Random Multiplicative Processes and Multifractals

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1. Introduction

Random multiplicative processes abound in a variety of natural phenomena./1,2/ Specific examples include the distribution of incomes, body weights, rainfall, etc., in addition to the examples that will be treated in this paper. While random multiplicative processes are relatively ubiquitous, the essential properties of such processes are not as well appreciated as random additive processes (such as a random walk). For the latter case, the existence of the central limit theorem provides crucial information for understanding the asymptotic properties of a sum of random variables. From this theorem, one knows that short-range correlations in the sequence of random variables do not affect the asymptotic properties of the sum, and that one-parameter scaling holds. On the other hand, multiplicativity gives rise to multifractal scaling, and this property has been seen in a wide variety of contexts./3-9/

In this talk, I begin by discussing some basic asymptotic properties of a relatively trivial example of a random multiplicative process, in order to introduce some of the new features not present in a random additive processes. Next, I discuss three specific (and apparently disjoint) realizations of random multiplicative processes, and attempt to relate them by their scaling properties. Various aspects of this work were performed in collaboration with L. de Arcangelis, D. ben-Avraham, G. G. Batrouni, Z. Cheng, A. Coniglio, and B. Kahng.

In the example of the distribution of voltages (or currents) in a random resistor network, it is argued that a good qualitative picture is provided in terms of an uncorrelated multiplicative process, which leads to the voltages obeying a log-binomial distribution. The moments of this distribution do not scale in the conventional way, leading to multifractal scaling. This multifractal scaling is lost, however, if one takes the continuous limit, in order to obtain the more familiar log-normal distribution. Next, I discuss the problem of a random walk in a disordered medium containing traps and sources. The properties of this random walk are governed by a multiplicative process with short-range correlations, indicative of the degree of recurrence (or transience) in the underlying random walk. These short-range correlations give rise to different asymptotic properties than in an uncorrelated multiplicative process. Finally, I treat the kinetics of fragmentation, in which particles are repeatedly split up to yield a continuously evolving distribution of fragment sizes. The general conditions under which this distribution is described by the log-normal form, found in classical works on this subject, and when the distribution has a conventional scaling form, are examined.

2. A Pedagogical Example

Consider a sequence of independent, identically distributed random variables consisting of the numbers 2 and 1/2 with equal probability. If the number of elements in the

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sequence is N, what is the average value of the product, $\langle P \rangle$, of these N numbers? Notice the close analogy with a random walk, where one studies the properties of the sum of the random variables. Naively, one might guess that $\langle P \rangle \simeq 1$, since a typical sequence of variables contains approximately equal numbers of 2's and 1/2's. However, the probability that the product has the value $(1/2)^k \cdot 2^{N-k}$ equals $2^{-N}\binom{N}{k}$, so that averaging over all possible outcomes of the product yields $\langle P \rangle = (5/4)^N$. Thus we see that the mean value of the distribution of products, and the typical, or most probable value, P_{mp}, are very different as $N \to \infty$. It is also amusing to note that if 1/2's are twice as likely to occur as 2's in the sequence of random variables, then $\langle P \rangle = 1$, while $P_{\mathrm{mp}} = ((1/2)\cdot(1/2)\cdot 2)^{N/3} = 2^{-N/3}$. These features are contained in Furstenberg's theorem, which elucidates the eigenvalue spectrum of a product of random matrices./10/

The essential reason for the large disparity between $\langle P \rangle$ and P_{mp} is the relatively important role played by rare events. For example, a sequence consisting of all 2's occurs with an exponentially small probability, but the value of the product is exponentially large. Consequently, this extreme event makes a finite contribution to $\langle P \rangle$, and an even more dominant contribution to the higher moments of the product, as we shall discuss below. The fact that the average is dominated by rare events has important ramifications for numerical studies of systems governed by a random multiplicative process. If one can sample a small fraction of the total number of states of the system, then by definition, one will measure the most typical value of an observable. It is only when one has the resources to observe a finite fraction of all the states that the measurement will converge to the true average value of the observable. However, this task is impossible in a macroscopic system, so that simulations can only provide information about the most probable, rather than the average value in a multiplicative system./11/

Higher moments of the product, $\langle P^k \rangle$, can be computed from the probability distribution given above. A simple calculation gives $\langle P^k \rangle = (2^{k-1} + 2^{-k-1})^N$, which shows that the higher moments do not scale simply. That is, $\langle P^k \rangle$ cannot be written in the form $\langle P^k \rangle \sim \exp(a(k)N)$, with $a(k)$ linearly dependent on k. This loss of scaling stems from the long tail in the underlying distribution, which has as a consequence that each moment is governed by a distinct scale. However, as the order of the moment goes to ∞, the rare events come to dominate completely, and a conventional scaling picture is restored.

For this random multiplicative process, one is naturally tempted to take the continuum limit in order to have a convenient mathematical way to compute the higher moments of the distribution of products. Furthermore, it is natural to then expand the resulting distribution about the maximum value to arrive at the classical log-normal distribution. This is an ill-founded approximation for a random multiplicative process, however. The average value and higher moments of the product are dominated by the events at the tail of the distribution, and these tails are inadequately described by the Gaussian approximation. Thus it is important to keep in mind the limitations of standard approximation methods, when applied to multiplicative processes.

An additional intriguing feature of a random multiplicative process is the sensitivity of $\langle P \rangle$ to short-range correlations in the sequence of variables that are being multiplied. As an example, suppose that there are "no immediate reversals" in the sequence of variables. That is, when a 2 first appears in the sequence, the next element must also be a 2. Only after the second appearance of a 2 does the sequence become uncorrelated again. This nearest-neighbor correlation is equivalent to replacing the sequence of N correlated variables, which may be either 2 or 1/2, by a sequence of $N/2$ independent variables, which may be either 4 or 1/4. For this correlated sequence, $\langle P \rangle = (\sqrt{17/8})^N \gg (5/4)^N$ as $N \to \infty$. The increase in $\langle P \rangle$ compared to the uncorrelated process stems from the relatively larger role played by rare events. This simple but remarkable result shows that there is no analog of a central limit theorem for a multiplicative process.

We now discuss the applications of some of these basic ideas to physical problems which are governed by an underlying multiplicative process.

3. Voltage Distribution in Random Resistor Networks

Consider a random resistor network in which the bonds are identical resistors, and in which a fixed potential drop V is imposed across opposite edges of the network. The current-carrying properties of this system have been extensively investigated, both because of applications to transport in random media and because of intrinsic theoretical interest. One basic property of a random resistor network is its conductance, G, which vanishes in a power law fashion as the percolation threshold is approached. This quantity can be expressed in terms of the power dissipated across all the bonds of the network as $G = \sum v_{ij}^2$, where v_{ij} is the voltage drop across bond ij, and the conductance of each bond has been taken to be unity. Thus the conductance is the second moment of the distribution of voltage drops across the bonds in the network. From a statistical point of view, more fundamental information is contained in the distribution of voltage drops itself./5,6,9,12,13/

This distribution is also intimately connected with the geometrical structure of the conducting backbone of the percolating cluster. Near the percolation threshold, this backbone is self-similar, as it contains both "blobs" and singly connected bonds on all length scales. A simple hierarchical model has been introduced/5/ to capture these essential features, in which a single bond is iteratively replaced by a "unit cell" consisting of 2 bonds in parallel and a single bond at each end of the parallel combination. The voltage distribution can be computed exactly on this model, and it is clear that the individual bond voltages are governed by a multiplicative process in which a change of the length scale leads to a multiplicative change in the value of the bond voltage. Formally, the voltage distribution in an N^{th}-order hierarchy is given by the coefficient of $V = V_1^j V_2^{n-j}$ in the binomial expansion of $[2(V_1 + V_2)]^N$, where $V_1 = 1/5$, and $V_2 = 2/5$ for the hierarchical model. Thus the voltage distribution is a log-binomial, and this logarithmic feature gives rise to multifractal scaling properties, which account for the observations on random resistor networks at the percolation threshold very accurately.

Although the hierarchical model provides an appealing picture for the voltage distribution of percolating networks, it is inadequate in several respects. At a qualitative level, the hierarchical model gives a symmetric voltage distribution, on a logarithmic voltage scale, while the corresponding distribution for percolating clusters is strongly asymmetric. One attempt to remedy this discrepancy is by a generalization/5/ in which an arbitrary set of "elemental" voltages is iterated in a hierarchical fashion. This gives a distribution which is asymmetric for finite N, with the asymmetry vanishing only as $N \to \infty$. However, the degree of asymmetry is much weaker than what is observed in the random resistor network, even when the initial set of voltages are strongly asymmetric. This suggests that there is an additional mechanism, in addition to random scalar multiplication, which controls the small voltage tail of the voltage distribution./12,13/

4. Random Walk in a Random Multiplicative Environment

Consider a random walk in a random medium containing partially absorbing traps of strength t and also "sources" of strength s. This might be thought of as an idealized description of the core of a nuclear reactor, where radioactive neutron-producing nuclei play the role of sources, and neutron-absorbing moderator plays the role of traps. It is convenient to think of the "mass" of the random walk being multiplied by a factor t or s, respectively, upon encounters with a trap or a source. Thus the total mass of the walk after N steps is governed by random multiplication. However, the sequence of t's and s's in a given random walk trajectory in a low-dimensional space will be correlated, reflecting the propensity for the walker to visit a particular defect many times before reaching another defect. As mentioned in the introduction, these correlations in the first passage probability are relevant in determining the distribution of masses after N steps. We will use this model to illustrate the role of correlations in the asymptotic properties of random, but correlated, multiplicative processes.

For simplicity, consider a periodic distribution in which a single source is immediately followed by 2 traps on a one-dimensional chain. For this system, all possible sequences of s's and t's for N-step walks are generated by the matrix product $\mathrm{M}^N\binom{1}{0}$, where $\mathrm{M}=\begin{pmatrix}0 & s\\ 2t & t\end{pmatrix}$. By summing over all N, the generating function is then $G(s,t) = (1\ 1)(1-\mathrm{M})^{-1}\binom{1}{0} = (1+t)/(1-t-2st)$. The N^{th} term in the series representation of $G(s,t)$ can be found by residue calculus. To make contact with the example discussed in Sec. 2, consider the case $s = 2$ and $t = 1/2$. For these weights, the average value of the product is unity if the factors in the product are uncorrelated. For the random walk process, however, the average mass at the N^{th} step, *i.e.*, the average of the product of t's and s's, varies asymptotically as μ^N, with $\mu = (\sqrt{33/16} - 1/4)^{-1} \simeq 0.8431$. The difference between the two results for the product stems from the fact that there is a relatively higher probability of generating a sequence consisting of all t's in the random walk than in the uncorrelated process.

These results can be extended, in principle, to longer unit cells. In this case, correlation effects will become relatively more important, and it will be interesting to quantify how the increased correlation in the first passage probability is reflected in the asymptotic properties of the average mass. Studies in higher dimensions, and random distributions of sources and traps are called for.

5. Fragmentation Processes

Fragmentation is a ubiquitous kinetic phenomenon that underlies processes such as polymer degradation,/14/ breakup of liquid droplets,/15/ and the crushing of rocks./16/ For the latter case, a classical result is that the distribution of fragment sizes can have a log-normal form. From a simple-minded point of view, this arises because the size of a fragment is multiplied by some factor which is less than unity in a single breakup event. We now study the general conditions on the nature of a single breakup event that determines whether the distribution of fragment sizes is log-normal, or whether it assumes a simple scaling form.

To answer this question, we outline the rate equation approach for describing fragmentation. The evolution of the system is described by the integro-differential equation/17/

$$\frac{\partial}{\partial t}c(x,t) = -a(x)\,c(x,t) + \int_x^\infty c(y,t)\,a(y)\,f(x|y)\,dy\,, \tag{1}$$

where $c(x,t)$ is the concentration of x-mers at time t, $a(x)$ is their overall rate of breakup, and $f(x|y)$ is the rate at which x is produced from the breakup of y. We consider homogeneous kernels for which $a(x) = x^\lambda$, thus defining the homogeneity index λ. Homogeneity also implies that $f(x|y)$ has the form $y^{-1}\, b(x/y)$, while mass conservation imposes $\int_0^1 x\, b(x)\, dx = 1$.

We are interested in the distribution of fragment sizes in the small size limit, and there are two generic cases to consider./18/ One is where the reduced breakup kernel, $b(x)$, has a sharp cutoff at a non-zero value of x, corresponding to a definite lower limit on the relative reduction in size of a fragment in a single breakup event. The second general case is that of no cutoff, *e.g.*, $b(x) \sim x^\nu$, as $x \to 0$. To determine the behavior in the small size limit, we make the scaling ansatz for the cluster size distribution, $c(x,t) \sim s^{-2}\phi(x/s)$, where s is a typical cluster size./19,20/ Substituting this ansatz into the rate equations, and then computing moments of the resulting scaling equation leads to the moment relation,

$$m_{\alpha+\lambda} = \omega\, \frac{1-\alpha}{L_\alpha - 1}\, m_\alpha, \tag{2}$$

where $m_\alpha = \int_0^\infty x^\alpha\, \phi(x)\, dx$, and $L_\alpha = \int_0^1 x^\alpha\, b(x)\, dx$, are the moments of the scaling function and the reduced breakup kernel, respectively, and ω is a constant.

For a kernel, $b(x)$, which is zero below a cutoff value x_0, with $0 < x_0 < 1$, $L_{-\alpha}$ has the controlling factor $x_0^{-\alpha}$ for large α. By iterating (2), one finds, after some straightforward algebra, that the controlling factor of $m_{-\alpha}$ has the form $\exp(\frac{\alpha^2}{2\lambda} \ln x_0^{-1})$, which is the Mellin transform of the log-normal distribution. Thus by taking the inverse Mellin transform, the controlling factor of $\phi(x)$ is the log-normal form, $\phi(x) \sim \exp(-\frac{\lambda}{2\ln x_0}(\ln^2 x))$, as $x \to 0$. On the other hand, if there is no cutoff in the breakup kernel, a similar calculation yields that $\phi(x) \sim b(x) \sim x^\nu$, as $x \to 0$.

Thus, reducing the size of a fragment by a factor which is limited from below by x_0 at each breakup event leads to a log-normal tail in the cluster size distribution. However, if there is no limiting scale factor, a power law distribution is obtained. This dichotomy is reminiscent of the situation of Lévy flights, where a power law distribution of jump sizes leads to asymptotic distributions which are different from that of a simple random walk.

6. Conclusions

A wide variety of statistical mechanical problems are governed by random multiplicative processes. The theory of such processes is much less developed, and also apparently much less well-appreciated than the theory of random additive processes, such as random walks. Multiplicativity appears to be an essential ingredient in leading to multifractal scaling properties. The study of correlated multiplicative processes and their corresponding scaling properties should prove an interesting new area for further investigations.

7. Literature

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