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Escape of a Uniform Random Walk from an Interval

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We study the first-passage properties of a random walk in the unit interval in which the length of a single step is uniformly distributed over the finite range $[-a, a]$. For a of the order of one, the exit probabilities to each edge of the interval and the exit time from the interval exhibit anomalous properties stemming from the change in the minimum number of steps to escape the interval as a function of the starting point. As a decreases, first-passage properties approach those of continuum diffusion, but non-diffusive effects remain because of residual discreteness effects.

KEY WORDS: First-passage, random walk, uniform steps, diffusion approximation, radiation boundary condition

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1. INTRODUCTION

Consider a discrete-time random walk in which the length of each step is uniformly distributed in the range $[-a, a]$. We term this process the uniform random walk (URW). The walker is initially at an arbitrary point x in the unit interval [0,1] and the endpoints are absorbing. For the URW, an absorbing boundary is defined such that if the walk crosses an endpoint of the interval, the walk is trapped exactly at this endpoint. We are interested in the first-passage properties of this URW.

One motivation for this study comes from the problem of DNA sequence recognition by a mobile protein.[1,2] The protein molecule typically seeks its target by a combination of diffusion along the DNA chain, and also detachment and subsequent reattachment of the protein at a point many base pairs away, and the basic

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quantity of interest is the time required for the protein to find its target on a finite DNA chain.[1–3] The two mechanisms of diffusion and detachment/reattachment can be viewed as a random walk along the chain with a variable step length distribution.[3] This is the viewpoint that we shall adopt for this work. A second motivation for our work is that when individual step lengths are drawn from a continuous distribution, the resulting random walk exhibit a variety of interesting properties beyond those of the discrete random walk. These include, among others, both unusual first-passage properties[4,5] as well as extreme-value phenomena.[6]

Here we study the related problem of first-passage properties of the URW in a finite interval. Perhaps the most basic such feature is the exit probability $R(x)$, defined as the probability for a walk that starts at x to eventually cross the boundary at $x = 1$. Since the probability to exit via the left boundary is $1 - R(x)$, we need only consider exit to the right boundary, for which we reserve the term exit probability. A related quantity is the mean exit time $t(x)$, defined as the average time to exit the interval at either boundary when the walk starts from x. This exit time from an interval has been investigated for a random walk with general single-step hopping probabilities[7–10] and also in an econophysics context.[4] As we shall see, the interplay between the maximum step size and the interval length leads to commensuration effects that are absent when the step length distribution extends over an infinite range.

For both pure diffusion and the classical random walk with the step length $\Delta x = 1/N$, where N is an arbitrary integer, it is well known that the exit probability from an absorbing interval is $R(x) = x$.[11,12] Similarly, the mean exit time is $t(x) = x(1 - x)/2D$, where the diffusion coefficient for the discrete random walk is $D = \langle(\Delta x)^2\rangle/2$, with $\langle(\Delta x)^2\rangle$ the mean-square length of a single step. We now determine these first-passage properties—the exit probability and the exit time—for the URW.

2. GENERAL FEATURES

To help visualize the general behavior, we performed numerical simulations of the URW by a probability propagation algorithm; this approach is orders of magnitude more efficient than direct Monte Carlo simulation of an ensemble of random walkers. In probability propagation, we first divide the unit interval into N discrete points. We correspondingly discretize the URW as follows: a URW that moves uniformly within the range $[-a, a]$ at each step is equivalent to its discretized counterpart hopping equiprobably to any one of the aN discrete points on either side of the current site. When an element of probability hops outside the interval, this element is considered to be trapped at the boundary where the element left the interval. We continue this propagation until less than 10^{-6} of the initial probability remains in the interval. Running the propagation further led to insignificant corrections. The simulations were generally performed with

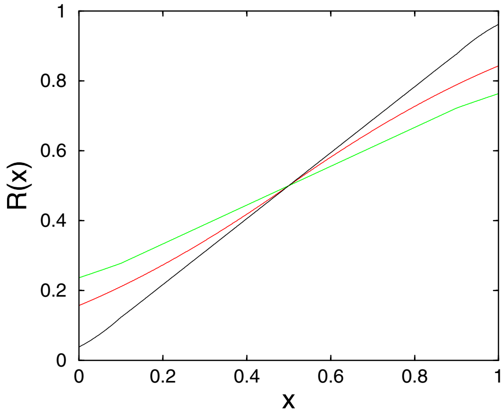

Fig. 1. Probability of exit at the right boundary, $R(x)$, for $a = 0.9$, 0.5, and 0.1 (slopes increasing, respectively), where a is the width of the single-step distribution. The data for $a = 0.1$ are based on probability propagation, while the other two datasets are obtained analytically.

$N = 5000$. We found negligible differences in our results when the interval was discretized into $N = 10,000$ points.

When a is of the order of 1, non-diffusive features arise because the walk can traverse the interval in just a few steps. In the limit $a \to \infty$, $R(x) = 1/2$ for all x; that is, either boundary is reached equiprobably, independent of the initial position. Conversely, for $a \ll 1$, the first-passage properties of the URW approach those of continuum diffusion. Thus $R(x) \approx x$, except when the starting point is close to a boundary (Fig. 1). In this boundary region, $R(x)$ undergoes a series of transitions in which the nth derivative is discontinuous whenever x or $1 - x$ passes through na. These transitions become more apparent by calculating $R'(x)$ (Fig. 2); this quantity amplifies the barely perceptible anomalies in $R(x)$ in Fig. 1. For small a, the qualitative behavior of R' is reminiscent of the Gibbs' overshoot phenomenon[13] when expanding a square wave in a Fourier series.

It is worth noting that the URW in an absorbing interval is not a martingale[14] and hence $R(x) \neq x$ in general. That is, the URW is not a "fair" process. The unfairness arises because when a walk crosses the boundary and ostensibly lands outside the interval, the walk is reassigned to be trapped at the edge of the interval. Thus the mean position of the probability distribution is not conserved. Another consequence of the unfairness is that the exit probability $R(x = 0)$ for fixed a, no matter how small, is non-zero. The possibility of starting at $x = 0$ and exiting at $x = 1$ can be viewed as an effective bias of the walk toward the middle of the interval. In the context of the gambler's ruin problem,[11] a gambler that is about to be ruined appears to be best served by making a reckless bet that is of the order of the total amount of capital in the game.

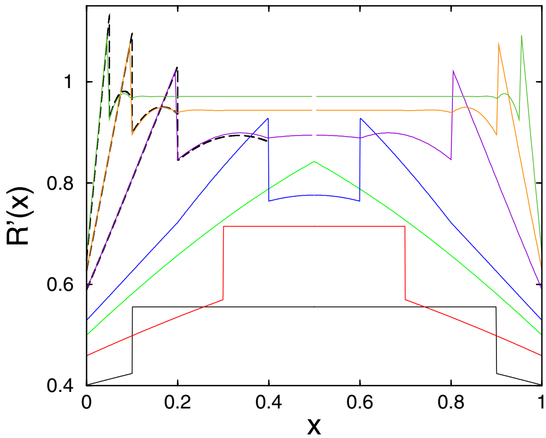

Fig. 2. The derivative $R'(x)$ for $a = 0.9, 0.7, 0.5, 0.4, 0.2, 0.1$, and 0.05 (rising curves, respectively). Shown dashed are the results of an asymptotic approximation that is discussed in Sec. 4.

The mean exit time of the URW deviates strongly from the continuum diffusive form $t(x) = x(1-x)/2D$, when a becomes of the order of 1 (Fig. 3). Notice also that $t(x)$ does not go to zero as $x \to 0$ or $x \to 1$. This limiting behavior again reflects the fact that there is a non-negligible probability for a walk that starts at one edge of the interval to exit via the opposite edge, a process that requires a non-zero time.

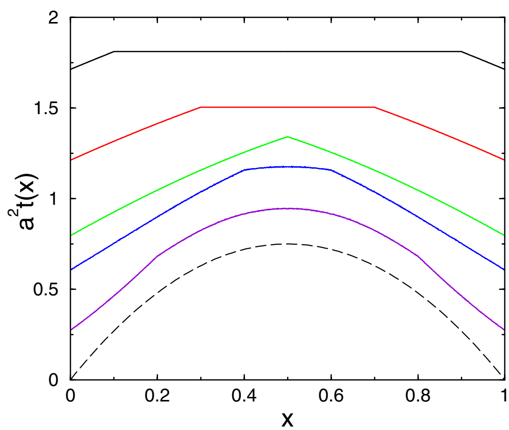

Fig. 3. Mean time to exit at either boundary, $t(x)$, times a^2 (proportional to the diffusion coefficient) versus x for $a \to 0$, as well as for $a = 0.2, 0.4, 0.5, 0.7$ and 0.9 (bottom to top). The data for $a < 0.5$ is based on simulation, while the remaining data is obtained analytically.

3. FIRST-PASSAGE PROPERTIES

In general, the exit probability may be determined from the backward equation that expresses the exit probability $R(x)$ in terms of the exit probability after one step of the random walk has elapsed.[12] This backward equation has the generic form

$$R(x) = \int dx'\, p(x \to x')R(x')\,. \tag{1}$$

That is, the exit probability starting from x equals the probability of making a single step to x' times the probability of exit from x', integrated over all possible values of x'. In a parallel fashion, the mean exit time can generically be written as

$$t(x) = \int dx'\, p(x \to x')[t(x') + 1]\,. \tag{2}$$

That is, the exit time starting from x equals one plus the exit time from x', when integrated over all possible values of x', with each term weighted by $p(x \to x')$. Note that the trailing factor of 1 can be taken outside the integral sum since $\int dx'\, p(x \to x') = 1$. We now apply these two formulae to determine first-passage properties for the URW by studying, in turn, the cases $a > 1$, $a \in [1/2, 1]$, $a \in [1/3, 1/2]$, $a < 1/3$, and finally $a \to 0$.

3.1. $a > 1$

When $a > 1$, the support of the probability distribution necessarily extends beyond the unit interval after one step. The residue that remains in the interval is also uniformly distributed. These facts allow us to obtain $R(x)$ and $t(x)$ by simple probabilistic reasoning. After one step, the walk jumps past $x = 1$ with probability $\frac{x+a-1}{2a}$, and jumps to the left of the origin with probability $\frac{a-x}{2a}$ (Fig. 4). Because the remaining probability of $1/2a$ is uniformly distributed within [0,1], there is a 50% chance that this residue will eventually exit via either end. Thus the exit probability is

$$R(x) = \frac{x+a-1}{2a} + \frac{1}{2}\frac{1}{2a} = \frac{2(x+a)-1}{4a}\,. \tag{3}$$

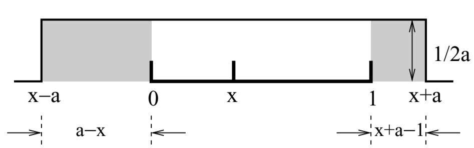

Fig. 4. Probability distribution of the uniform random walk after a single step. The shaded regions are the portion of the probability distribution outside the unit interval.

As expected, the exit probability approaches 1/2, independent of the starting point, as the average step length becomes large. Also notice that $R(1) = \frac{1}{2} + \frac{1}{4a}$; as $a \to 1$ from above, the probability of exiting the right boundary when starting at $x = 1$ is only 3/4.

The survival probability for the walk to remain within the interval after a single step is simply $1/2a$, and the survival probability after n steps is then $S(n) = (1/2a)^n$. Since the first-passage probability for the walk to first exit the interval at the nth step is $F(n) = S(n-1) - S(n)$, the mean exit time is

$$\begin{aligned}\langle t \rangle &= \sum_{n=1}^{\infty} n[S(n-1) - S(n)] \\ &= \sum_{n=0}^{\infty} S(n) = \frac{1}{1 - 1/2a}\,. \end{aligned} \tag{4}$$

As $a \to \infty$, $\langle t \rangle \to 1$, while as $a \to 1$ from above, $\langle t \rangle \to 2$. This same value for the exit time can also be obtained by solving the backward equation for $t(x)$ itself (see below).

3.2. $a \in [1/2, 1]$

When $1/2 < a < 1$, the unit interval naturally divides into an inner subinterval $(1-a, a)$ and outer subintervals $(0, 1-a)$, and $(a, 1)$ (Fig. 5). If the walk begins in $(1-a, a)$, then the probability distribution of the walk necessarily extends beyond [0,1] after a single step, and the exit probability is again given by Eq. (3).

On the other hand, when the walk starts in the outer subintervals the recursion formula (1) for the exit probability becomes

$$R(x) = \begin{cases} \dfrac{1}{2a}\displaystyle\int_0^{x+a} dx'\,R(x') & x \in [0, 1-a], \\[2ex] \dfrac{1}{2a}\displaystyle\int_{x-a}^{1} dx'\,R(x') + \dfrac{x+a-1}{2a} & x \in [a, 1]. \end{cases} \tag{5}$$

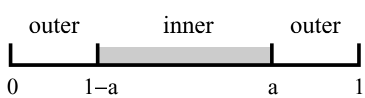

Fig. 5. The unit interval for $1/2 < a < 1$. A walk starting in $(1-a, a)$ can leave the interval via either end in a single step.

Differentiating these equations gives

$$2aR'(x) = R(x+a) \qquad x \in (0, 1-a)\,, \tag{6a}$$

$$2aR'(x) = 1 - R(x-a) \qquad x \in (a, 1)\,. \tag{6b}$$

When $x \in (0, 1-a)$, the second derivative is

$$4a^2R''(x) = 2aR'(x+a) = 1 - R(x), \tag{7}$$

where we use the fact that if $x \in (0, 1-a)$, then $x + a \in (a, 1)$. The solution to (7) is

$$R(x) = 1 + c_1 \sin\left[\frac{1}{2a}(x - c_2)\right]. \tag{8}$$

To determine the constants c_1 and c_2, we first substitute (8) into (6a) and also use the fact that $R(x + a) = 1 - R(x)$ for $x \in (0, 1 - a)$ to obtain $c_2 = (1 - a)/2 + \pi a/2$. Second, we match (8) and (3) at $x = 1 - a$ to find c_1. The exit probability for $x \in (0, 1 - a)$ therefore is

$$R(x) = 1 + \frac{\sin\left[\frac{1}{2a}\left(x - \frac{1-a}{2}\right) - \frac{\pi}{4}\right]}{\left(1 - \frac{1}{4a}\right)\sin\left(\frac{a-1}{4a} + \frac{\pi}{4}\right)}\,. \tag{9}$$

The sinusoidal segment of $R(x)$ is visually close to a linear function (Fig. 1), and the difference between these two functional forms becomes more clearly visible upon plotting $R'(x)$ versus x (Fig. 2).

We now compute the mean exit time. Again, there are two cases to consider: either the walk begins within $(a, 1 - a)$ or it begins in the complementary outer subintervals. Let us denote by $t_{\rm in}(x)$ and $t_{\rm out}(x)$ as the mean exit times when the walk starts at a point x in the inner and in the outer subintervals, respectively. Then the backward Eq. (2) for $t_{\rm in}$ becomes

$$\begin{aligned} t_{\rm in}(x) &= 1 + \frac{1}{2a}\int_0^1 t(x')\,dx' \\ &= 1 + \frac{1}{a}\int_0^{1-a} t_{\rm out}(x')\,dx' + \frac{1}{2a}\int_{1-a}^{a} t_{\rm in}(x')\,dx'. \end{aligned} \tag{10}$$

For the last line, we break up the integral into a contribution from the outer subinterval, with two equal contribution from $(0, 1 - a)$ and $(a, 1)$, and the inner subinterval $(1 - a, a)$. Notice also from the first line that $t_{\rm in}(x)$ is independent of x. Thus we define $t_{\rm in}(x) = \mathcal{T}$, with $\mathcal{T}$ dependent only on a.

Similarly, the backward equation for $t_{\rm out}$ is

$$t_{\rm out}(x) = 1 + \frac{1}{2a}\int_0^{x+a} t(x')\,dx'\,. \tag{11}$$

Differentiating gives $t'_{\rm out}(x) = t_{\rm out}(x+a)/2a$. Notice that if $x \in (0, 1-a)$, then $x+a$ is necessarily in $(a, 1)$. Correspondingly, the backward equation for $t_{\rm out}(x+a)$ gives $t'_{\rm out}(x+a) = -t_{\rm out}(x)/2a$. Thus $t''_{\rm out}(x) = -t_{\rm out}(x)/4a^2$, with solution

$$t_{\rm out}(x) = \tau_1 \cos\left(\frac{x}{2a}\right) + \tau_2 \sin\left(\frac{x}{2a}\right)\,. \tag{12}$$

To complete the solution, we need to determine the three unknown constants $\mathcal{T}$, τ_1, and τ_2. The solution is straightforward and the details are given in Appendix A. From this solution, quoted in Eq. (A.6), we obtain the mean exit times plotted in Fig. 3. As a decreases, $t(x)$ quickly approaches the parabolic form of the diffusive limit, but $t(0)$ and $t(1) = t(0)$ remain strictly greater than zero when a is non zero.

3.3. $a \in [1/3, 1/2]$

For any $a < 1/2$, the exit probability now obeys the generic recursion formulae:

$$R(x) = \begin{cases} \dfrac{1}{2a}\displaystyle\int_0^{x+a} dx'\,R(x') & x \in [0, a] \\ \dfrac{1}{2a}\displaystyle\int_{x-a}^{x+a} dx'\,R(x') & x \in [a, 1-a] \\ \dfrac{1}{2a}\displaystyle\int_{x-a}^{1} dx'\,R(x') + \dfrac{x+a-1}{2a} & x \in [1-a, 1]. \end{cases} \tag{13}$$

For example, the middle equation states that the exit probability starting at x equals the exit probability after making one step to x'—which is uniformly distributed in the range $[-a, a]$ around x—times the exit probability from x'. The first and third equations account for the modified range of the single-step distribution if the walk leaves the interval.

We differentiate these integral equations to obtain the more compact form

$$\begin{aligned} 2aR'(x) &= R(x+a) \quad x \in (0, a); \\ 2aR'(x) &= R(x+a) - R(x-a) \quad x \in (a, 1-a); \\ 2aR'(x) &= 1 - R(x-a) \quad x \in (1-a, 1). \end{aligned} \tag{14}$$

To solve these equations for the cases where $a \in [1/3, 1/2]$, we should consider the five subintervals $(0, 1-2a)$, $(1-2a, a)$, $(a, 1-a)$, $(1-a, 2a)$, and $(2a, 1)$

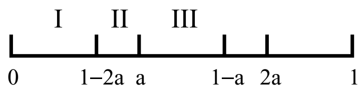

Fig. 6. The unit interval with the subregions used to determine the exit probabilities for the case $a \in (1/3, 1/2)$.

(Fig. 6). By symmetry, we only need to study the range $x < 1/2$, and we now examine, in turn, subintervals III, I, and II.

`Subinterval III`: For $x \in (a, 1-a)$, Eq. (15) connects $R'(x)$ with $R(x+a)$ and $R(x-a)$. In turn, the equations for $R'(x+a)$ and $R'(x-a)$ involve $R(x)$ with x again within $(a, 1-a)$. Thus,

$$4a^2 R''(x) = 2aR'(x+a) - 2aR'(x-a) = 1 - 2R(x)\,, \tag{15}$$

with solution

$$R(x) = \frac{1}{2} + c_1 \sin\left[\frac{1}{\sqrt{2}a}\left(x - \frac{1}{2}\right)\right] \quad x \in (a, 1-a)\,. \tag{16}$$

This form automatically satisfies the symmetry condition $R(1/2) = 1/2$.

`Subinterval I`: We obtain the exit probability for $x \in (0, 1-2a)$ by integrating (14) and also using the fact that the argument $x+a$ in $R(x+a)$ lies within $(a, 1-a)$. Thus we use the result of Eq. (16) to give

$$R(x) = \frac{x}{4a} - \frac{c_1}{\sqrt{2}} \cos\left[\frac{1}{\sqrt{2}a}\left(x + a - \frac{1}{2}\right)\right] + c_2 \quad x \in (0, 1-2a)\,. \tag{17}$$

To determine the constants c_1 and c_2, we use the general antisymmetry condition, $R(y) = 1 - R(1-y)$, to write (15) in the form

$$2aR'(x) = 1 - R(1-x-a) - R(x-a) \quad x \in (a, 1-a)\,. \tag{18}$$

Now we substitute the solutions (16) and (17) into Eq. (18) and find $c_2 = \frac{3}{4} - \frac{1}{8a}$. Thus

$$R(x) = \frac{x - 1/2}{4a} - \frac{c_1}{\sqrt{2}} \cos\left[\frac{1}{\sqrt{2}a}\left(x + a - \frac{1}{2}\right)\right] + \frac{3}{4}\,. \tag{19}$$

`Subinterval II`: Finally, for $x \in (1-2a, a)$, Eqs. (14) and (15) show that the subintervals $(1-2a, a)$ and $(1-a, 2a)$ are coupled only to each other. Using the antisymmetry of the exit probability about $x = 1/2$, we have

$$4a^2 R''(x) = 1 - R(x) \quad x \in (1-2a, a)\,, \tag{20}$$

with solution $R(x) = 1 + c_3 \sin((x - c_4)/2a)$. We determine c_4 from the condition $R(x) = 1 - R(1-x)$ to then give

$$R(x) = 1 + c_3 \sin\left[\frac{1}{2a}\left(x - \frac{1-a}{2}\right) - \frac{\pi}{4}\right]\,. \tag{21}$$

To obtain the remaining two constants c_1 and c_3, we match Eqs. (19) and (21) at $1-2a$, and (21) and (16) at a. These lead to

$$c_1 = \frac{\frac{1}{8a} - \frac{3}{4} + \frac{1}{2}\tan\alpha}{\frac{1}{\sqrt{2}}\cos\beta - \sin\beta\tan\alpha} \tag{22}$$

$$c_3 = \frac{1}{2\cos\alpha} + \frac{\frac{1}{8a} - \frac{3}{4} + \frac{1}{2}\tan\alpha}{\frac{1}{\sqrt{2}}\cos\alpha\cot\beta - \sin\alpha},$$

with

$$\alpha = \frac{3a-1}{4a} + \frac{\pi}{4}, \qquad \beta = \frac{1-2a}{2\sqrt{2a}}. \tag{23}$$

For the special case of $a = 1/3$, subinterval II disappears so that the solution consists of (17) and (16) only, and the relevant constant in (22) simplifies to

$$c_1 = \left(4\sqrt{2}\cos\frac{1}{\sqrt{8}} - 8\sin\frac{1}{\sqrt{8}}\right)^{-1} \tag{24}$$

3.4. $a < 1/3$

It is straightforward to treat smaller values of a, but the bookkeeping of the various subintervals becomes increasingly tedious. However, it is still possible to infer general properties of the exit probability. From Eqs. (14) and (15), we have $2aR'(x = a^+) = R(2a) - R(0)$ while $2aR'(x = a^-) = R(2a)$. Thus $R'(x)$ has a jump of magnitude $R(0)/2a$ when x passes through a, as illustrated in Fig. 2 $a \leq 0.4$. Similarly, consider $R''(x)$ near $x = 2a$. By (15), $R''(x)$ is coupled to $R'(x+a)$ and $R'(x-a)$, and the latter derivative has a jump when its argument equals a. Thus $R''(x)$ has a jump of magnitude $R(0)/4a^2$ when x passes through $2a$. This pattern continues so that for $a \in (\frac{1}{n+1}, \frac{1}{n})$, the nth derivative of $R(x)$ has a jump discontinuity as x passes through na, while all lower derivatives are continuous. Thus $R(x)$ becomes progressively smoother and more linear in visual appearance for x deeper in the interior of the interval.

The analytical solution for $R(x)$ can, in principle, be obtained from the backward equations (14)–(15), for the exit probability. When $a < 1/2$, these equations naturally partition the unit interval into two classes of subintervals as shown in Fig. 7. Nearest-neighbor shaded subintervals are coupled only to each other by

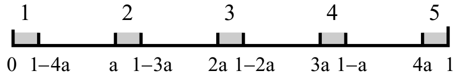

Fig. 7. The unit interval (to scale) for the case $\frac{1}{5} < a < \frac{1}{4}$ and the two associated classes of subintervals. The shaded subintervals are labeled sequentially as defined in the text.

these backward equations, and similarly for the complementary subintervals. It is convenient to define $r_k(x)$ as the exit probability for a walk that begins at $x + (k-1)a$ in the kth shaded subinterval; that is $r_k(x) = R(x + (k-1)a)$ for $x \in ((k-1)a, ka)$. With these conventions, the backward equations for the exit probability for a starting point in one of the shaded or in one of the unshaded subintervals have the matrix form:

$$2a\mathbf{r}'(x) = \begin{pmatrix} 0 & 1 & 0 & \cdots & & \\ -1 & 0 & 1 & 0 & \cdots & \\ 0 & -1 & 0 & 1 & 0 & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \\ & & 0 & -1 & 0 & 1 \\ & & & 0 & -1 & 0 \end{pmatrix} \mathbf{r}(x) + \mathbf{I}\,, \tag{25}$$

where $\mathbf{r}(x)$ is the column vector with components $[r_1(x), r_2(x) \ldots r_n(x)] = [R(x), R(x+a) \ldots R(x+(n-1)a)]$ and $\mathbf{I}$ is the column vector with the n components $(0, 0, \ldots, 0, 1)$.

To solve these equations, we note that the eigenvalues of the above matrix are given by $\lambda_j = 2i\cos(\pi j/(n+1))$, with $j = 1, 2, \ldots, n$.[15] Thus $r_n(x)$ is, in general, a linear superposition of the eigenvectors of the matrix; these are sinusoidal functions with arguments $\lambda_j/2a$. The actual form of $R(x)$ is then obtained by fixing the various constants in this eigenvector expansion through matching the components r_k at appropriate boundary points.

4. EXIT PROBABILITY FOR $a \to 0$

In the limit $a \to 0$, the diffusion approximation becomes increasingly accurate so that $R(x)$ is very nearly equal to x, except in a small region of the order of a near each boundary (Fig. 1). This deviation is more clearly evident when plotting $R'(x)$ versus x (Fig. 2). As a gets small, this plot also suggests that a good approximation to $R'(x)$ will be obtained by solving the exact equation for $R'(x)$ in the boundary region and treating $R'(x)$ as a constant in the interior of the interval.

At a zero-order level of approximation, we assume that $R'(x)$ is position dependent for $x \in (0, a)$ and $x \in (1-a, 1)$ and is constant otherwise. Then within $(0, a)$, the backward equation $2aR'(x) = R(x+a)$ means that $R'(x)$ is a linear function (and similarly for $x \in (1-a, 1)$). Thus we make the ansatz

$$R(x) \approx \begin{cases} \frac{1}{2}(1-s) + sx & x \in (a, 1-a) \\ r_0 + r_1 x + r_2 x^2 & \text{otherwise}\,, \end{cases} \tag{26}$$

with s and the r_i to be determined. We expect the slope s of $R(x)$ in the interior of the interval to approach 1 as $a \to 0$ to recover $R(x) \to x$ in this limit. The form of the constants in the first line also ensure the obvious special case $R(1/2) = 1/2$. Similarly, the linear form for $R'(x)$ in the boundary regions roughly corresponds to what is seen in Fig. 2.

We determine the 4 unknowns in the above asymptotic approximation for $R(x)$ by the following conditions: (i) $2aR'(x) = R(x + a)$ must be satisfied in the region $x \in (0, a)$ (this gives two conditions—one for the linear term and one for the constant term), (ii) $R(x)$ is continuous, (iii) the discontinuity in $R'(x)$ at $x = a$ equals $R(0)/2a$, as follows from Eqs. (14) and (15). Applying these conditions gives, after some simple calculation,

$$s = \frac{1}{1+a}; \quad r_0 = \frac{1}{2}\frac{a}{1+a}; \quad r_1 = \frac{3}{4}\frac{1}{1+a}; \quad r_2 = \frac{1}{4a}\frac{1}{1+a}.$$

A much better approximation is obtained by treating $R(x)$ exactly in the domains $(0, a)$ and $(a, 2a)$, and then assuming that $R'(x)$ is constant otherwise. Thus for $x > 2a$, $R(x)$ is given by the first line of Eq. (26), while within $(0, a)$ and $(a, 2a)$, the governing backward equations for $R(x)$ are

$$2aR'(x) = \begin{cases} R(x+a) & x < a \\ R(x+a) - R(x-a) & x > a \end{cases} \tag{27}$$

For $x < a$, we iterate the first equation to give $4a^2R''(x) = R(x + 2a) - R(x)$ and make use of the assumption that $R'(x) = s$ for $x > 2a$. This leads to the approximation

$$R(x) = d_1 \cos\left(\frac{x}{2a}\right) + d_2 \sin\left(\frac{x}{2a}\right) + s(x+2a) + \frac{1}{2}(1-s). \tag{28}$$

Similarly, to obtain $R(x)$ in the region $(a, 2a)$, we integrate the backward equation $2aR'(x) = R(x + a) - R(x - a)$ and again use the fact that the argument $x + a$ in $R(x + a)$ is beyond $2a$, so that $R(x + a)$ is a linear function. This integration leads to

$$R(x) = -d_1 \sin\left(\frac{x-a}{2a}\right) + d_2 \cos\left(\frac{x-a}{2a}\right) + d_3\,. \tag{29}$$

We determine the 4 coefficients in these two forms for $R(x)$ by requiring that at $x = a$, R is continuous and the first derivative has a jump of magnitude $R(0)/2a$, while at $x = 2a$, both R and R' are continuous. The resulting formulae are given in Appendix B. Fig. 2 shows the result of this small-a approximation for $R'(x)$. The agreement between this asymptotic approximation and the numerical results is extremely good.

As a further test of the accuracy of this approach, we show the numerically-obtained dependence of $R(0)$ on a together with our zeroth- and first-order approximations for $R(0)$ (Fig. 8). As already mentioned in Sec. 2, $R(0)$ is greater than zero for any $a > 0$ because there is a non-zero chance that a walk exactly at

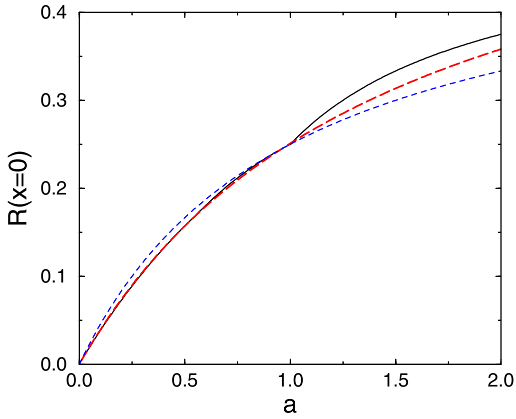

Fig. 8. Exit probability $R(x = 0)$ versus a, based on 5×10^8 walks for 200 equally-spaced values of a between 0 and 2. Shown dotted is the result of our zeroth-order asymptotic approximation $R(0) = a/[2(1 + a)]$, while the dashed curve is the result of the first-order approximation.

the left boundary can still exit via the right boundary. In the limit of small a, our asymptotic approach for $R(0)$ closely approximates the data.

5. CONCLUDING REMARKS

An enigmatic feature of the uniform random walk (URW) is that its first-passage properties in a finite interval are not described in terms of a radiation boundary condition.[12,16,17] This boundary condition is $c' = c/\kappa$, where κ is the extrapolation length. This condition can be interpreted as partial absorption and partial reflection when a walk hits the boundary. In the case of a URW that starts at one absorbing boundary, there is a non-zero chance for the walk to exit via the opposite boundary. This incomplete absorption should be equivalent to partial reflection from the initial boundary, which should be described by a radiation boundary condition. Indeed, the probability distribution of a URW in a semi-infinite interval $x > 0$ with absorption at $x = 0$ closely approximates that obtained for pure diffusion with a radiation boundary condition. However, in the finite interval the radiation boundary condition gives $R(x)$ as the linear function $R(x) = \frac{x+\kappa}{1+2\kappa}$ which does not account for the anomalous behavior observed near the edges of the interval.

What we do find is that the first-passage properties of the URW in a finite interval exhibit curious commensuration effects as a passes through $1, \frac{1}{2}, \frac{1}{3} \cdots$. For $a \in (\frac{1}{n+1}, \frac{1}{n})$, the nth derivative of $R(x)$ has a jump discontinuity as x passes through na, while all lower derivatives are continuous. The exit time has corresponding singular behaviors. For small n, we have computed the exit probability and the mean exit time exactly by a direct probabilistic approach. In the limit $a \to 0$, the diffusion approximation becomes increasingly accurate, except when the starting point is close to either boundary where the exit probability continues to exhibit non-diffusive effects. In the limit of small a, we constructed an approximation of treating the exit probability exactly within the boundary region, an approach that gives extremely accurate results.

APPENDIX A: MEAN EXIT TIME FOR $a > 1/2$

To complete the solution for the mean exit time, we substitute $t_{\mathrm{in}} + a$ and t_{out} quoted in Eq. (12) into Eqs. (10) and (11). The former equation becomes

$$\mathcal{T} = 1 + \frac{1}{a}\int_0^{1-a} t_{\mathrm{out}}(x')\,dx' + \frac{2a-1}{2a}\,\mathcal{T}\,. \tag{A.1}$$

This can be rewritten as

$$\mathcal{T} = 2a + 2T\,, \tag{A.2}$$

where

$$\begin{aligned} T &= \int_0^{1-a} t_{\mathrm{out}}(x')\,dx' \\ &= \int_0^{1-a} \tau_1 \cos\left(\frac{x'}{2a}\right) + \tau_2 \sin\left(\frac{x'}{2a}\right)\,dx' \\ &= 2a\tau_1 \sin\left(\frac{1-a}{2a}\right) - 2a\tau_2 \cos\left(\frac{1-a}{2a}\right)\,. \end{aligned} \tag{A.3}$$

For Eq. (11), we evaluate it at $x = 0$. This gives

$$t_{\mathrm{out}}(0) = 1 + \frac{1}{2a}\int_0^{a} t(x')\,dx'\,.$$

Then using Eq. (12) for $t_{\mathrm{out}}(0)$ and separating the integral into the inner and outer subintervals, we have

$$\tau_1 = 1 + \frac{1}{2a}\,T + \left(1 - \frac{1}{2a}\right)\mathcal{T}\,. \tag{A.4}$$

Finally, we equate t_{in} and t_{out} at $x = 1 - a$, where the inner and outer subintervals meet. This gives

$$\mathcal{T} = \tau_1 \cos\left(\frac{1-a}{2a}\right) + \tau_2 \sin\left(\frac{1-a}{2a}\right)\,. \tag{A.5}$$

The conditions (A.2), (A.4), and (A.5) provide the three independent equations

$$\mathcal{T} = 2a + 4a\tau_1 \sin\left(\frac{1-a}{2a}\right) - 4a\tau_2\left[\cos\left(\frac{1-a}{2a}\right) - 1\right]$$

$$\tau_1 = 1 + a\tau_1 \sin\left(\frac{1-a}{2a}\right) - \tau_2\left[\cos\left(\frac{1-a}{2a}\right) - 1\right] + \left(1 - \frac{1}{2a}\right)$$

$$\mathcal{T} = \tau_1 \cos\left(\frac{1-a}{2a}\right) + \tau_2 \cos\left(\frac{1-a}{2a}\right).$$

for the unknown coefficients $\mathcal{T}$, τ_1, and τ_2.

To express the solution succinctly, let $z \equiv \frac{1-a}{2a}$ and $\epsilon \equiv \left(1 - \frac{1}{2a}\right)$. Further, define

$$\alpha = 4a \sin z - \cos z \qquad \beta = 4a(\cos z - 1) + \sin z$$

$$\gamma = \sin z - 1 + \epsilon \cos z \quad \delta = \cos z - 1 - \epsilon \sin z\,.$$

Then the constants are

$$\tau_1 = \frac{2a\delta - \beta}{\gamma\beta - \alpha\delta} \quad \tau_2 = \frac{2a + \alpha\tau_1}{\beta} \quad \mathcal{T} = \tau_1 \cos z + \tau_2 \sin z. \tag{A.6}$$

The result of this solution is shown in Fig. 3.

APPENDIX B: COEFFICIENTS OF THE EXIT PROBABILITY FOR $a \to 0$

At $x = a$, $R(x)$ is continuous, while $R'(x)$ has a discontinuity of size $R(0)$. Similarly, at $x = 2a$, both $R(x)$ and $R'(x)$ are continuous. From Eqs. (28) and (29), we thus have the conditions:

$$d_1 \cos\frac{1}{2} + d_2 \sin\frac{1}{2} + 3sa + \frac{1}{2}(1-s) = d_2 + d_3$$

$$-d_1 \sin\frac{1}{2} + d_2 \cos\frac{1}{2} = \frac{1}{2}(1-s)$$

$$-d_1 \sin\frac{1}{2} + d_2 \cos\frac{1}{2} + d_3 = +2sa + \frac{1}{2}(1-s)$$

$$-d_1 \cos\frac{1}{2} - d_2 \sin\frac{1}{2} = 2sa\,. \tag{B.1}$$

The solution to these equations are:

$$d_1 = \frac{(1-u)/2}{(u-1)\left(\frac{1}{4a}\cos\frac{1}{2} + \sin\frac{1}{2}\right) + \left(\cos\frac{1}{2} - \frac{1}{4a}\sin\frac{1}{2}\right)v}$$

$$s = -\frac{d_1}{2a}\cos\frac{1}{2} - \frac{d_2}{2a}\sin\frac{1}{2}$$

$$d_2 = -\frac{vd_1}{u-1}$$
$$d_3 = 2sa + \frac{1}{2}(1-s) + d_1 \sin\frac{1}{2} - d_2 \cos\frac{1}{2}\,, \quad \text{(B.2)}$$

with

$$u = \cos\frac{1}{2} + \frac{1}{2}\sin\frac{1}{2} \quad v = \frac{1}{2}\cos\frac{1}{2} - \sin\frac{1}{2}\,. \quad \text{(B.3)}$$

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