

# Facilitated Asymmetric Exclusion

**Complex Driven Systems - From Statistical Physics to the Life Sciences**

*celebrating Royce's achievements in 40 years of research*

# Facilitated Asymmetric Exclusion

Complex Driven Systems - From Statistical Physics to the Life Sciences  
*celebrating Royce's achievements in 40 years of research*

Sid Redner ([physics.bu.edu/~redner](http://physics.bu.edu/~redner))

Alan Gabel & Paul Krapivsky (BU), [arXiv:1007.3217](https://arxiv.org/abs/1007.3217)  
support from NSF DMR0906504

## Basic Facts about the ASEP

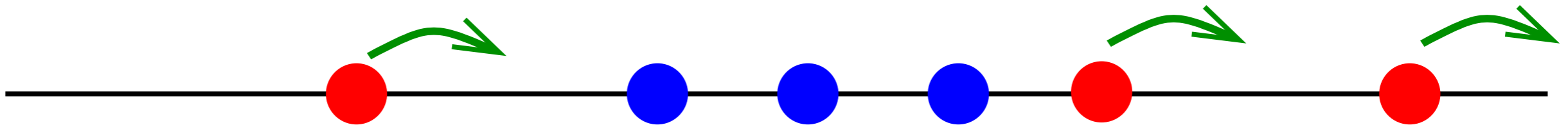
### Facilitated ASEP

steady state current  
island size distribution  
rarefaction wave discontinuity

## Summary & Outlook

# Asymmetric Exclusion Process (ASEP)

B. Schmittmann & R.K.P. Zia, *Phase Transitions and Critical Phenomena*, Vol. 17, eds. C. Domb and J. L. Lebowitz  
G. Schütz, *Phase Transitions and Critical Phenomena*, Vol. 19, eds. C. Domb and J. L. Lebowitz  
B. Derrida, *Phys. Repts.* **301**, 65 (1998);

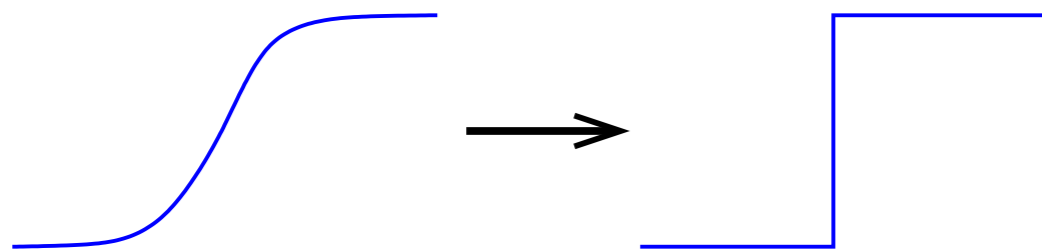


- particles confined to a 1d lattice
- particles move to the right only (asymmetric)
- only one particle per site (exclusion)
- hopping only if target is vacant (process)

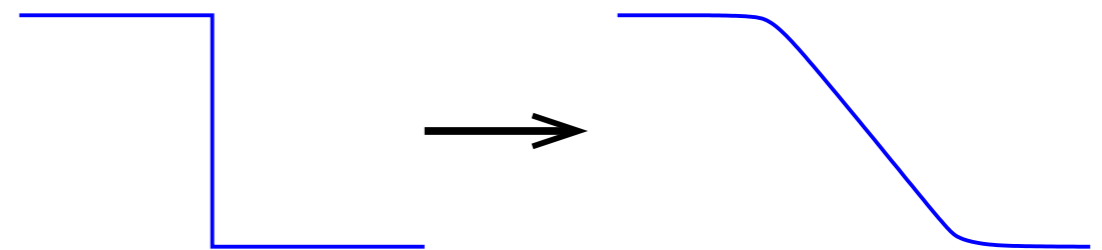
# Asymmetric Exclusion Process (ASEP)

- *simple*: only one parameter, the density  $\rho$
- *no correlations*: steady-state current  $j = \rho(1 - \rho)$
- inhomogeneity smoothing by Burgers eqn

$$\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho(1 - \rho)}{\partial x} = 0$$



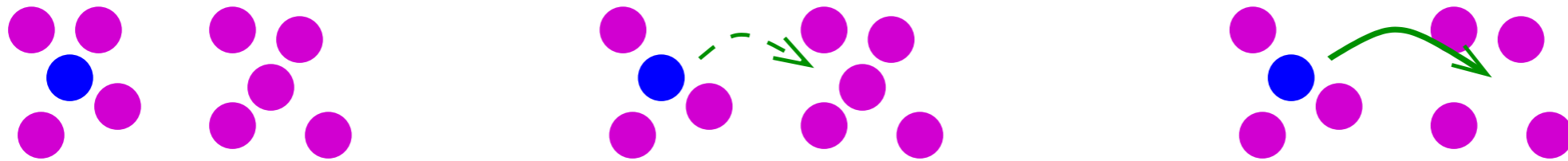
upslope  $\rightarrow$  shock wave



downstep  $\rightarrow$  rarefaction wave

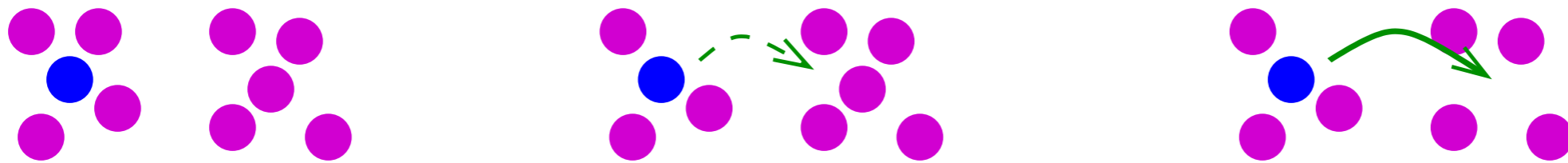
# Why and What is Facilitation?

particle motion in glasses: higher mobility in low-density regions



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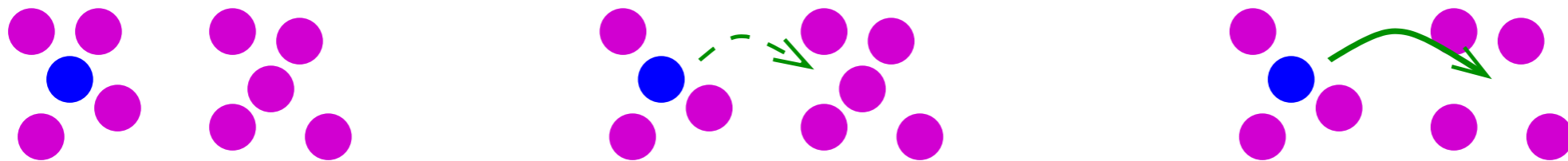
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facilitated exclusion: need stimulus to move

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particle motion in glasses: higher mobility in low-density regions



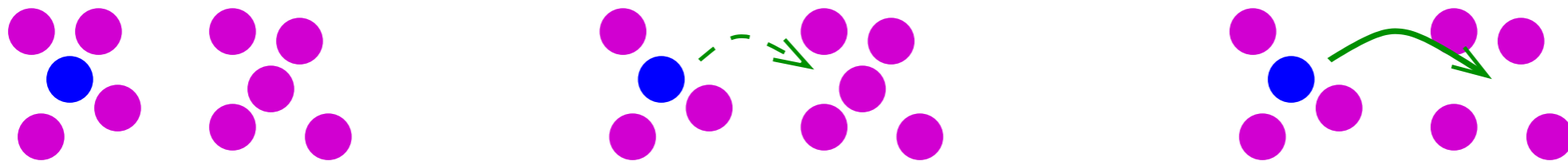
facilitated exclusion: need stimulus to move



*lazy: no stimuli*

# Why and What is Facilitation?

particle motion in glasses: higher mobility in low-density regions



facilitated exclusion: need stimulus to move



*lazy: no stimuli*



*mobile: kicked in the ass*



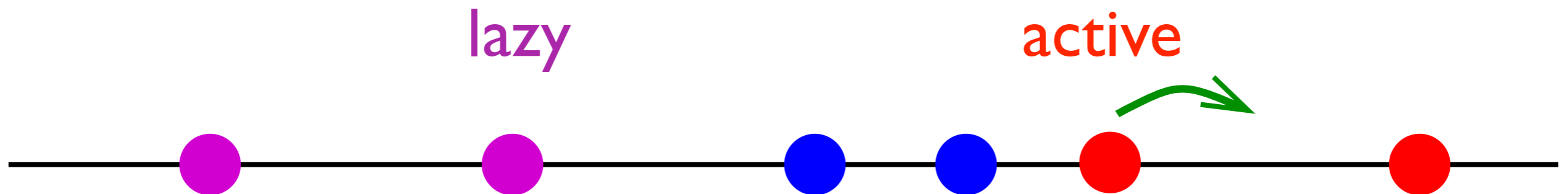
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L. B. Shaw, R. K. P. Zia, and K. H. Lee, Phys. Rev. E **68**, 021910 (2003)

U. Basu and P. K. Mohanty, Phys. Rev. E **79**, 041143 (2009)

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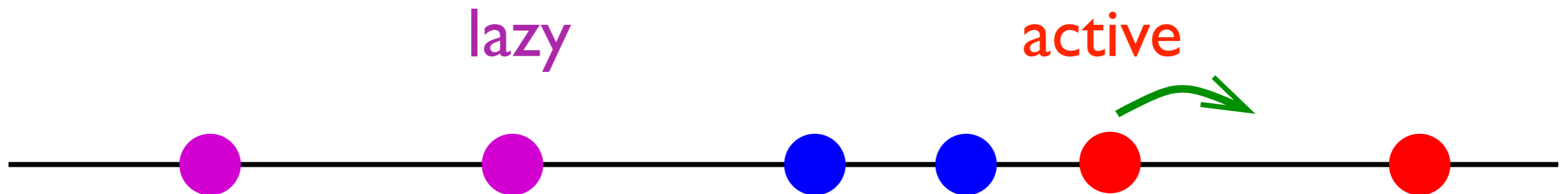
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only the triplet ● ● ○ is active

# Mapping to Dimer ASEP



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valid **only** for density  $> 1/2$



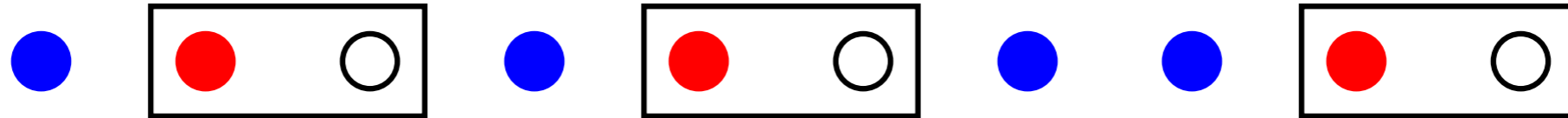
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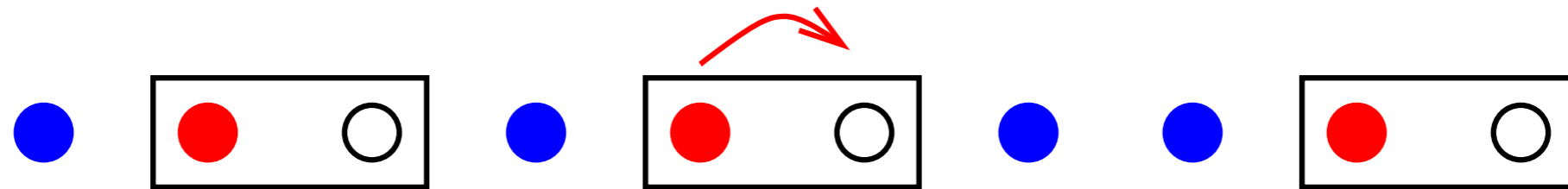
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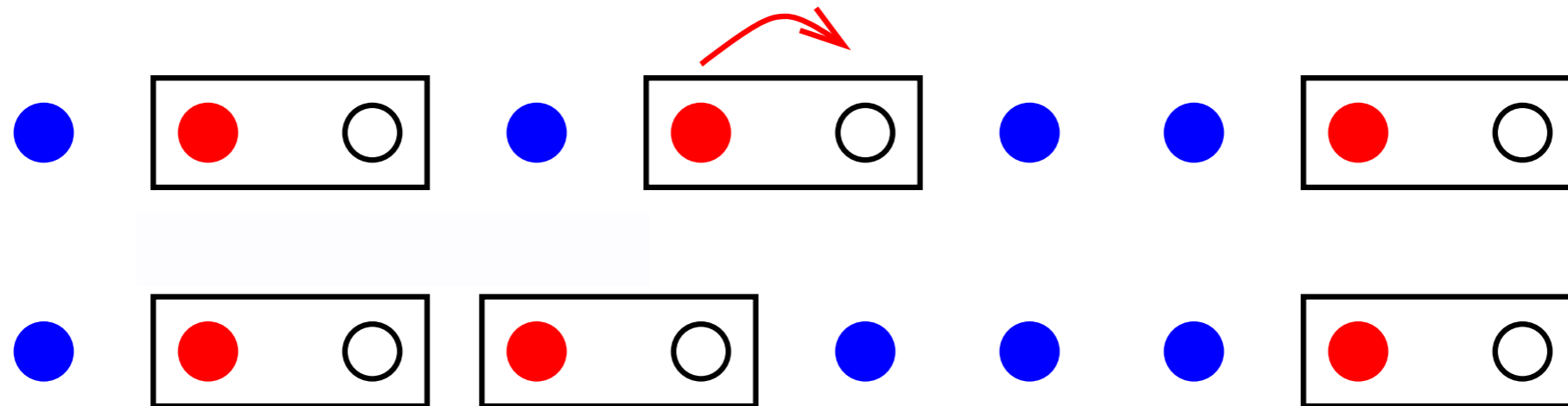
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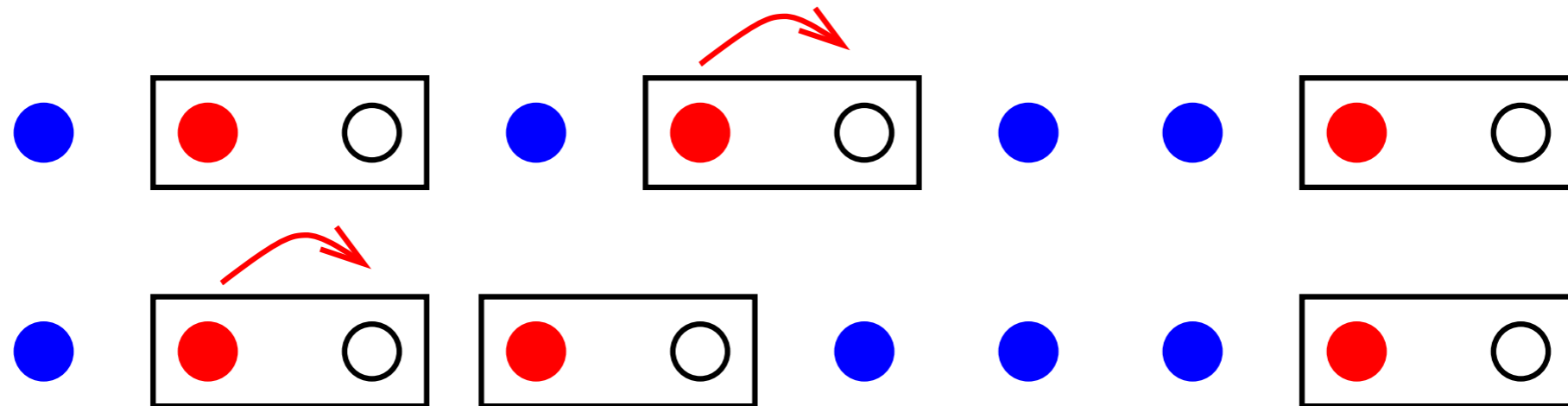
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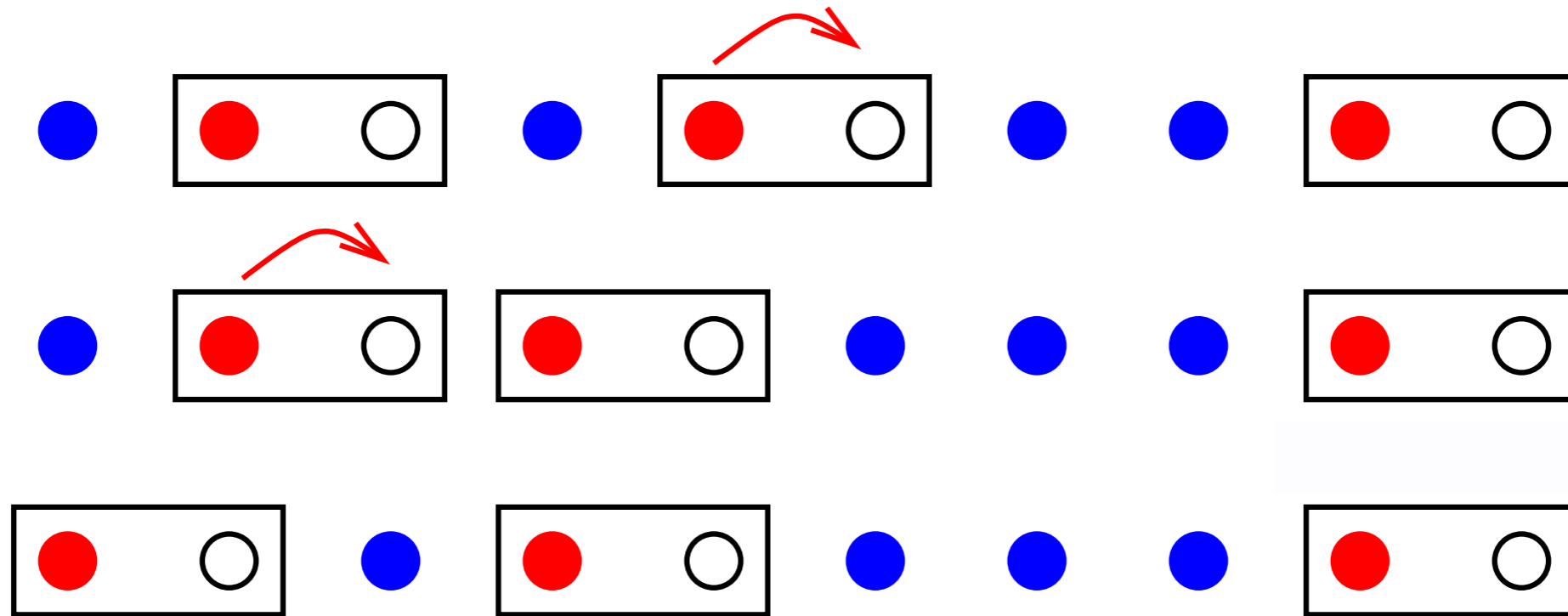
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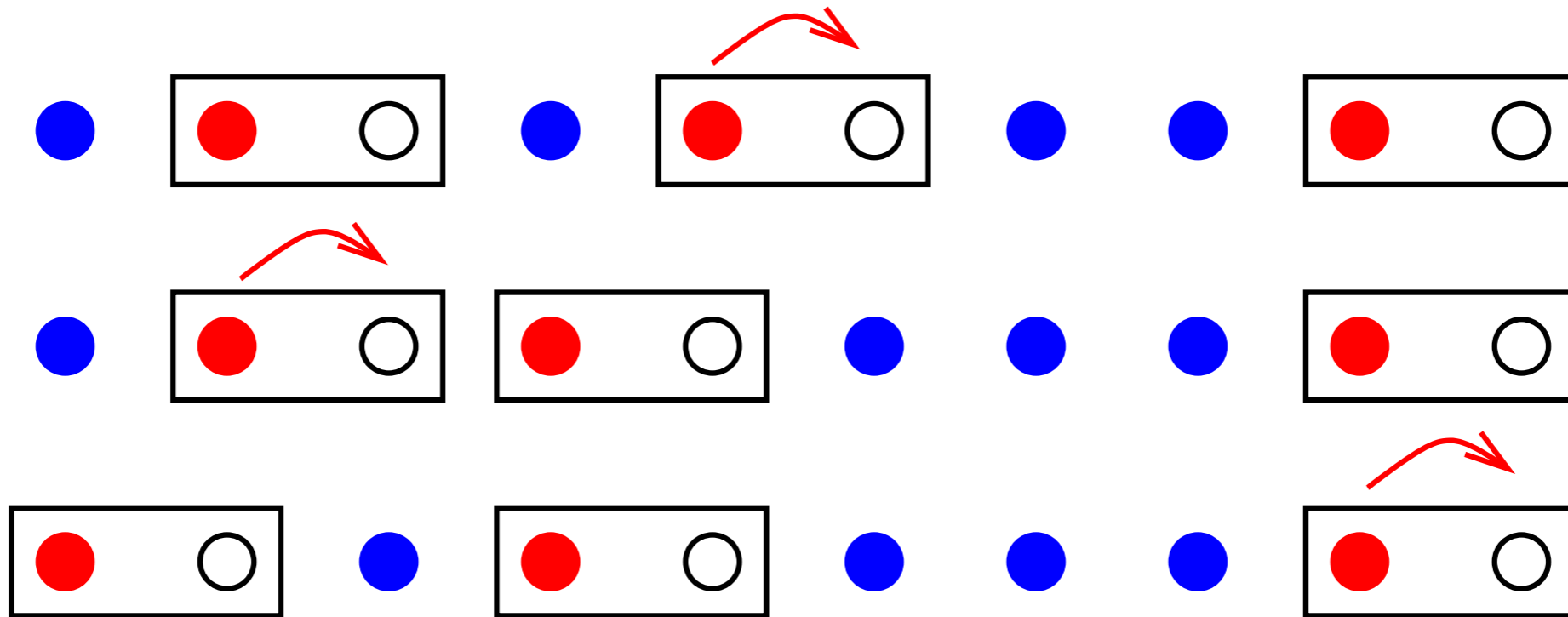
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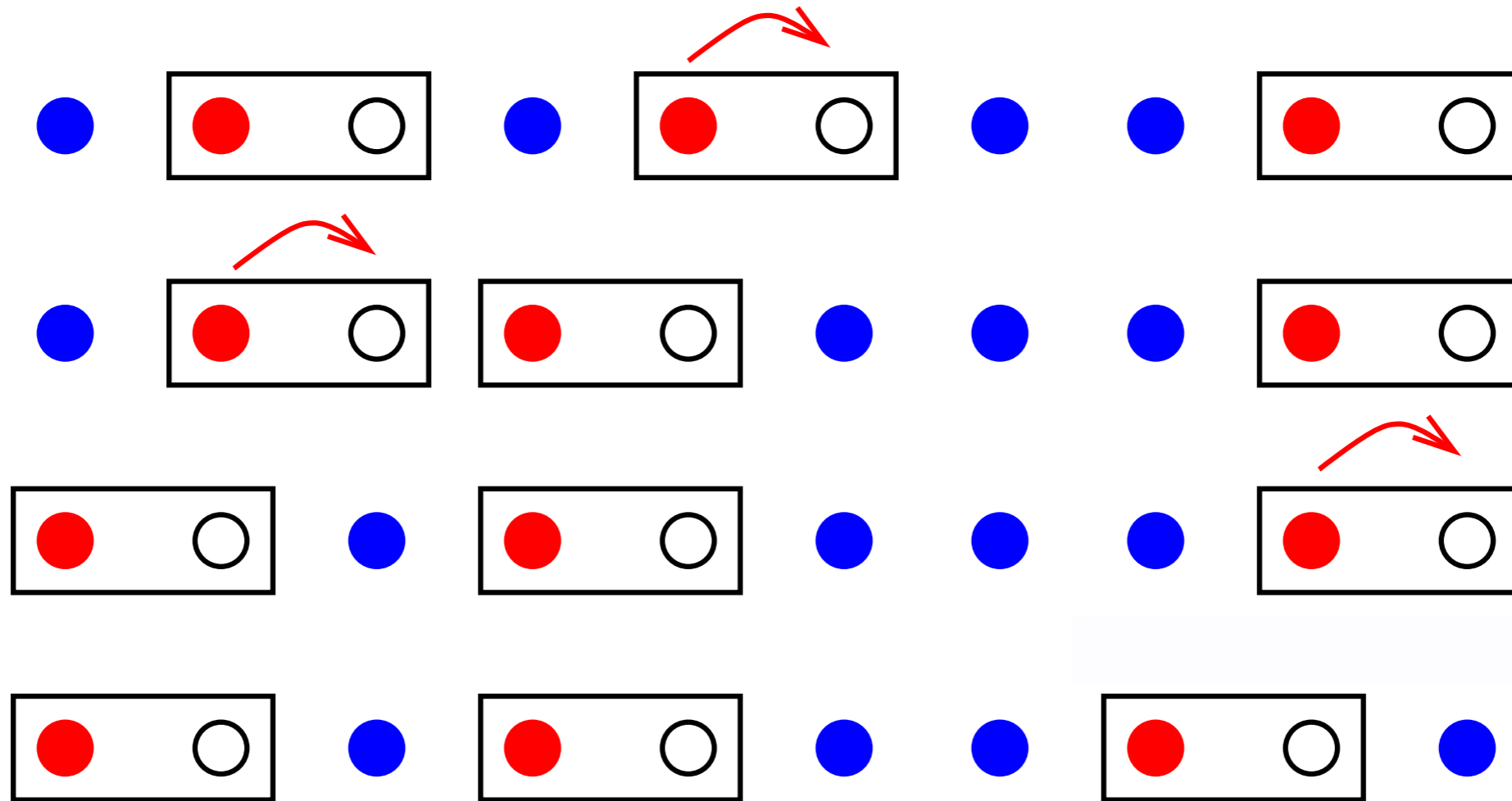
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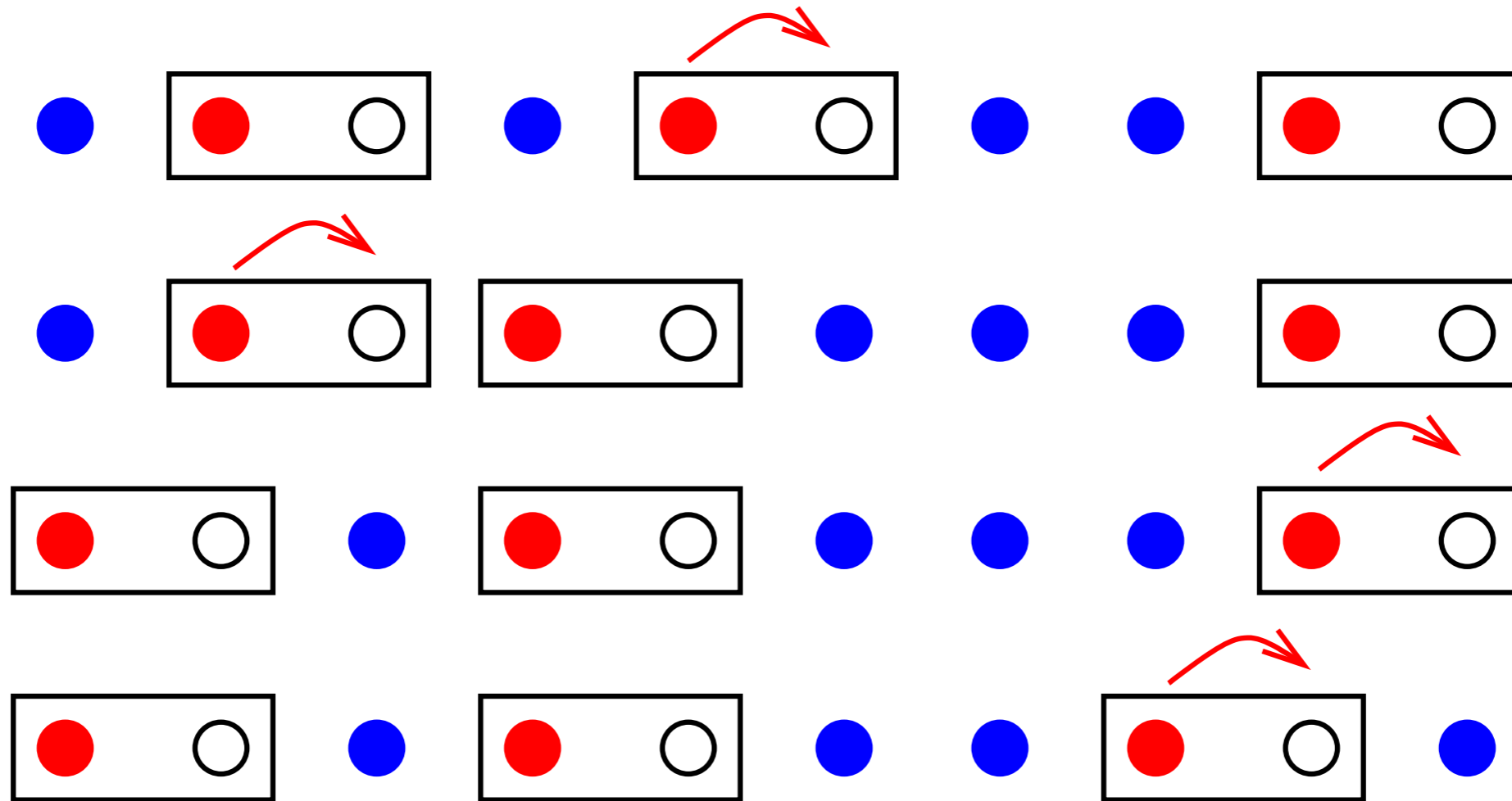
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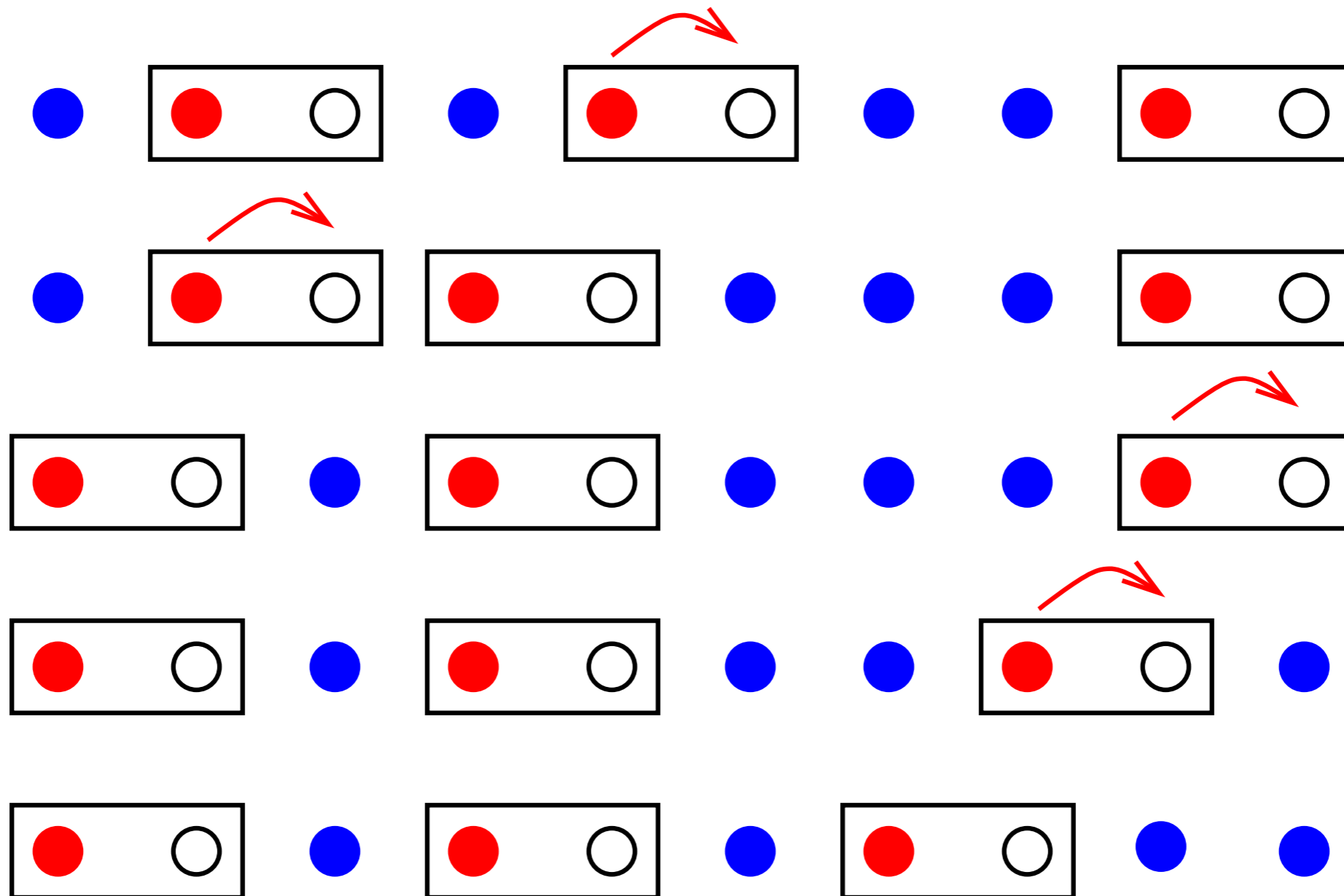
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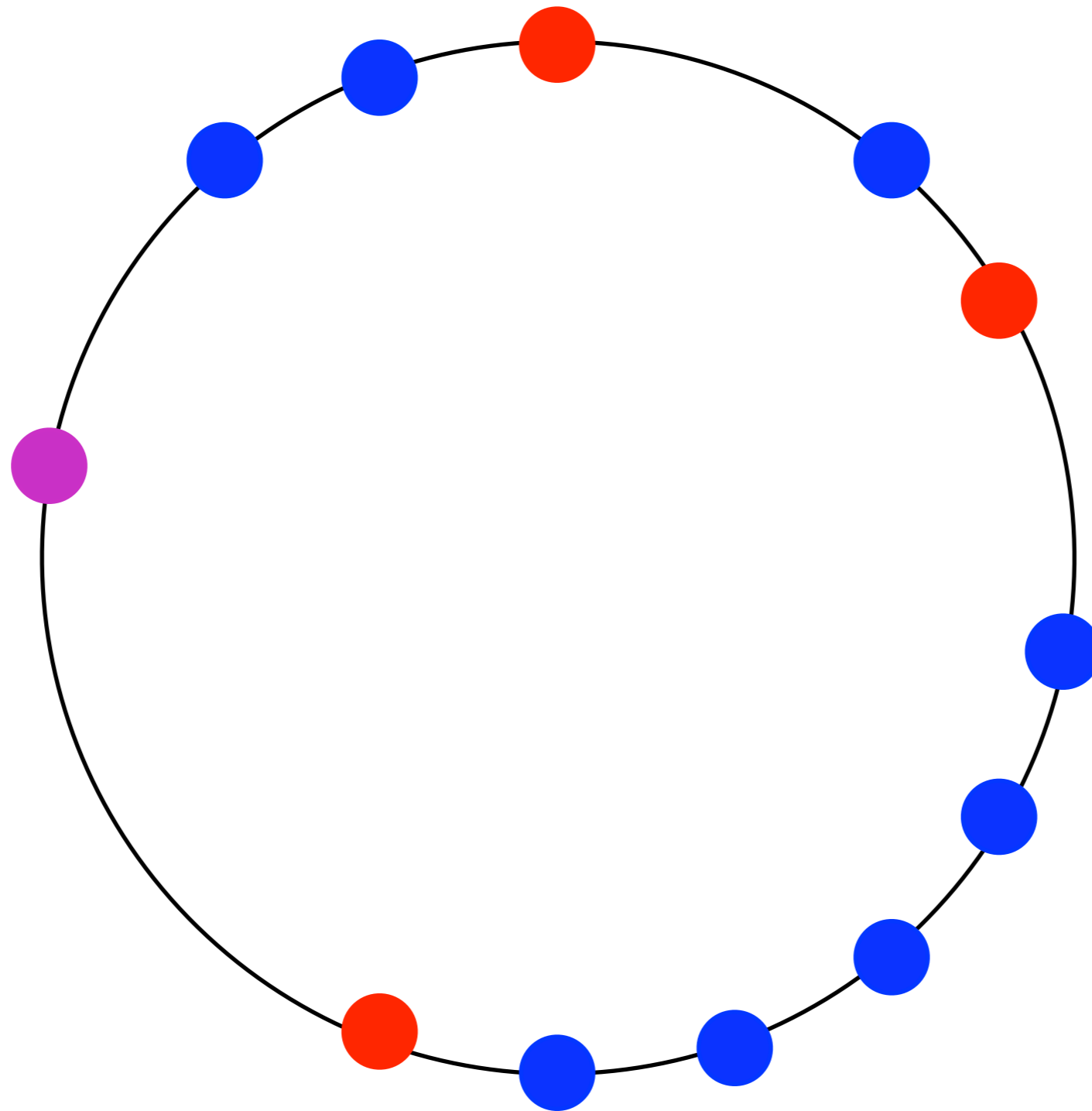


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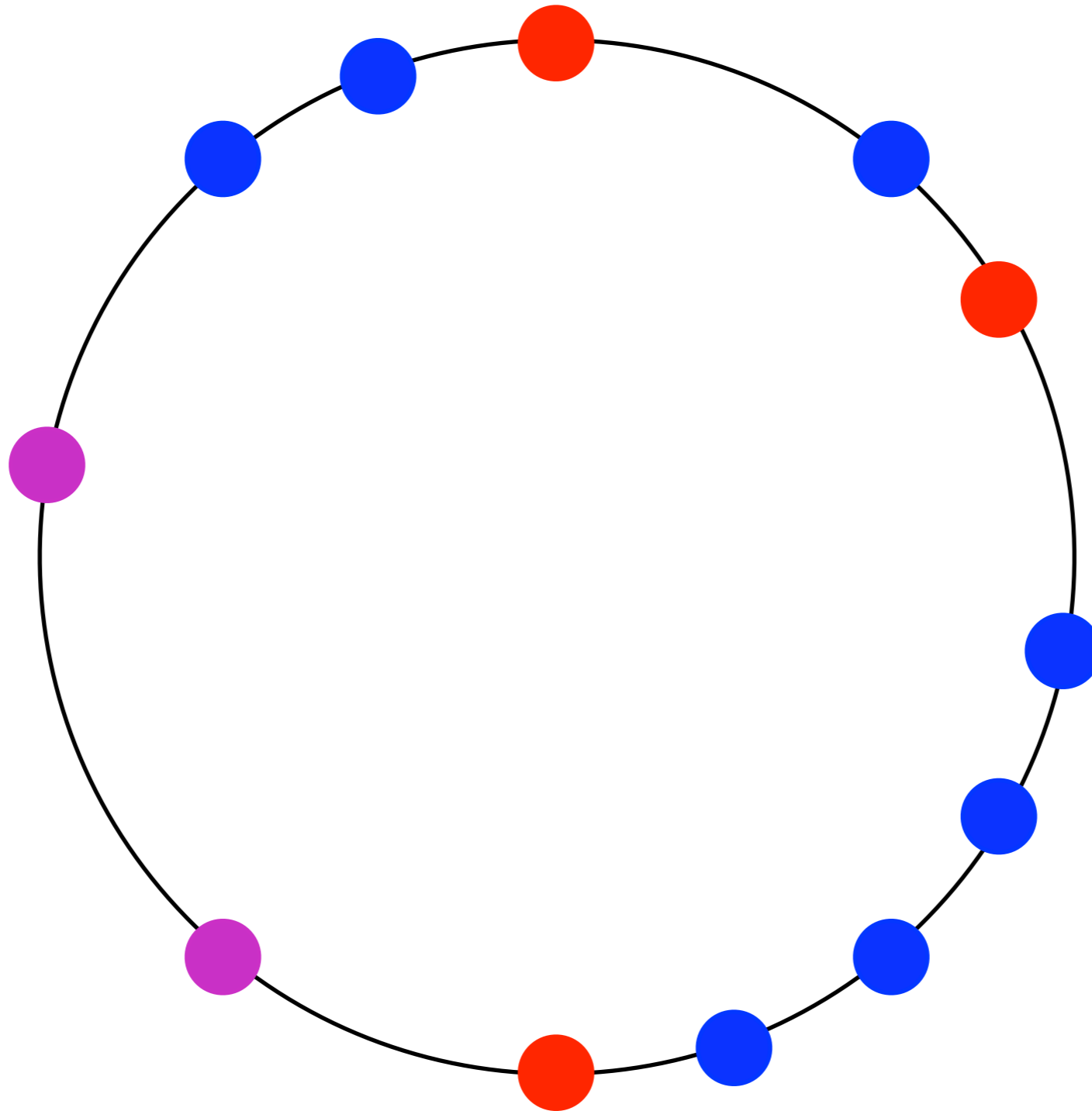
# Finite Ring (density $> 1/2$ )



4 islands

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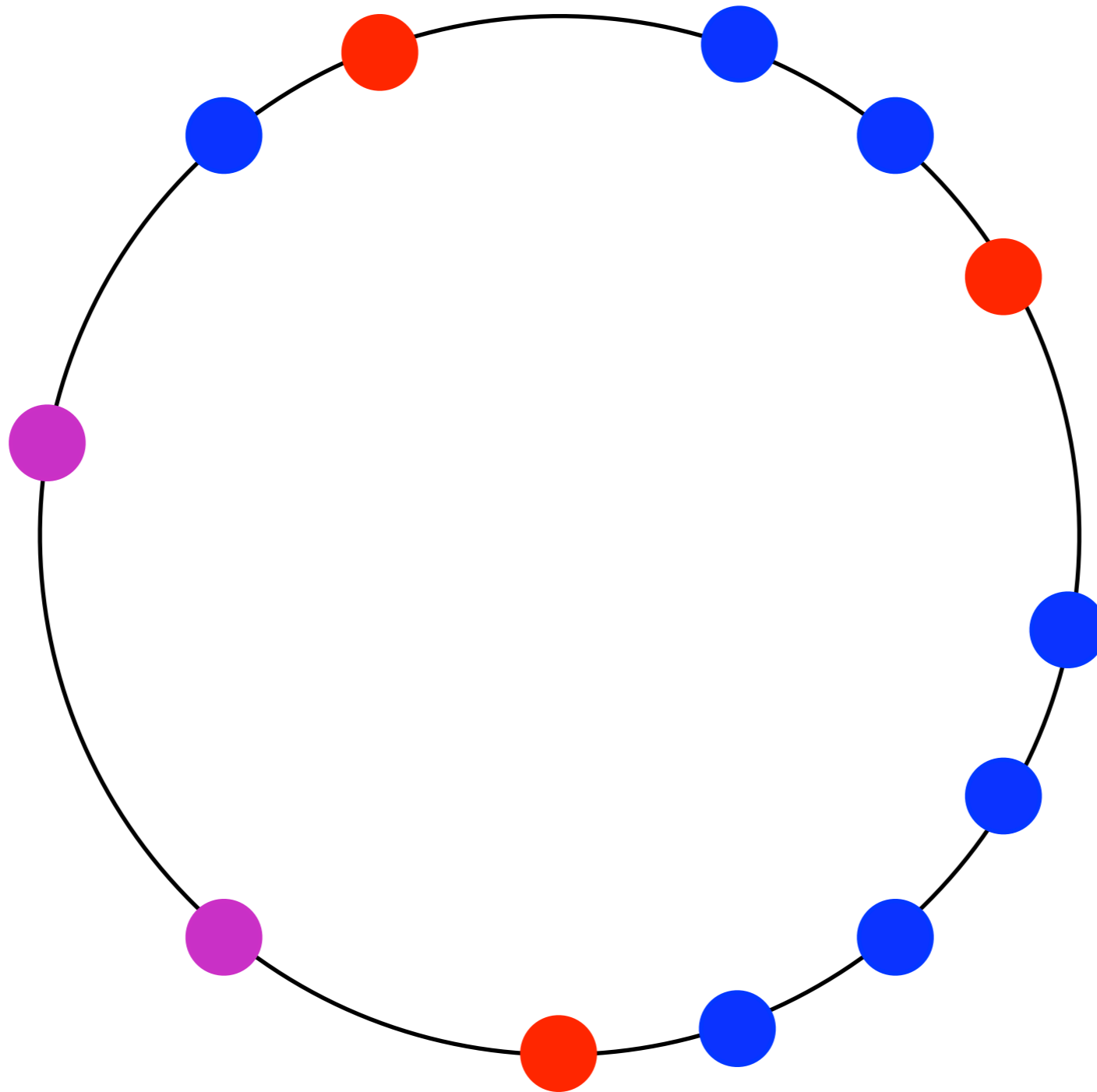
5 islands





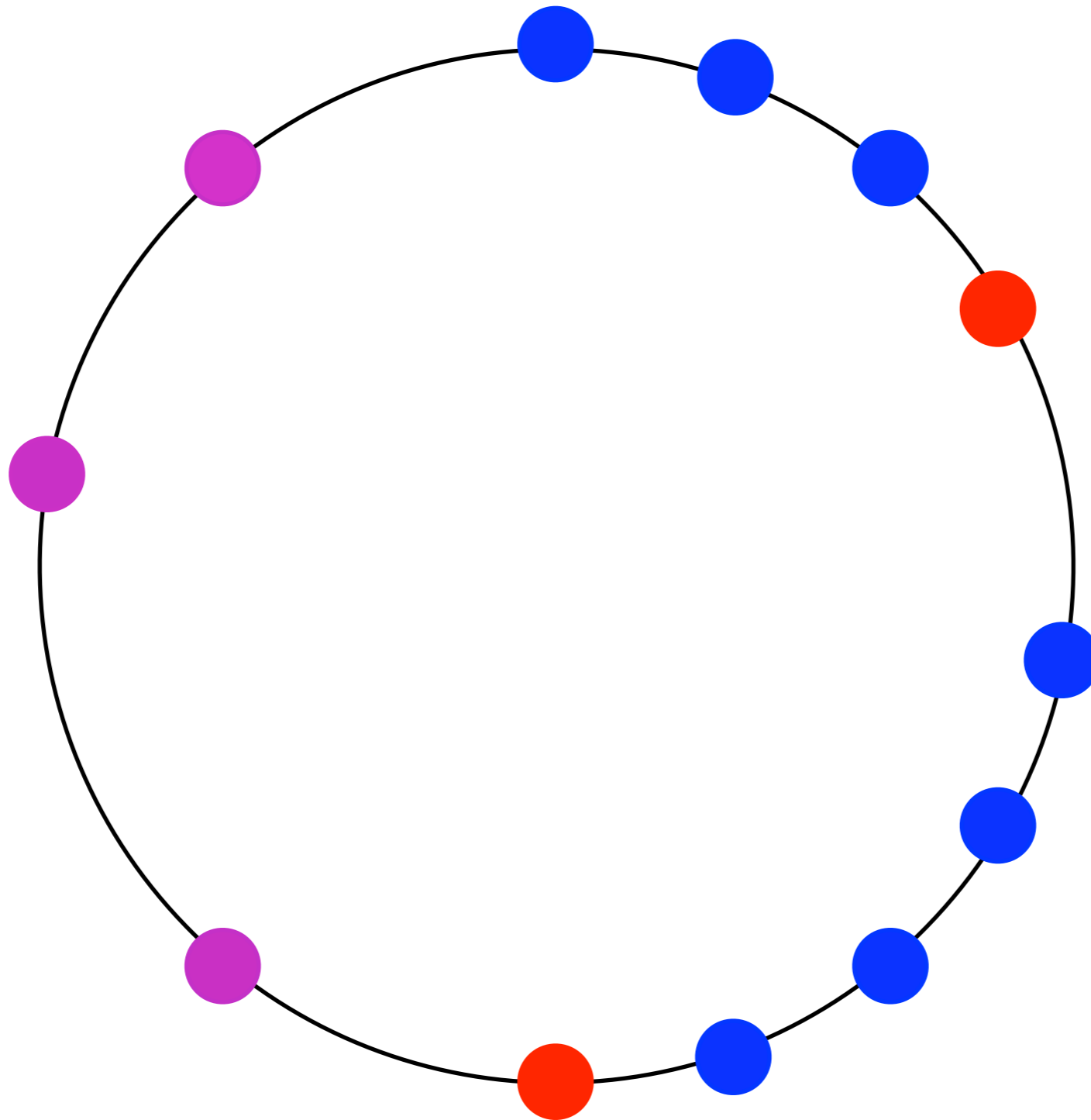
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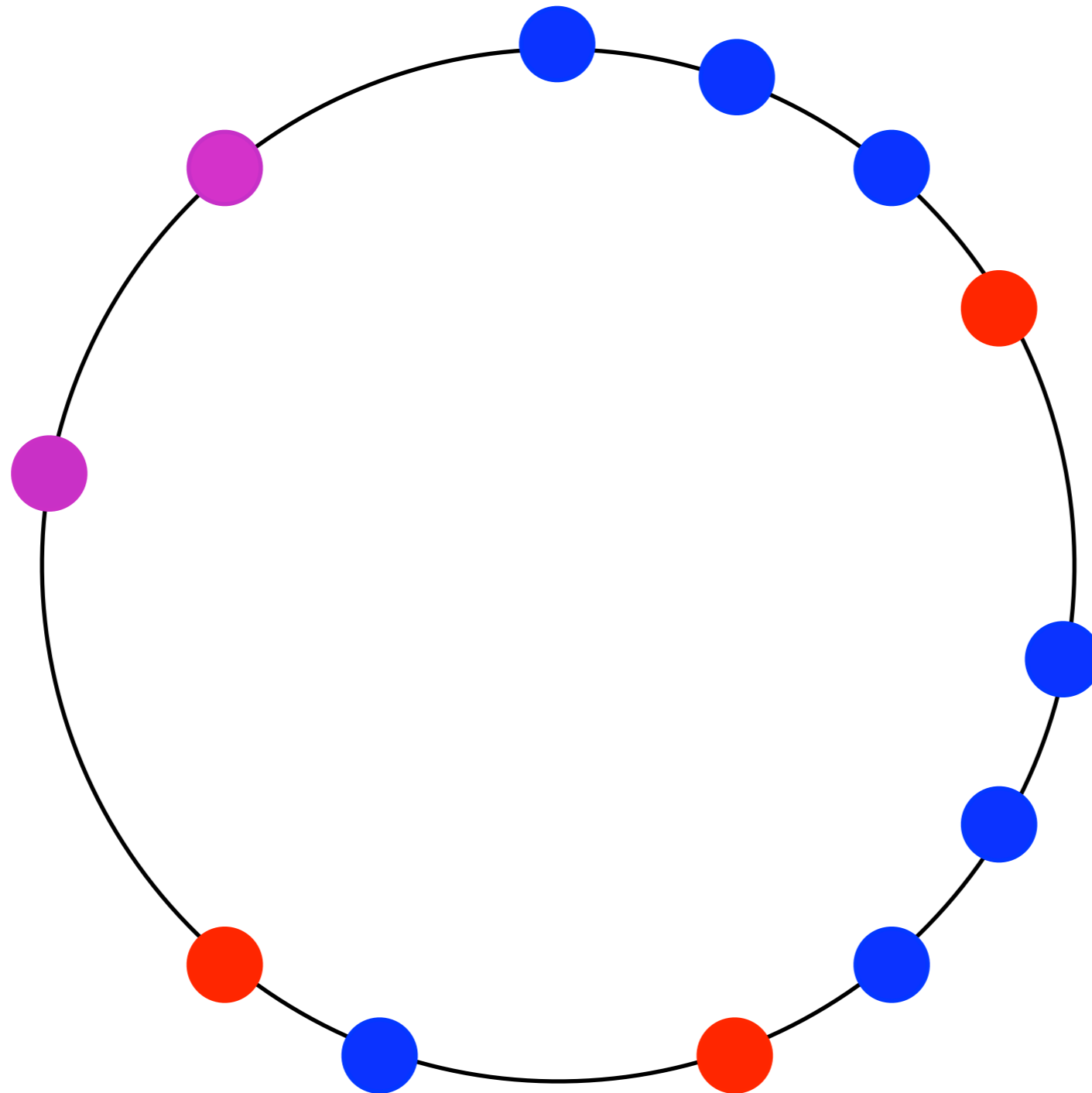
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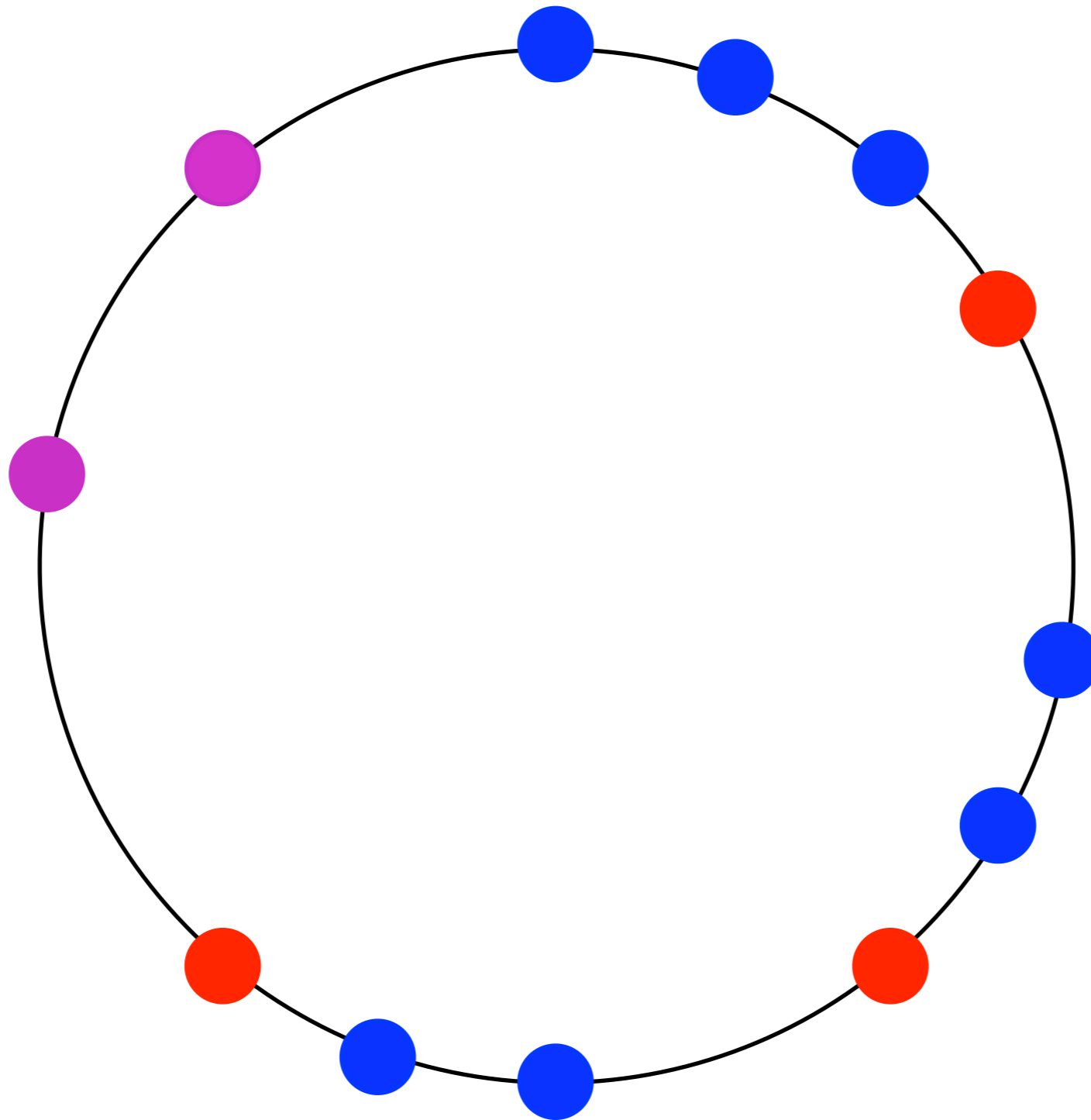
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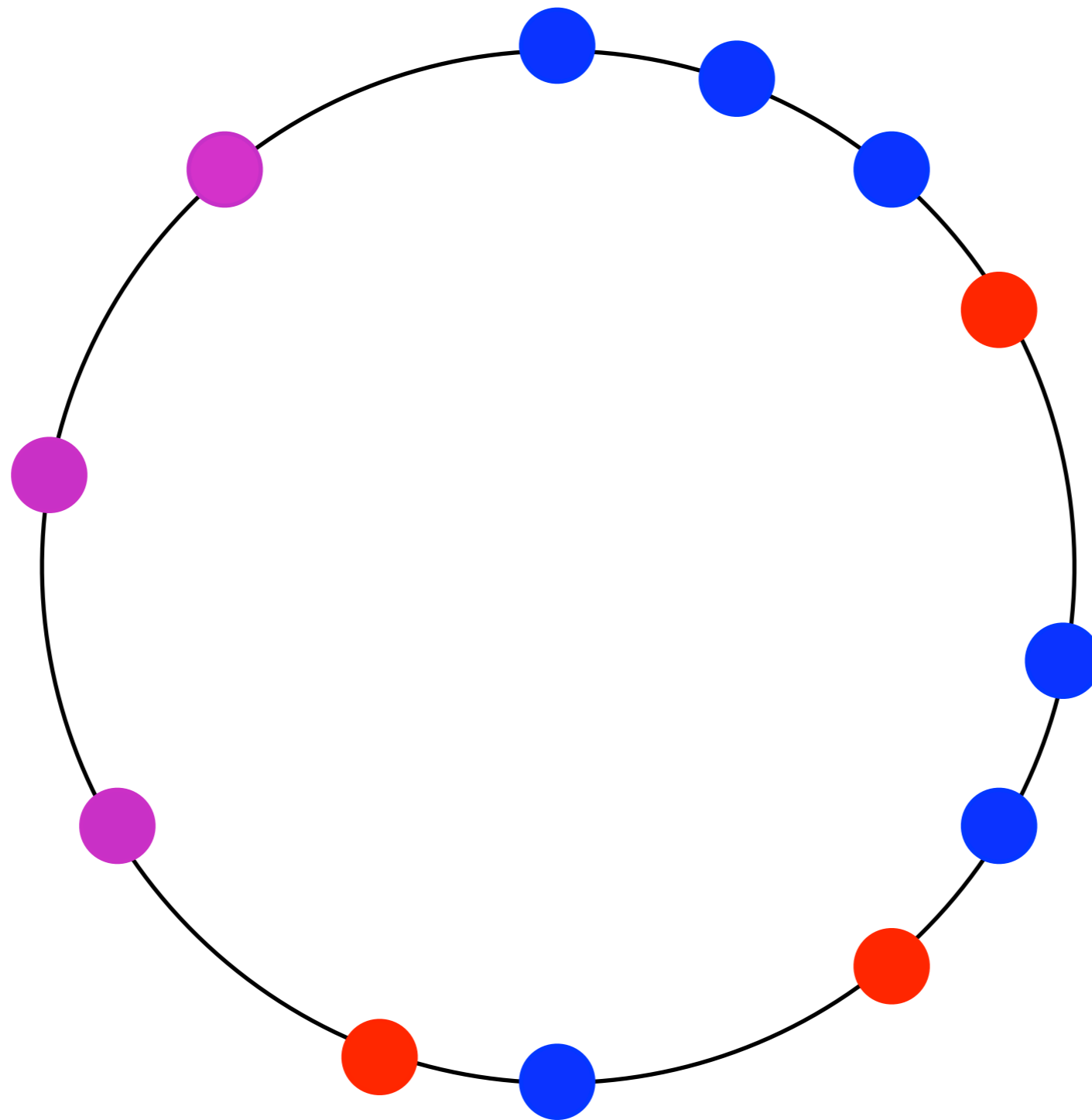


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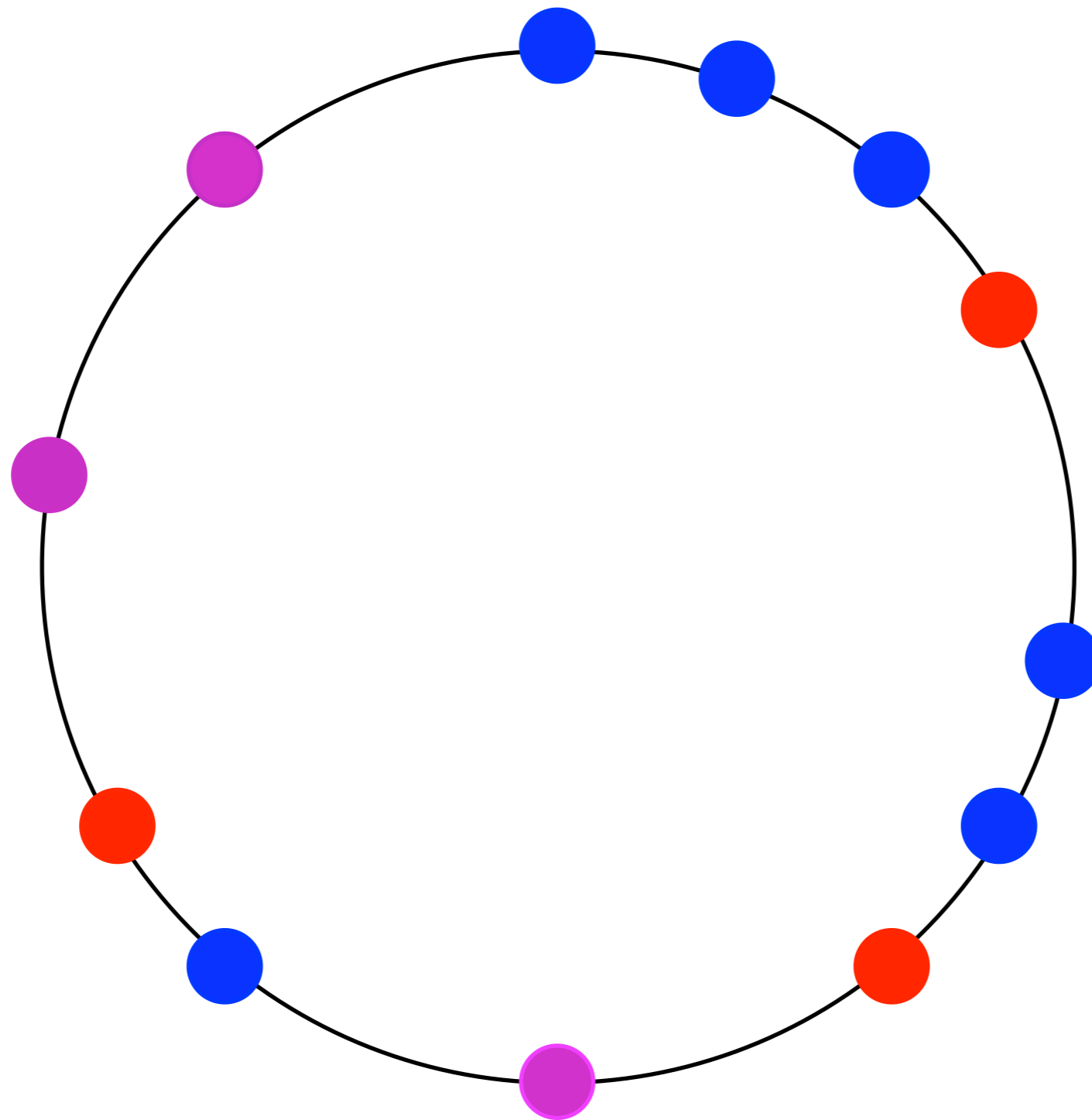


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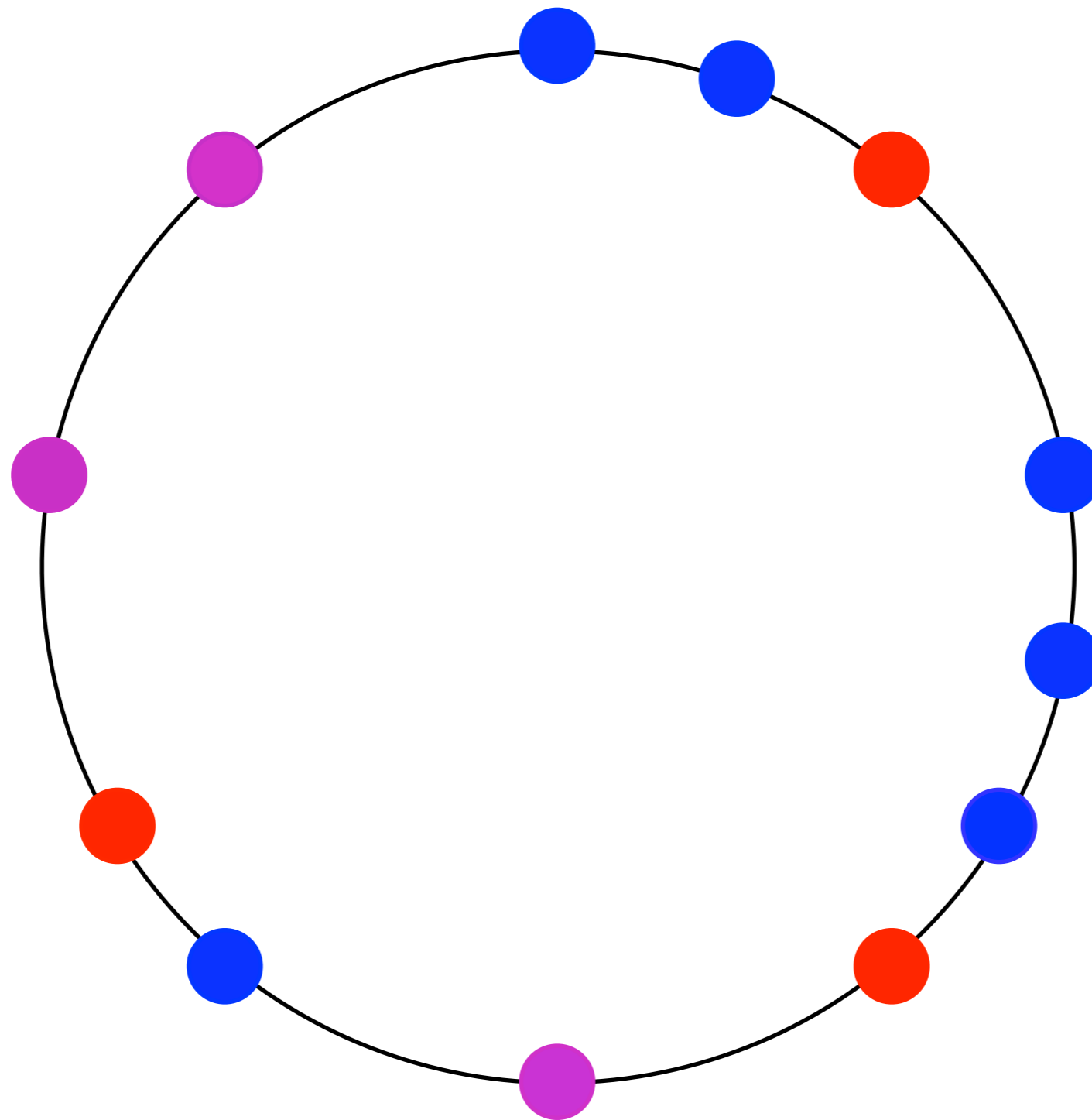
6 islands

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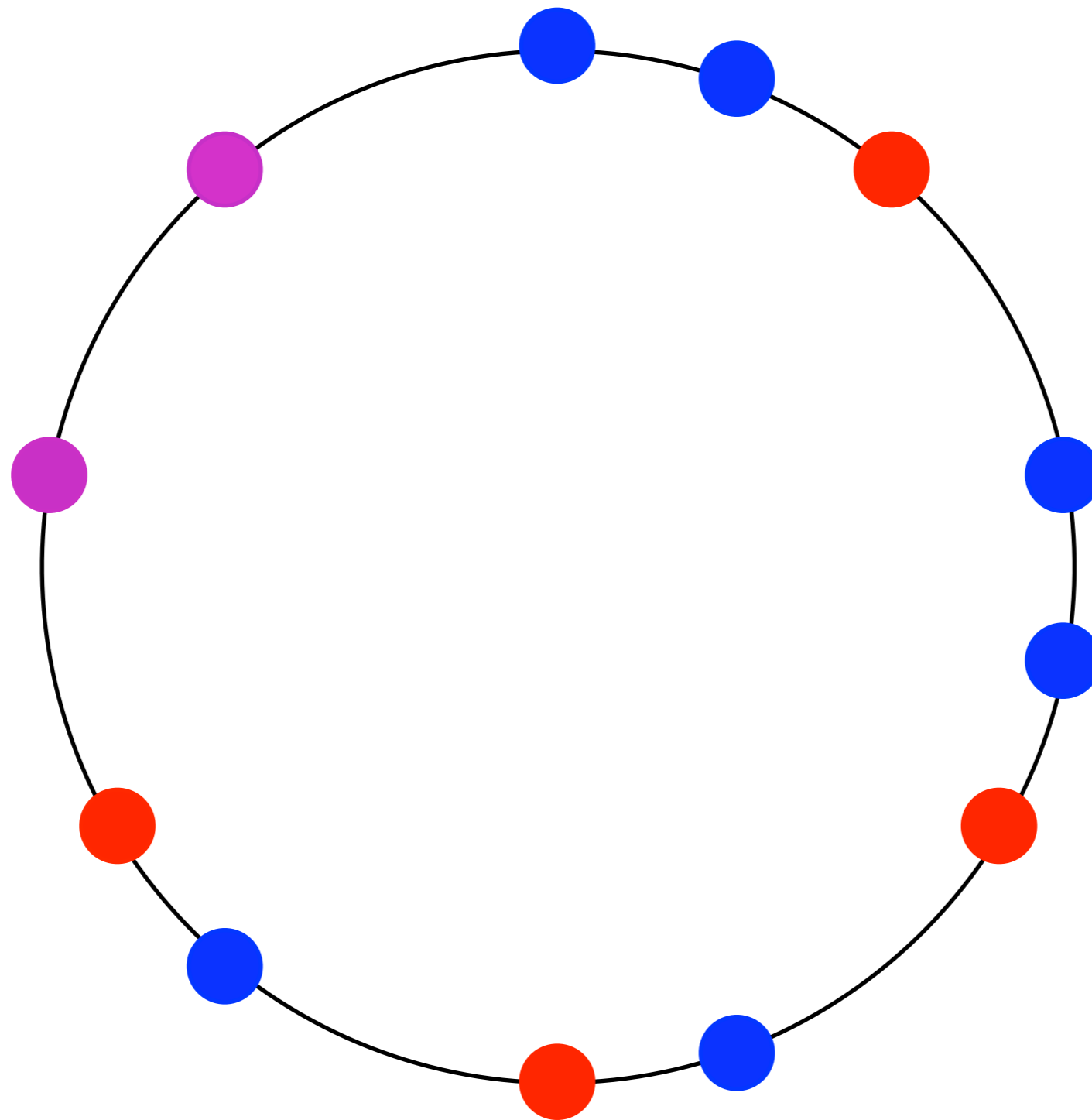
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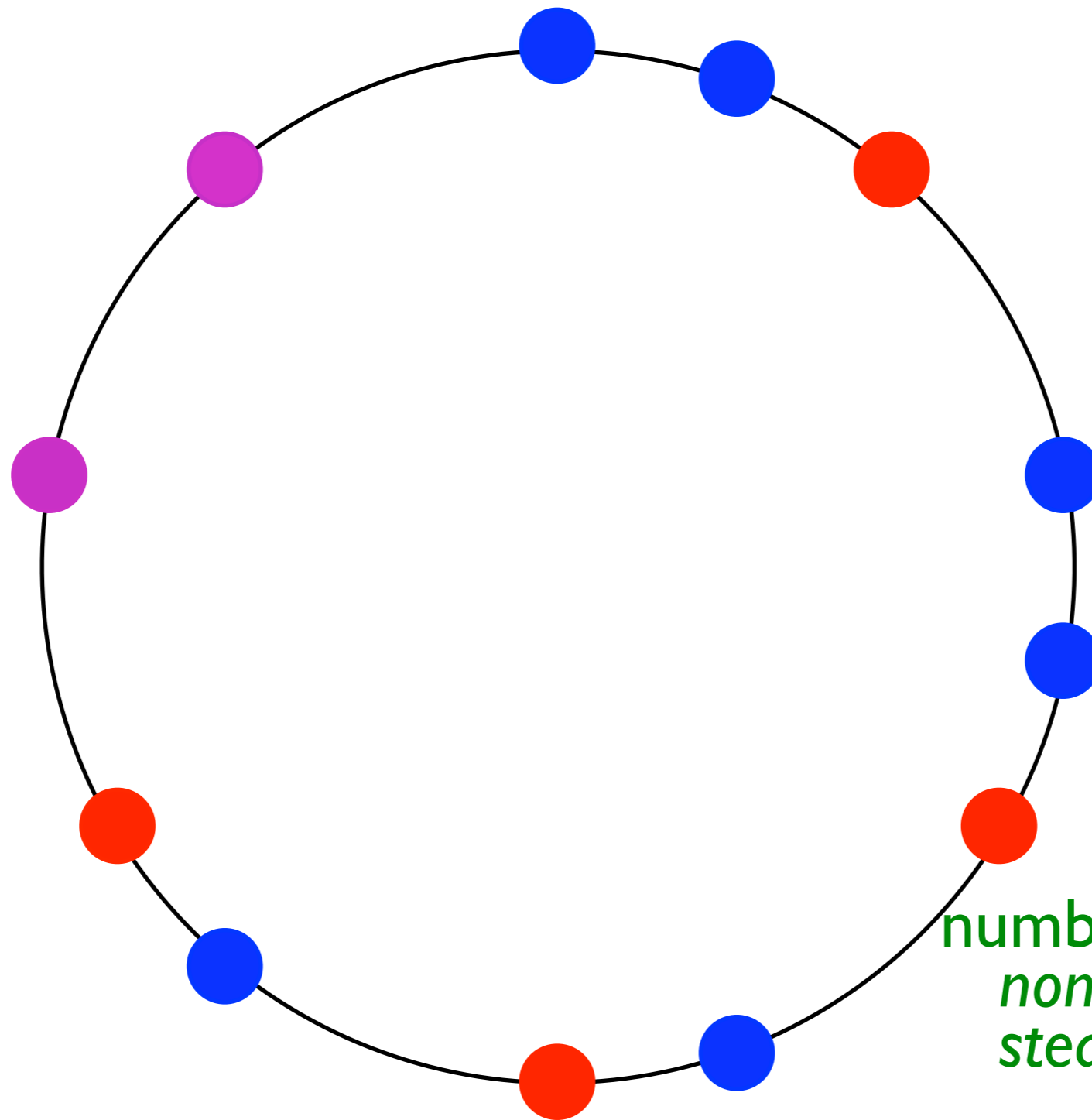


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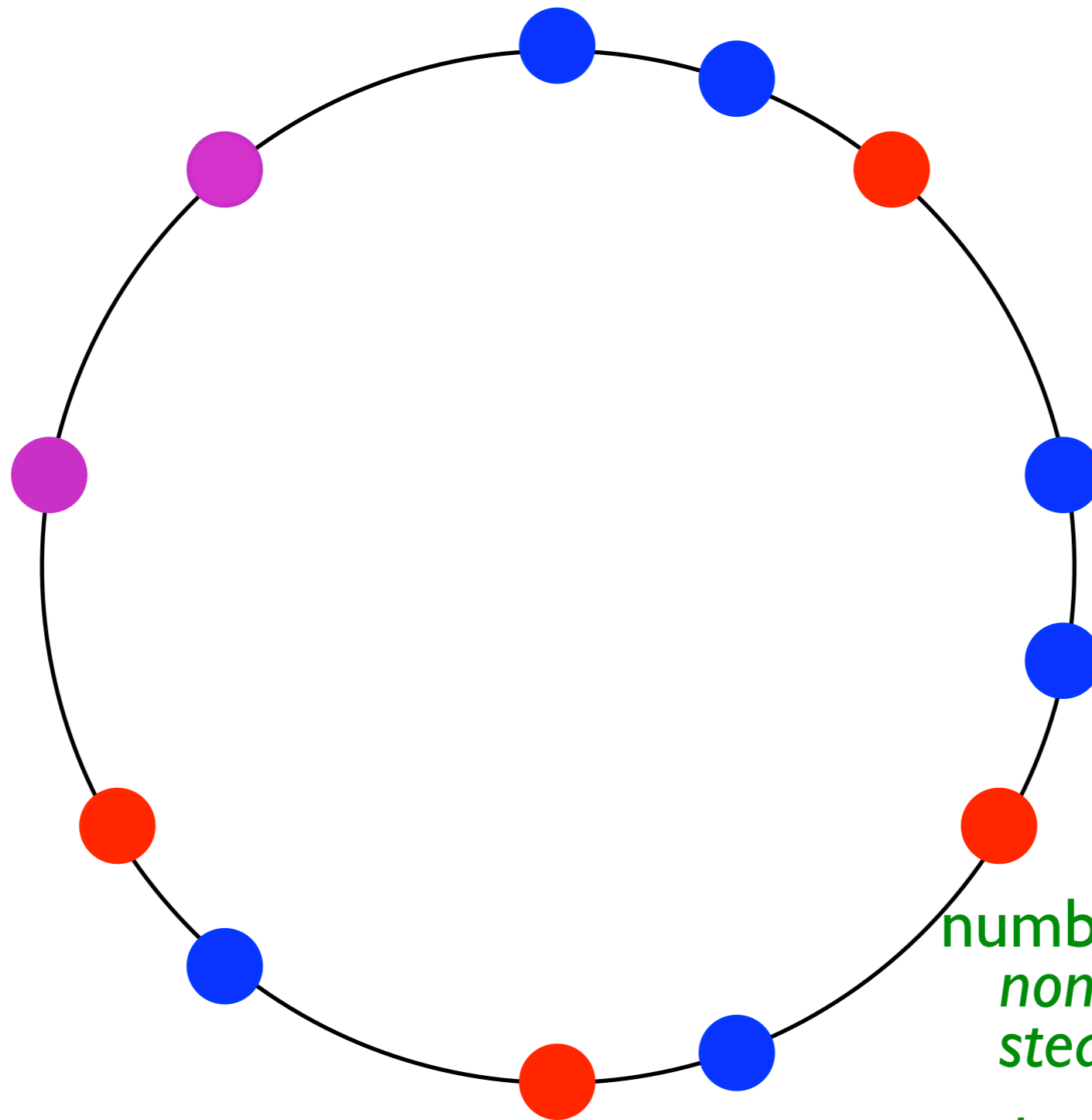
6 islands



number of islands:  
*non-decreasing until a  
steady state is achieved*

# Finite Ring (density $> 1/2$ )

6 islands



number of islands:  
*non-decreasing until a  
steady state is achieved*  
*isolated vacancies*

# Steady State on the Ring

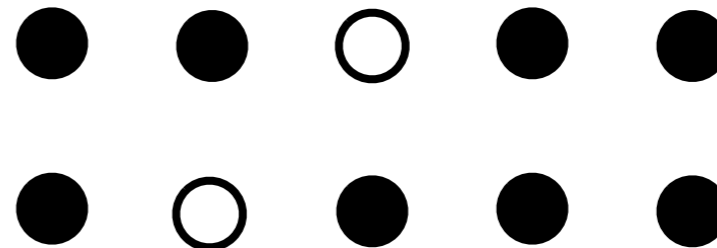
*claim:* all maximal-island states are equiprobable

$$P(C) \sum_{C'} R(C \rightarrow C') = \sum_{C'} P(C') R(C' \rightarrow C)$$

# of active  
leading triplets



# of active  
leading triplets



steady state for  $P(C) = \text{constant}$

# Steady State on the Ring

$$P(C) = C^{-1}$$

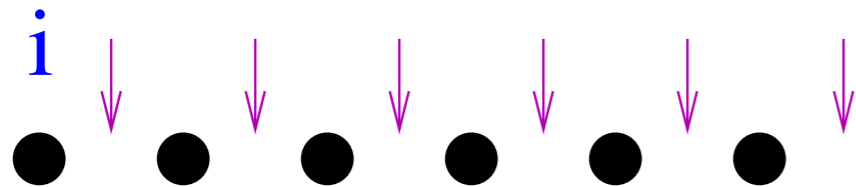
$C =$  number of maximal-island configurations with  $N$  particles &  $V$  vacancies

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if site  $i$  occupied:



$N$  particles

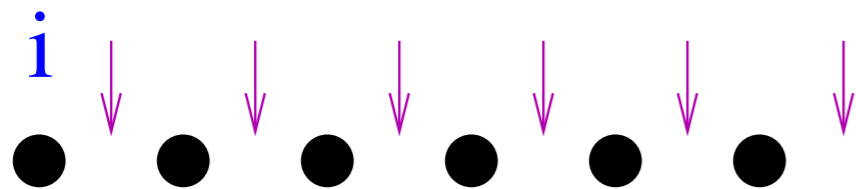
$N$  possibilities for  $V$  vacancies

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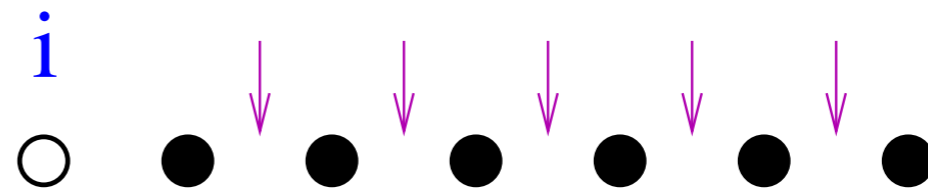
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if site  $i$  occupied:



$N$  particles  
 $N$  possibilities for  $V$  vacancies

if site  $i$  empty



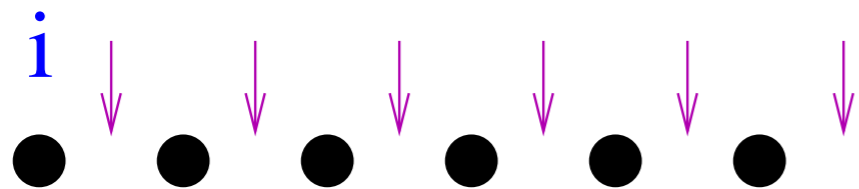
$N$  particles  
 $N-1$  possibilities for  $V$  vacancies

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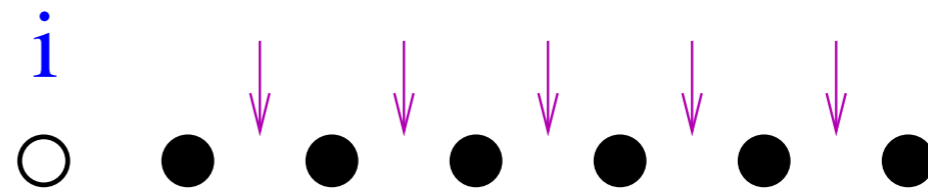
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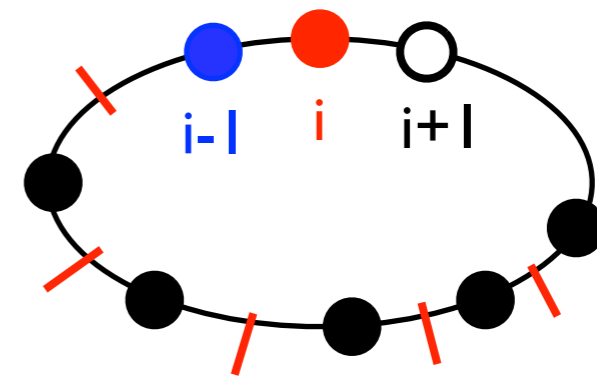
$N$  particles  
 $N-1$  possibilities for  $V$  vacancies

$$C = \binom{N}{V} + \binom{N-1}{V-1}$$

# Steady State Current

for flow between sites  $i$  and  $i+1$ :

- sites  $i-1$  &  $i$  occupied; site  $i+1$  empty
- number of consistent maximal-island configs.
- $N-2$  allowed locations for  $V-1$  vacancies



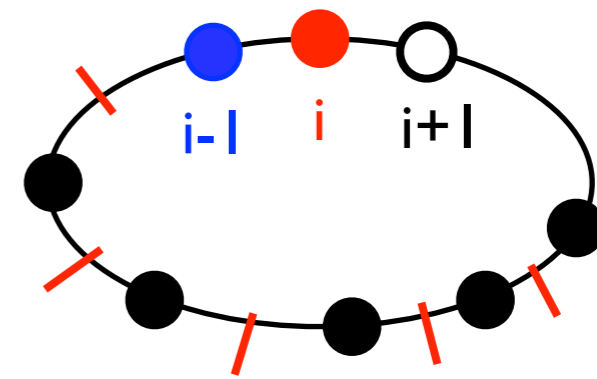
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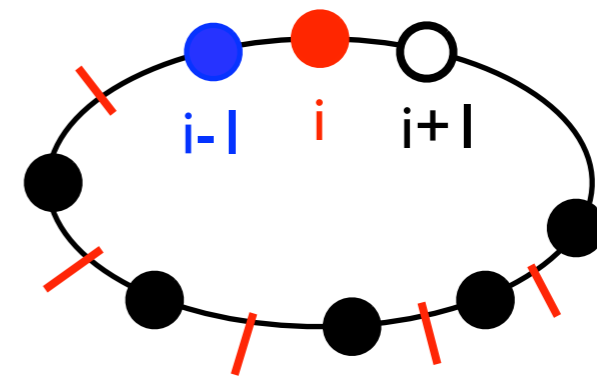
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$$\begin{aligned}
 J &= \frac{\binom{N-2}{V-1}}{\mathcal{C}} = \frac{\binom{N-2}{V-1}}{\binom{N}{V} + \binom{N-1}{V-1}} \\
 &= \frac{(1-\rho)(2\rho-1)}{\rho - L^{-1}} \\
 &\sim \frac{(1-\rho)(2\rho-1)}{\rho}
 \end{aligned}$$

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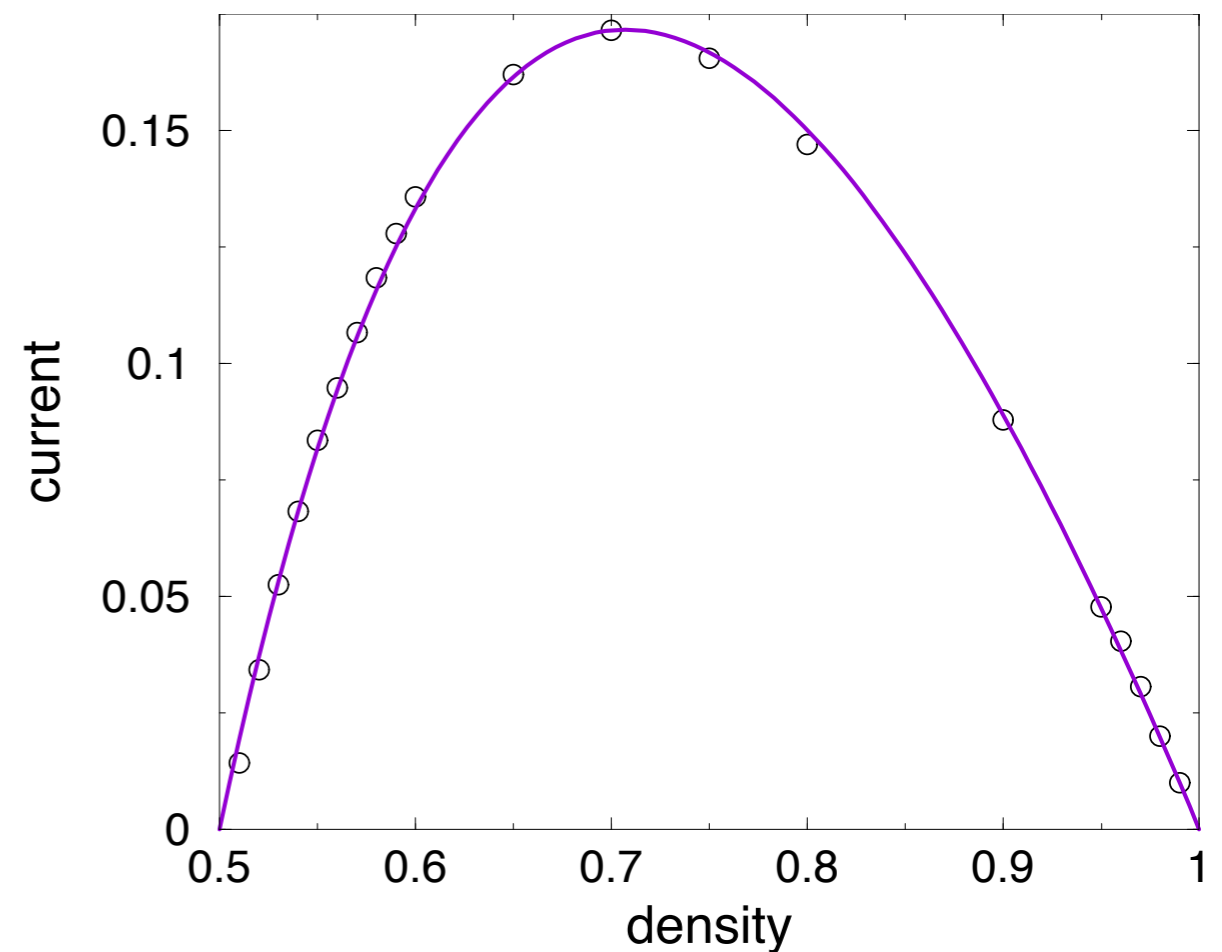
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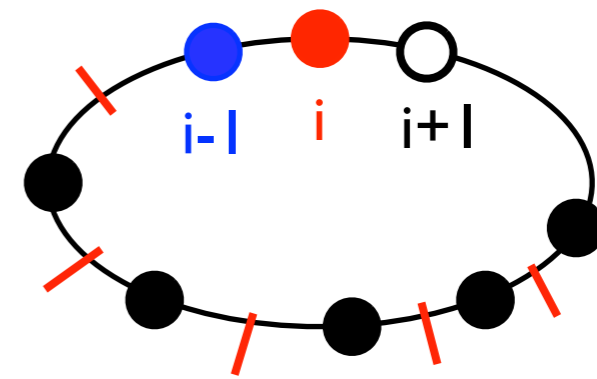
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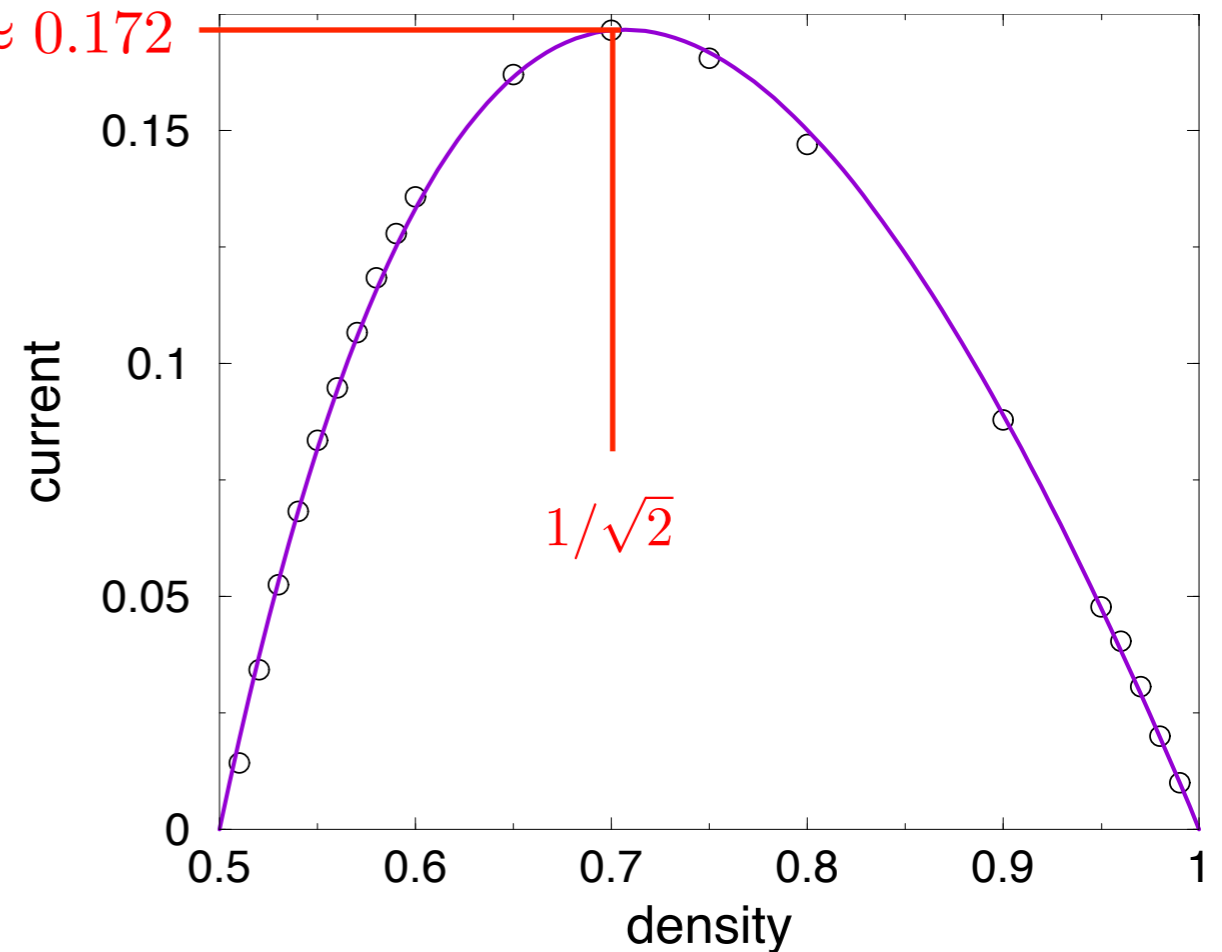
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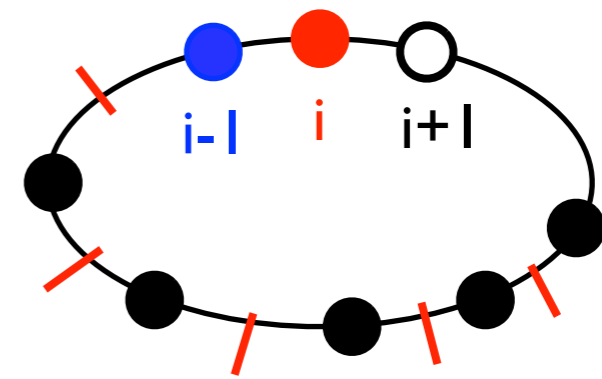
$$3 - 2\sqrt{2} \approx 0.172$$



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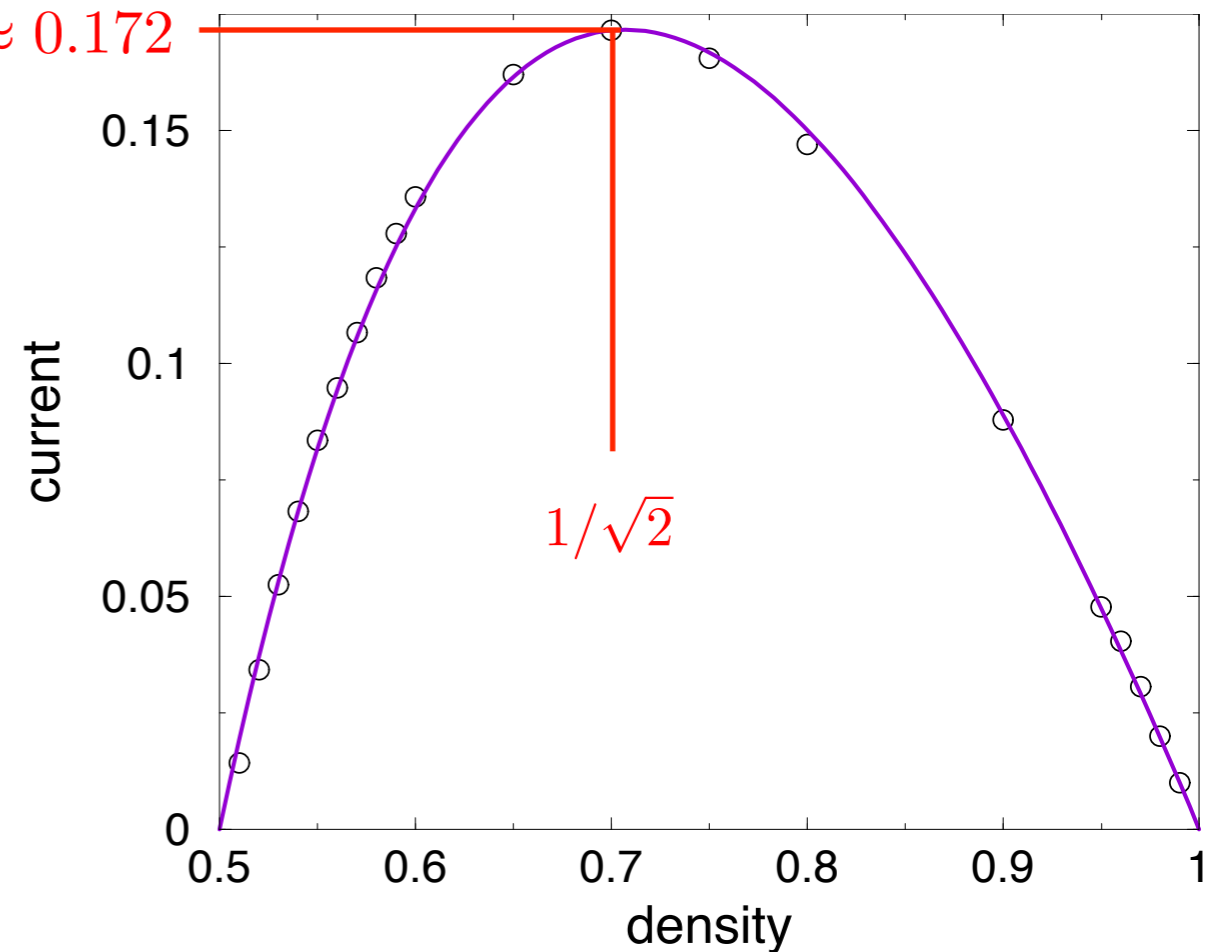
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much more: Gaussian current distribution

L. B. Shaw, R. K. P. Zia, and K. H. Lee, Phys. Rev. E **68**, 021910 (2003)

# Steady State Island-Size Distribution



- $n$  consecutive occupied sites & 2 empty
- $N-n$  other particles
- $N-n-1$  allowed locations for  $V-2$  vacancies

$$I_n = \frac{\binom{N-n-1}{V-2}}{\binom{N}{V} + \binom{N-1}{V-1}} \rightarrow \frac{(1-\rho)^2}{\rho} \left( \frac{2\rho-1}{\rho} \right)^{n-1}$$

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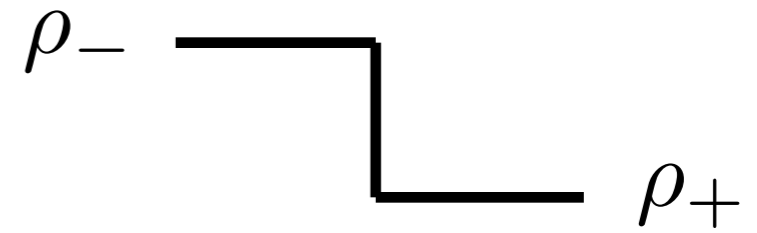
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$\ll \rho^n$

# Evolution of Density Downstep

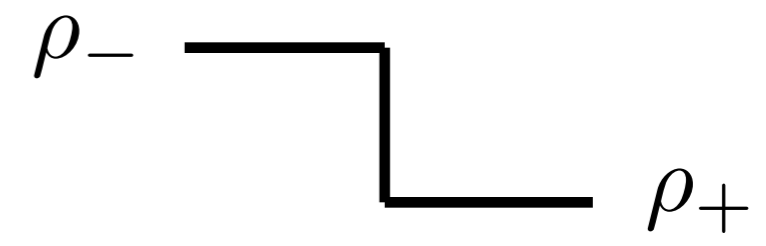
initial state:

$$\rho = \begin{cases} \rho_- & x < 0 \\ \rho_+ & x > 0 \end{cases}$$



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evolution equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x} = 0$$

scaling ansatz:

$$\rho(x, t) = f(z), \quad \text{with } z \equiv x/t$$

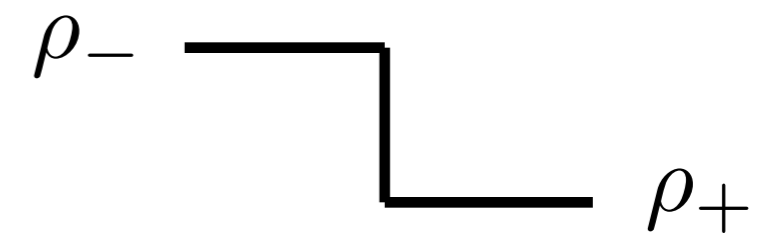
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$$f = \begin{cases} \rho_- & z < z_- \\ (2 + z)^{-1/2} & z_- < z < z_+ \\ \rho_+ & z > z_+ \end{cases}$$

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wave by continuity:

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left boundary of rarefaction wave by continuity:

$$(2+z_-)^{-1/2} = \rho_-$$

right boundary of rarefaction wave by mass conservation:

$$-\rho_- z_- + \rho_+ z_+ = \int_{z_-}^{z_+} \frac{dz}{\sqrt{2+z}} + J_- - J_+$$

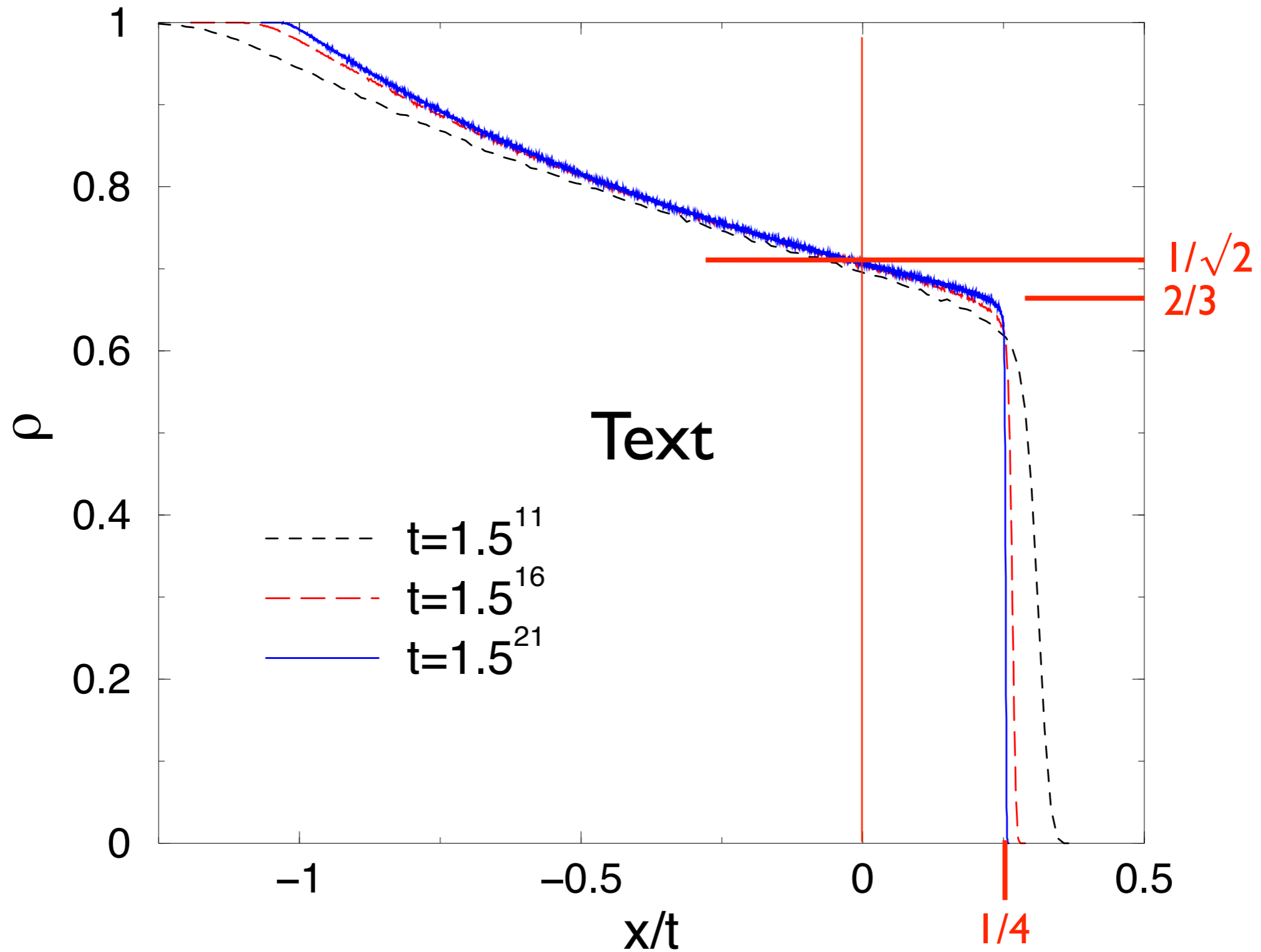
initial mass  
in  $[z_-, z_+]$

mass at time t  
in  $[z_-, z_+]$

net flux  
into  $[z_-, z_+]$

# Scaled Density Profile (for (1,0) IC)

*rarefaction wave with jump discontinuity*

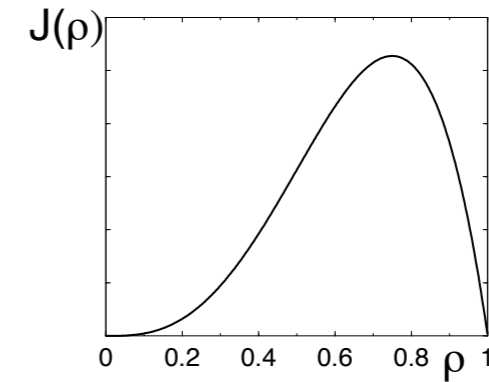


# Ubiquitous Rarefaction Discontinuity

generic current-density relation:

$$J(0)=J(1)=0,$$

single maximum in  $[0,1]$

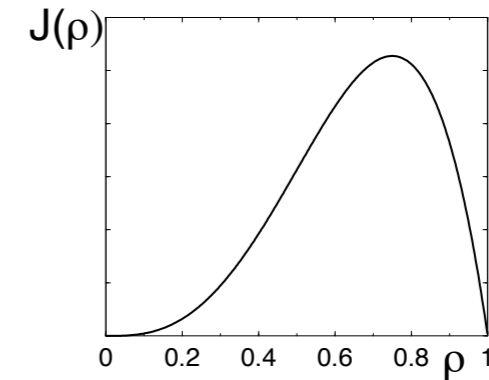


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continuity eqn + scaling  $\rightarrow f'(z) \left( \frac{\partial J}{\partial \rho} - z \right) = 0$

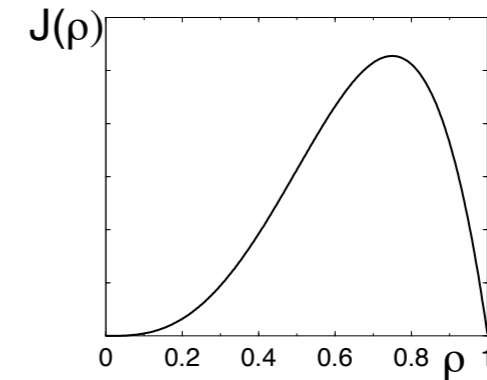
$$\rho(z) = \begin{cases} \rho_- & z < z_- \\ I(z) & z_- < z < z_+ \\ \rho_+ & z > z_+ \end{cases}$$

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derivative of  $\frac{\partial J}{\partial \rho} = z \rightarrow J_{\rho\rho} \rho_z = 1$

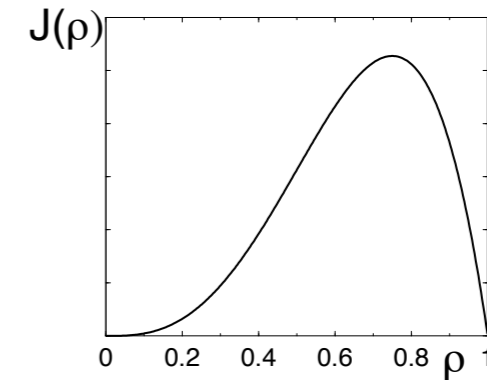


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derivative of  $\frac{\partial J}{\partial \rho} = z \rightarrow J_{\rho\rho} \rho_z = 1$

If  $J_{\rho\rho} < 0 \rightarrow \rho$  is a decreasing function of  $z$

If  $J_{\rho\rho} > 0 \rightarrow \rho$  must have jump discontinuity

# Summary & Outlook

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## Facilitated exclusion:

steady state: current and island sizes

generalizable to n-tuple facilitation

rarefaction wave: robust jump discontinuity

non-equilibrium phase transition for  $\rho \rightarrow 1/2$

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## Future:

more general mechanisms (distance facilitation)?

is exclusion even necessary?

diffusive corrections to hydrodynamic solutions?

higher dimensions?

# Crass Commercialism

Aimed at graduate students, this book explores some of the core phenomena in non-equilibrium statistical physics. It focuses on the development and application of theoretical methods to help students develop their problem-solving skills.

The book begins with microscopic transport processes: diffusion, collision-driven phenomena, and exclusion. It then presents the kinetics of aggregation, fragmentation, and adsorption, where basic phenomenology and solution techniques are emphasized. The following chapters cover kinetic spin systems, by developing both a discrete and a continuum formulation, the role of disorder in non-equilibrium processes, and hysteresis from the non-equilibrium perspective. The concluding chapters address population dynamics, chemical reactions, and a kinetic perspective on complex networks. The book contains more than 200 exercises to test students' understanding of the subject. A link to a website hosted by the authors, containing an up-to-date list of errata and solutions to some of the exercises, can be found at [www.cambridge.org/9780521851039](http://www.cambridge.org/9780521851039).

**Pavel L. Krapivsky** is Research Associate Professor of Physics at Boston University. His current research interests are in strongly interacting many-particle systems and their applications to kinetic spin systems, networks, and biological phenomena.

**Sidney Redner** is a Professor of Physics at Boston University. His current research interests are in non-equilibrium statistical physics and its applications to reactions, networks, social systems, biological phenomena, and first-passage processes.

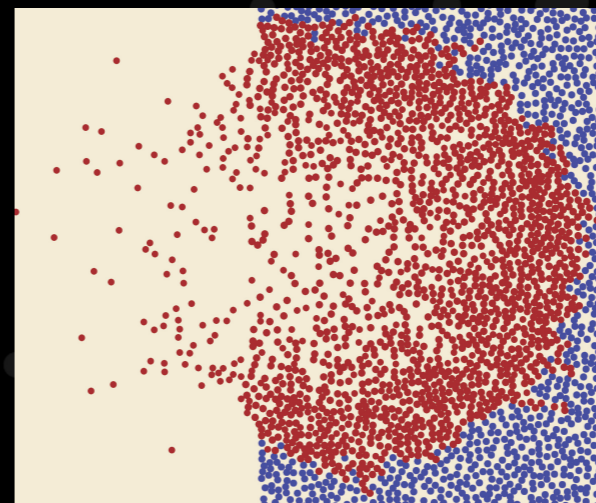
**Eli Ben-Naim** is a member of the Theoretical Division and an affiliate of the Center for Nonlinear Studies at Los Alamos National Laboratory. He conducts research in statistical, nonlinear, and soft condensed-matter physics, including the collective dynamics of interacting particle and granular systems.

Cover illustration: Snapshot of a collision cascade in a perfectly elastic, initially stationary hard-sphere gas in two dimensions due to a single incident particle. Shown are the cloud of moving particles (red) and the stationary particles (blue) that have not yet experienced any collisions. Figure courtesy of Tibor Antal.

Krapivsky  
Redner  
Ben-Naim

A Kinetic View of STATISTICAL PHYSICS

## A Kinetic View of STATISTICAL PHYSICS



Pavel L. Krapivsky  
Sidney Redner  
Eli Ben-Naim

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