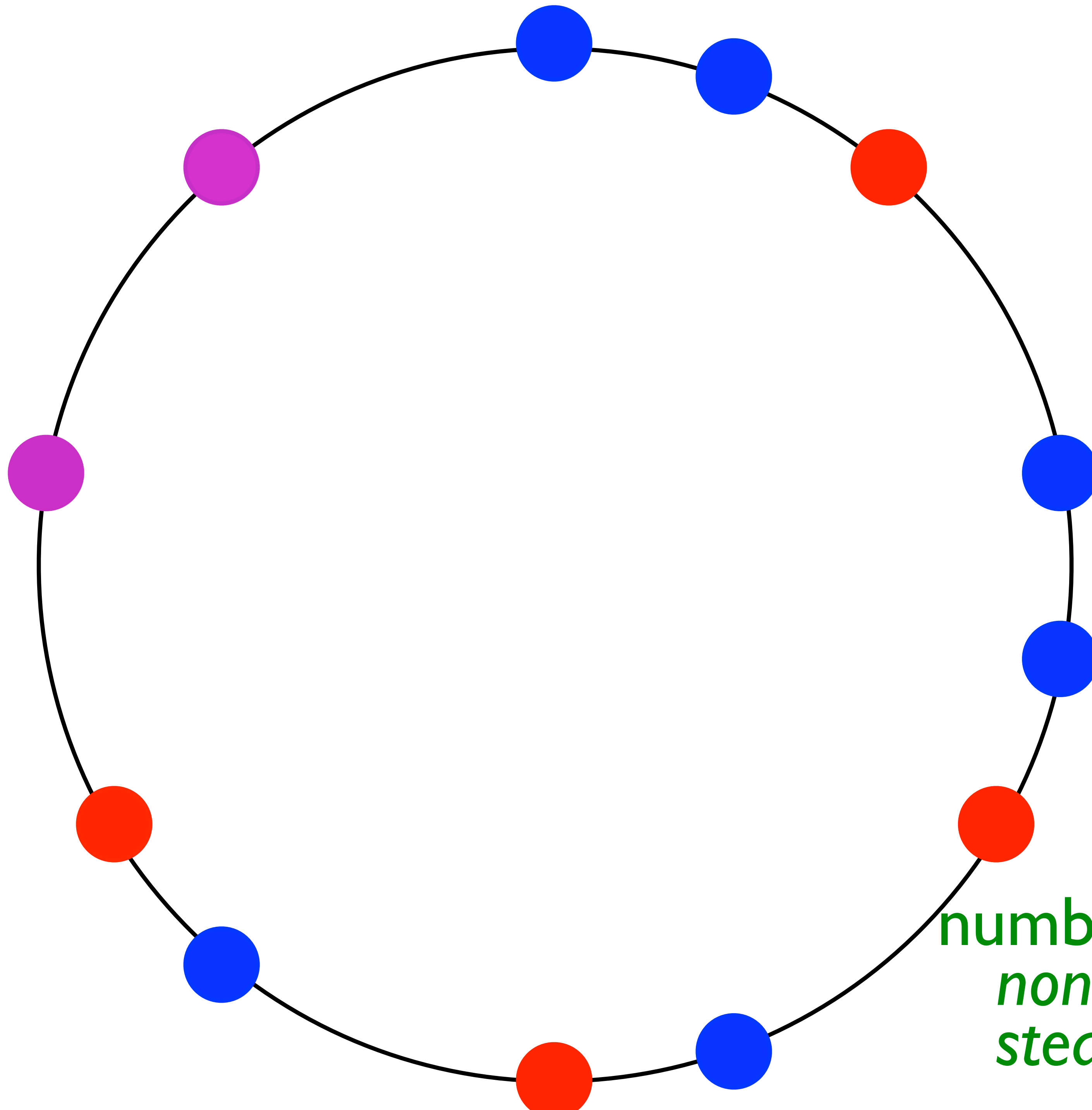


6 islands

number of islands:
non-decreasing until a steady state is achieved

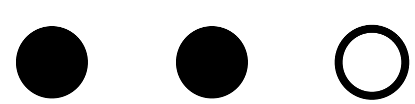
isolated vacancies

Steady State on the Ring

claim: all maximal-island states are equiprobable

$$P(C)\sum_{C'} R(C\to C') = \sum_{C'} P(C')\,R(C'\to C)$$

# of active leading triplets



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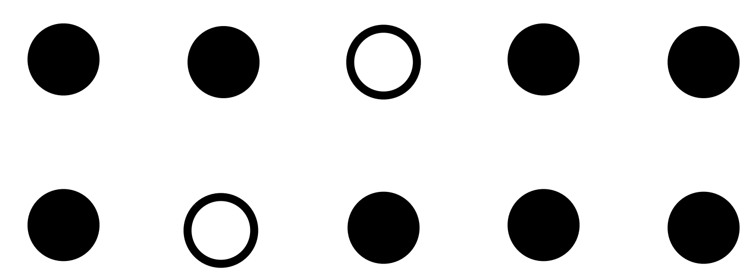

steady state for P(C)= constant

Steady State on the Ring

$$P(C) = \mathcal{C}^{-1}$$

$\mathcal{C} =$ number of maximal-island configurations with N particles & V vacancies

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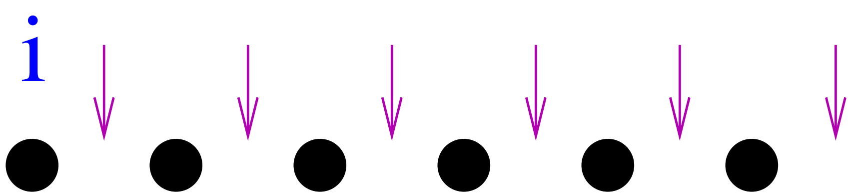
if site i occupied:



N particles
N possibilities for V vacancies

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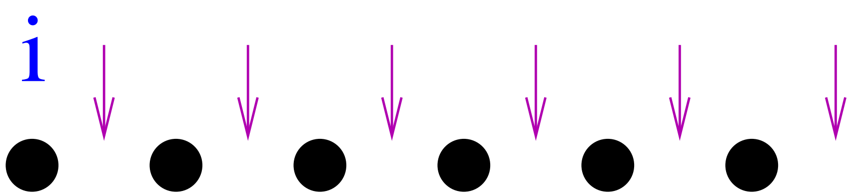
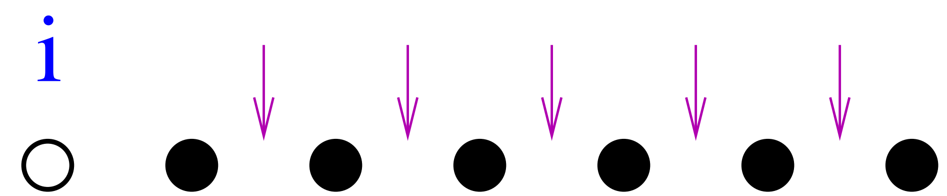


N particles
N possibilities for V vacancies

if site i empty



N particles
N-1 possibilities for V vacancies

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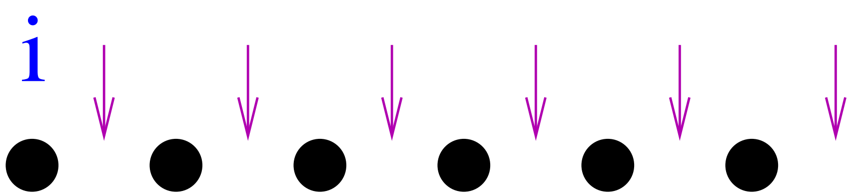
if site i occupied:



N particles
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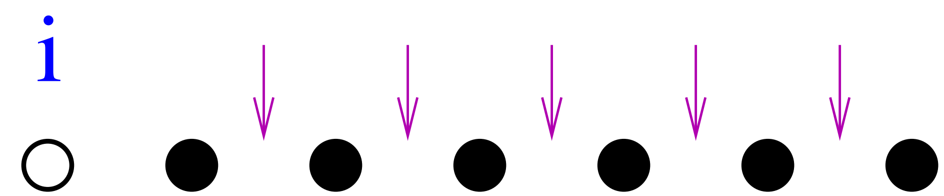


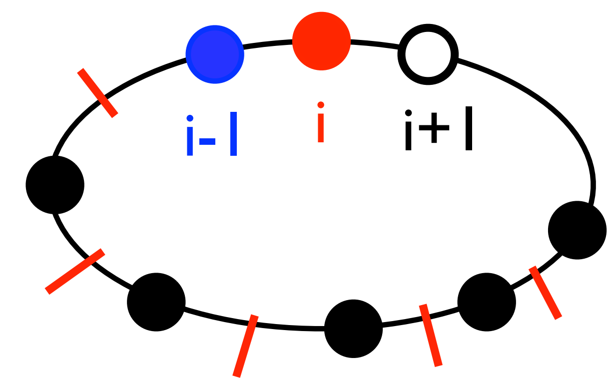
N particles
N-1 possibilities for V vacancies

$$\mathcal{C} = \binom{N}{V} + \binom{N-1}{V-1}$$

Steady State Current

for flow between sites i and i+1:

- sites i-1 & i occupied; site i+1 empty
- number of consistent maximal-island configs.
- N-2 allowed locations for V-1 vacancies



N-2 remaining particles

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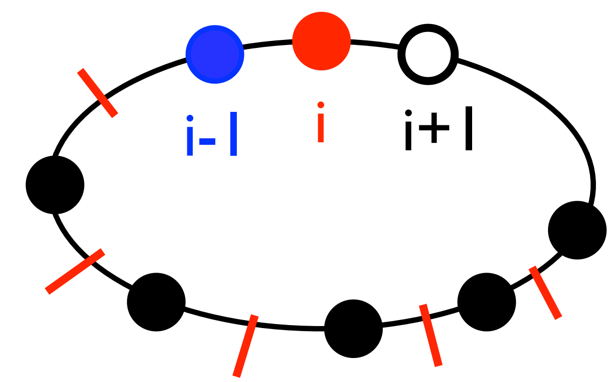

N-2 remaining particles

$$J = \frac{\binom{N-2}{V-1}}{\mathcal{C}} = \frac{\binom{N-2}{V-1}}{\binom{N}{V} + \binom{N-1}{V-1}}$$

$$= \frac{(1-\rho)(2\rho-1)}{\rho - L^{-1}}$$

$$\simeq \frac{(1-\rho)(2\rho-1)}{\rho}$$

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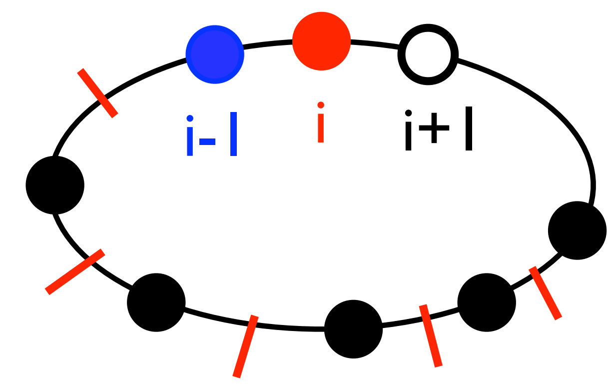

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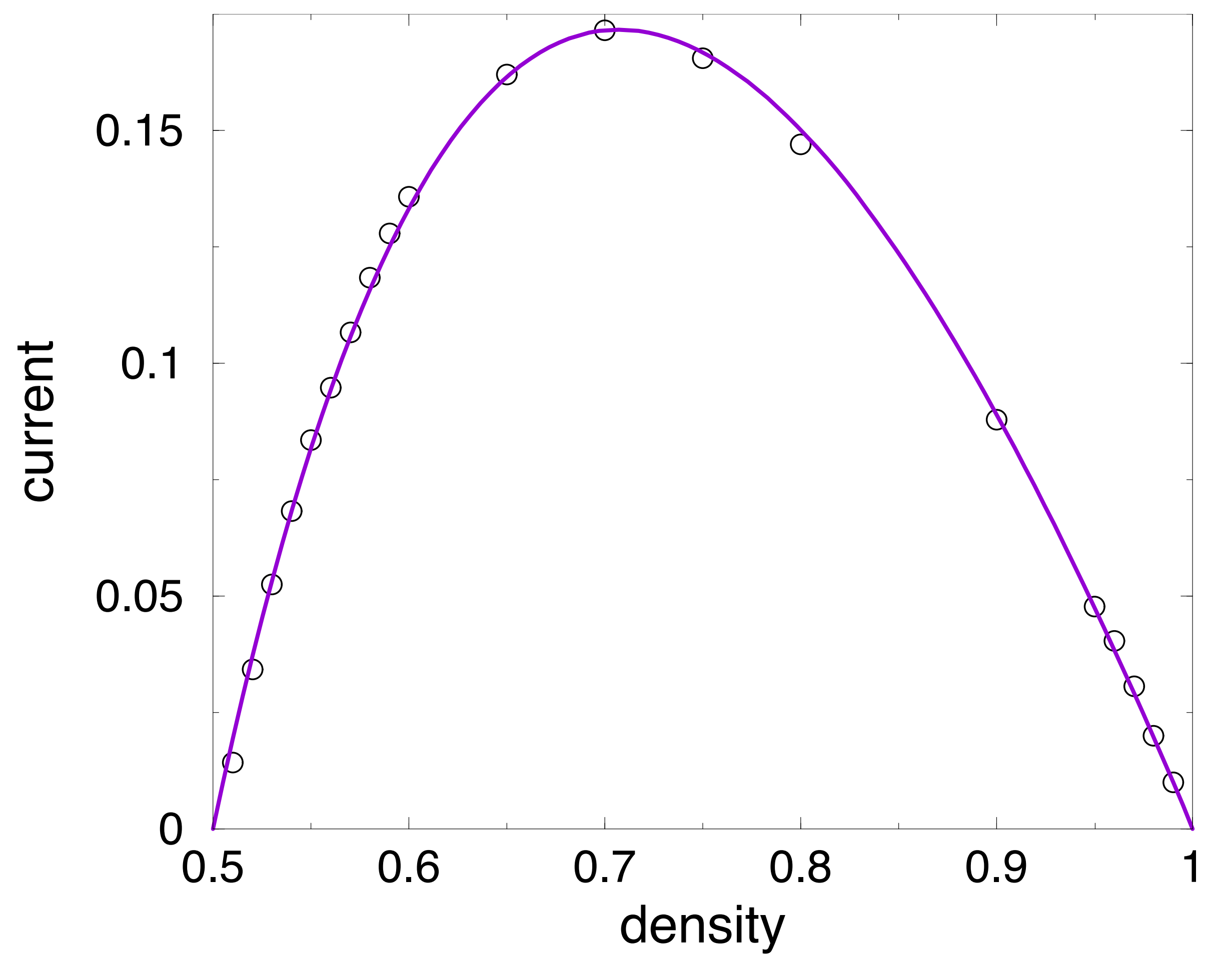

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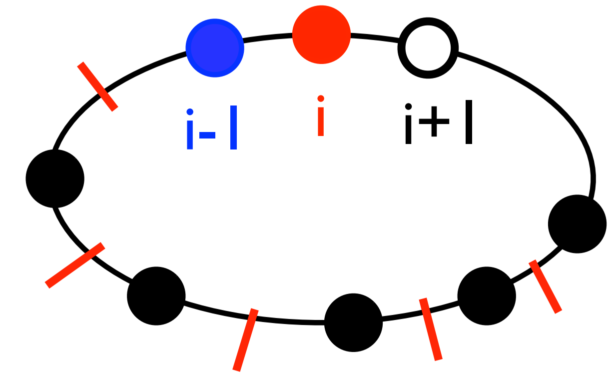

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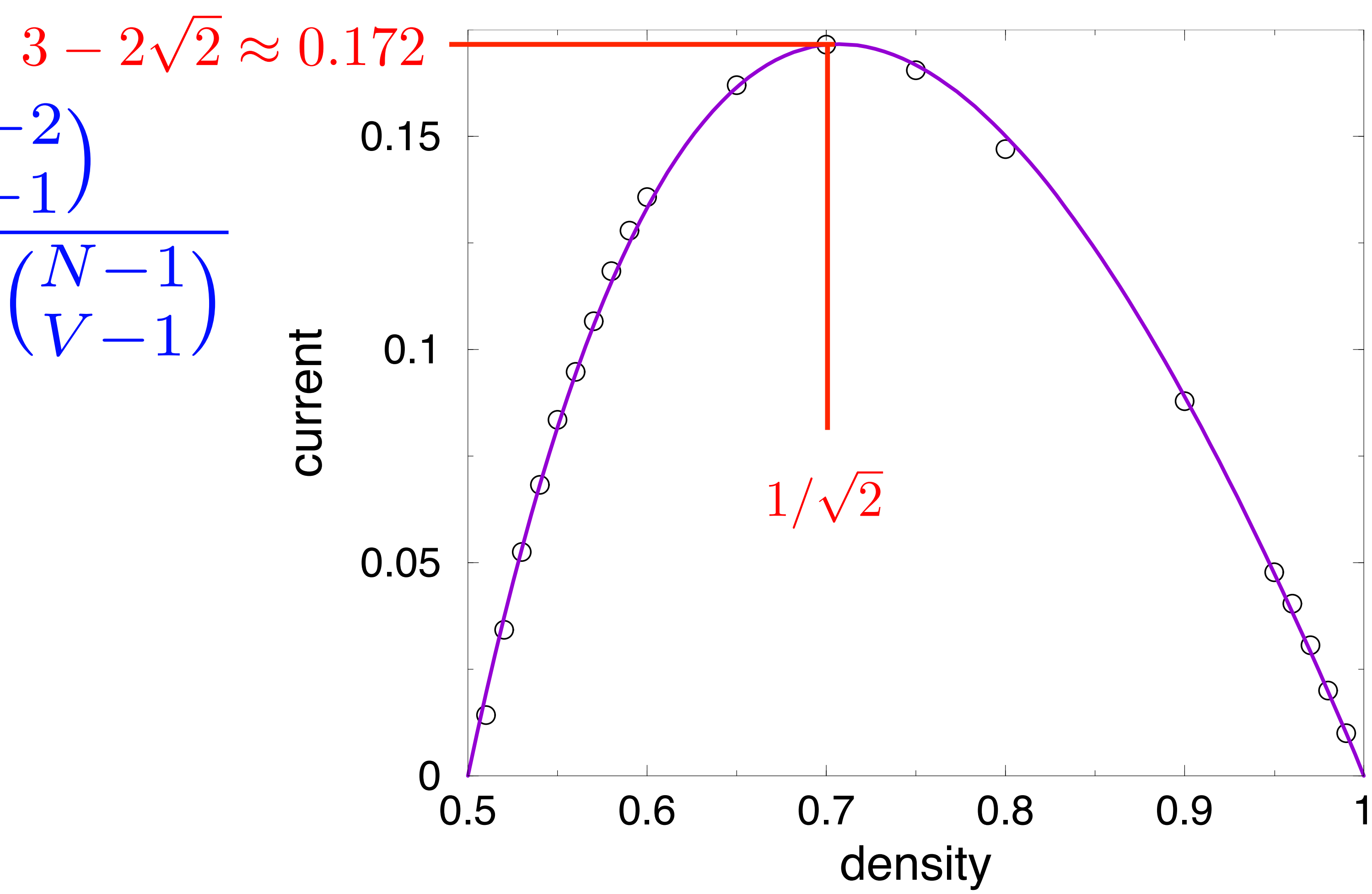

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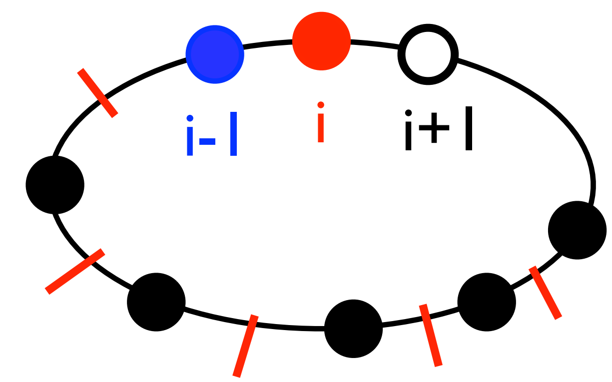

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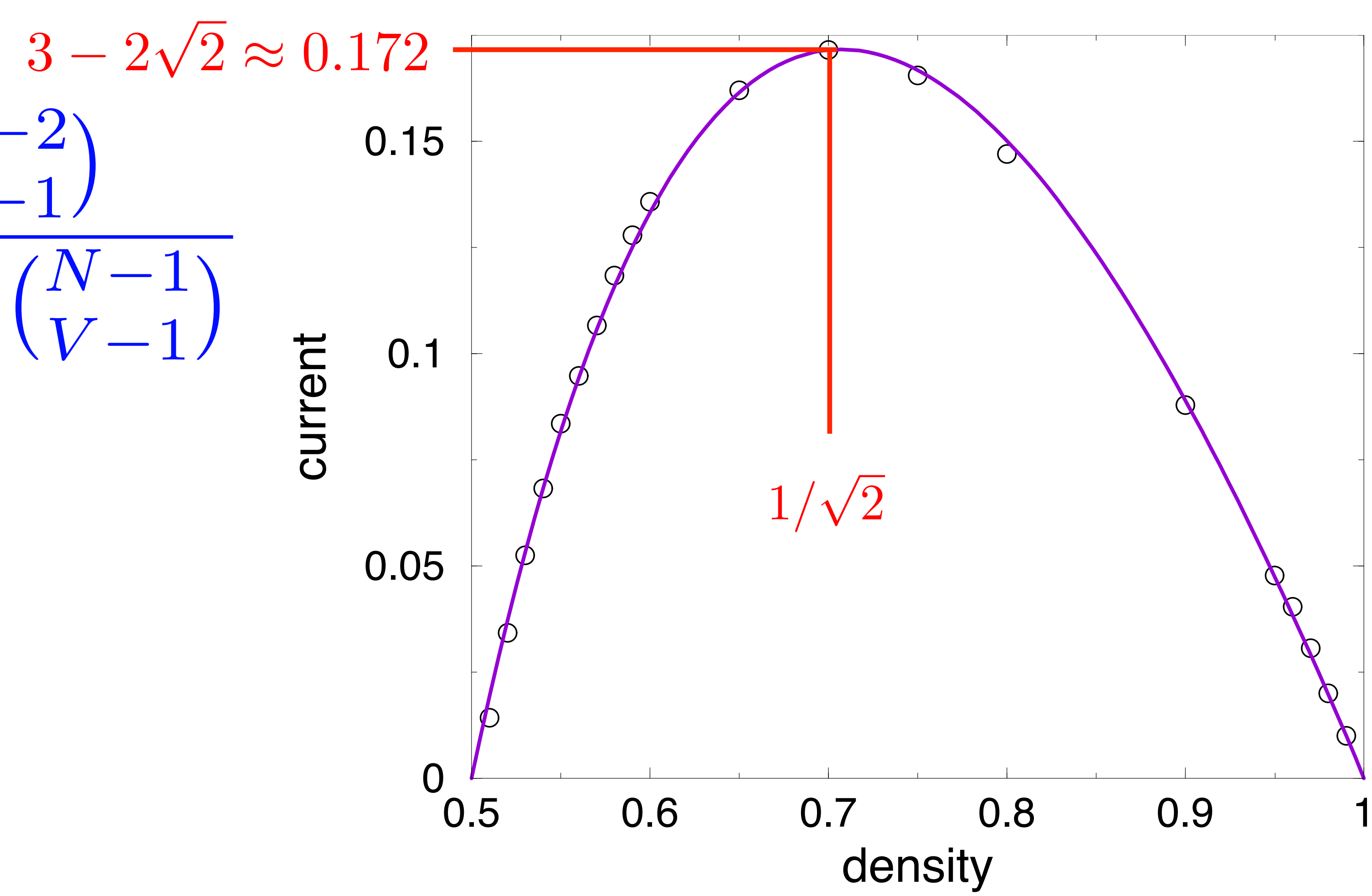
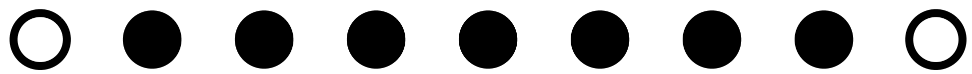


much more: Gaussian current distribution

L. B. Shaw, R. K. P. Zia, and K. H. Lee, Phys. Rev. E **68**, 021910 (2003)

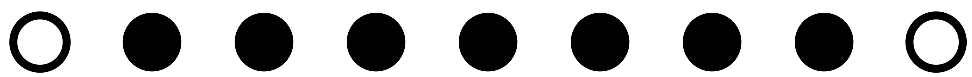
Steady State Island-Size Distribution



- n consecutive occupied sites & 2 empty
- N-n other particles
- N-n-1 allowed locations for V-2 vacancies

$$I_n = \frac{\binom{N-n-1}{V-2}}{\binom{N}{V} + \binom{N-1}{V-1}} \rightarrow \frac{(1-\rho)^2}{\rho}\left(\frac{2\rho-1}{\rho}\right)^{n-1}$$

Steady State Island-Size Distribution



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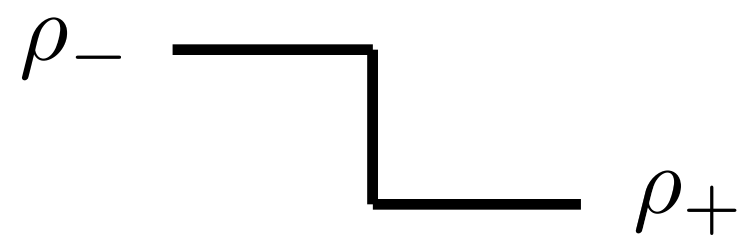
$$\ll \rho^n$$

Evolution of Density Downstep

initial state: $$\rho = \begin{cases} \rho_- & x < 0 \\ \rho_+ & x > 0 \end{cases}$$



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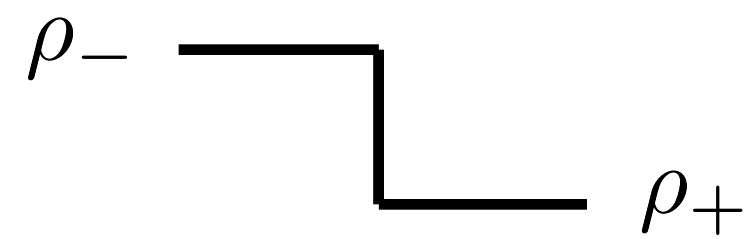

evolution equation: $$\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x} = 0$$

scaling ansatz: $$\rho(x,t) = f(z), \ \ \text{with} \ \ z \equiv x/t$$

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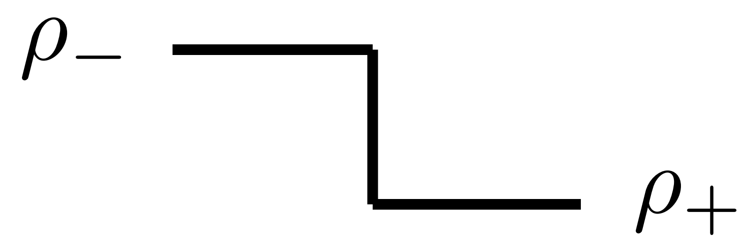

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$$f = \begin{cases} \rho_- & z < z_- \\ (2+z)^{-1/2} & z_- < z < z_+ \\ \rho_+ & z > z_+ \end{cases}$$

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right boundary of rarefaction wave by mass conservation:

$$\underbrace{-\rho_- z_- + \rho_+ z_+}_{\text{initial mass in } [z_-, z_+]} = \underbrace{\int_{z_-}^{z_+} \frac{dz}{\sqrt{2+z}}}_{\text{mass at time t in } [z_-, z_+]} + \underbrace{J_- - J_+}_{\text{net flux into } [z_-, z_+]}$$

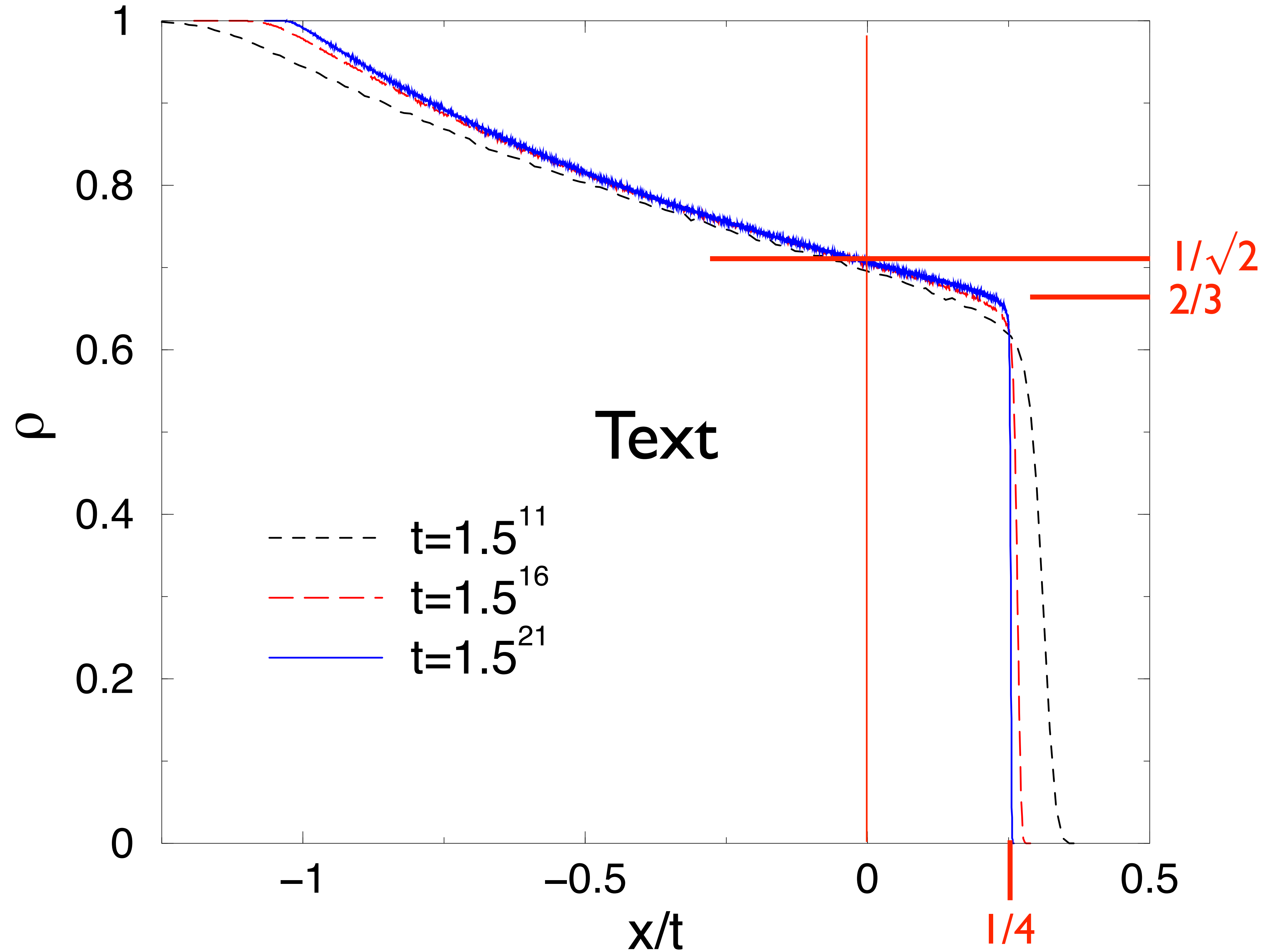
initial mass in [z_-,z_+]

mass at time t in [z_-,z_+]

net flux into [z_-,z_+]

Scaled Density Profile (for (1,0) IC)

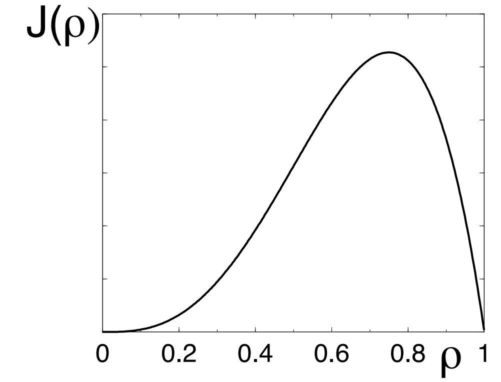
rarefaction wave with jump discontinuity



Ubiquitous Rarefaction Discontinuity

generic current-density relation:

- $J(0)=J(1)=0$,
- single maximum in [0,1]



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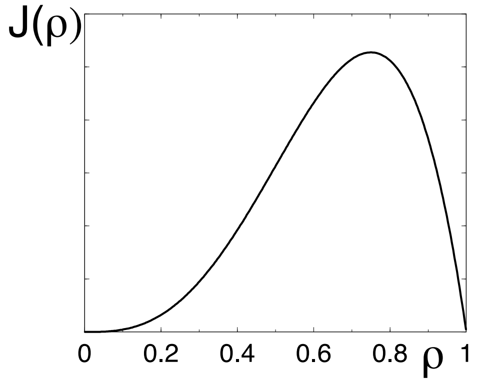

continuity eqn + scaling → $f'(z)\left(\frac{\partial J}{\partial \rho} - z\right) = 0$

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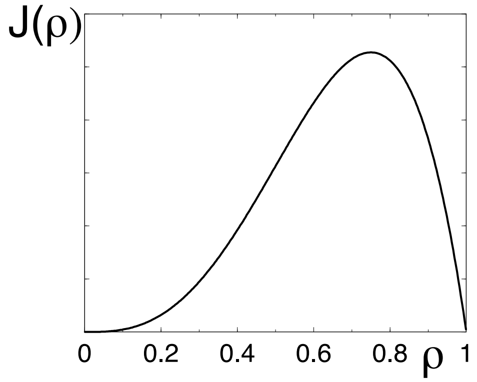

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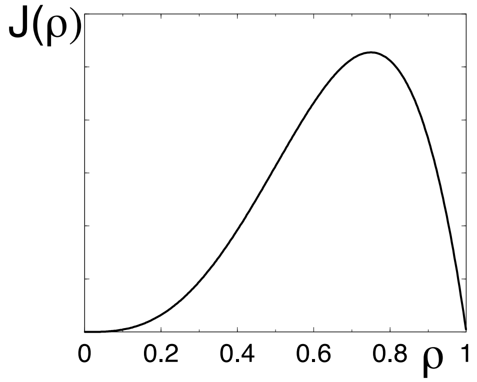

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derivative of $\frac{\partial J}{\partial \rho} = z \to \; J_{\rho\rho}\,\rho_z = 1$

If $J_{\rho\rho} < 0 \;\to\; \rho$ is a decreasing function of z

If $J_{\rho\rho} > 0 \;\to\; \rho$ must have jump discontinuity

Summary & Outlook

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Facilitated exclusion:

steady state: current and island sizes

generalizable to n-tuple facilitation

rarefaction wave: robust jump discontinuity

non-equilibrium phase transition for $\rho \to 1/2$

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Future:

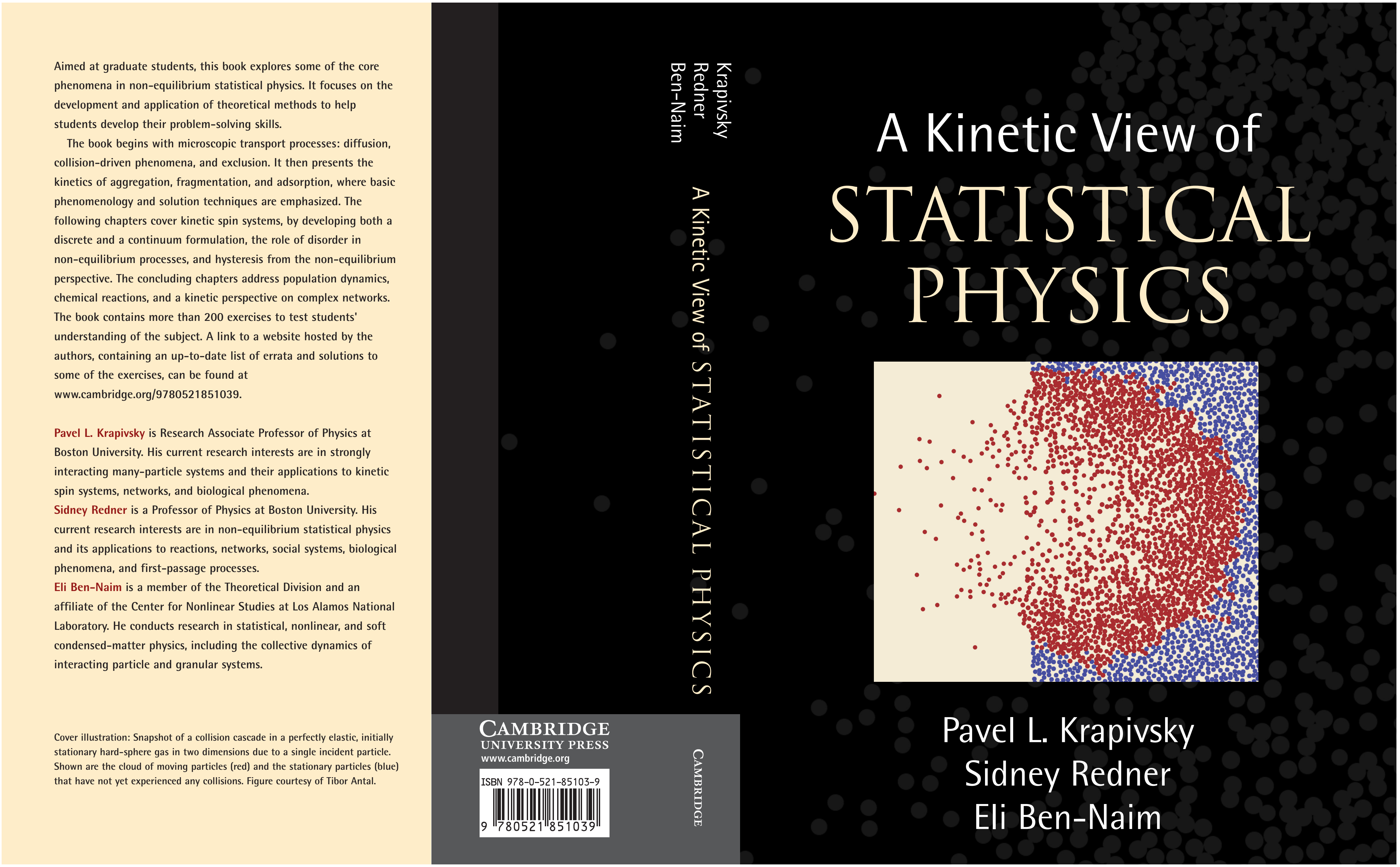
more general mechanisms (distance facilitation)?

is exclusion even necessary?

diffusive corrections to hydrodynamic solutions?

higher dimensions?

Crass Commercialism



to appear this month

1. Aperitifs
2. Diffusion
3. Collisions
4. Exclusion
5. Aggregation
6. Fragmentation
7. Adsorption
8. Spin Dynamics
9. Coarsening
10. Disorder
11. Hysteresis
12. Population Dynamics
13. Diffusive Reactions
14. Complex Networks

\+ *>200 problems & soln manual*