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LETTER TO THE EDITOR

Rupture in the bubble model

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Abstract. We study the asymptotic scaling of the rupture strength threshold $S(W)$, on the width W, of a quasi-one-dimensional structure, the 'bubble' model. This model can mimic many features of the percolation transition on Euclidean lattices, while still being simple enough to be exactly soluble. The dependence of the system length L on W can be tuned to give rise to different behaviours of the rupture strength S.

Three regimes are found: for $L \ll \exp(\alpha W)$, $S(W)$ diverges as $W \to \infty$ while for $L \gg \exp(\alpha W)$, $S(W)$ vanishes as $W \to \infty$. In the transition regime, $L \sim \exp(\alpha W)$, the dependence of $S(W)$ depends on microscopic details.

Rupture properties of inhomogeneous media are difficult to determine due to the complex interplay between the role of the quenched disorder and the growth aspects of the rupture (de Arcangelis *et al* 1985, Sornette *et al* 1988). This last aspect depends upon the existence of screening and enhancement effects occurring on large defects and which can have a long range (Gilabert *et al* 1987). A complete unifying picture of the failure properties of random systems does not yet exist but an important step is to recognise that the study of breakdown problems can be roughly divided into two main areas:

(i) the statistics and statistical mechanics of breakdown in random media

(ii) the patterns emerging in rupture related to crack growth, fractal ramification, etc (Stanley and Ostrowsky (1988), see in particular the papers on the problem of rupture).

In the first area, which is the one addressed in this letter, the question of the behaviour of the strength S of the system as a function of its size and as a function of disorder is one of the most important theoretically, and also for obvious practical applications. In this respect, only very partial results exist, either based on numerical simulations (de Arcangelis *et al* 1988), or on bounds obtained from local configuration analysis with extreme order statistics (Duxbury *et al* 1986, Machta and Guyer 1987, Kahng *et al* 1988) or also from studies of special systems which can be solved exactly (Galambos 1978, Sornette 1989a, b) but which are far from realistic samples.

In this letter we present a study of the different regimes of failure for a quasi-one-dimensional structure, the 'bubble' model which was introduced as an exactly soluble description of percolation (Kahng *et al* 1987). In this system, there are L bundles in series; each containing W bonds in parallel. When each bond in the system is randomly occupied with probability p, the model exhibits a non-trivial percolation transition at

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a value of p_c which is strictly within the interval [0, 1] in the thermodynamic limit for the case where $L \sim \exp(W)$. However, owing to the one-dimensional nature of the model, all percolation properties, as well as electrical transport properties, can be obtained exactly.

To motivate our interest in this type of model, let us briefly summarise some known analytical results about failure on other related models.

(i) In one-dimensional (1D) systems, i.e. in models of links associated in series with randomly distributed breakdown thresholds, failure is associated with the statistics of extremes (Galambos 1978): the weakest part of a system submitted to a given load fails first and this corresponds to the macroscopic failure. In general, the strength of such a system decreases with its size as a power law.

(ii) In the other limit where W links with random rupture thresholds are associated in parallel, a central limit theorem holds (Galambos 1978, Sornette 1989a) and the system strength increases as the size of the system $S = W\theta$ where θ is a constant.

(iii) Hierarchical models can be thought of as self-similar mixtures of associations of links in series and in parallel. Exact results on the failure properties in hierarchical models have recently been presented (Sornette 1989b). An interesting phase diagram is found as a function of the disorder (the so-called order 'm' of the Weibull distribution of link failure thresholds is taken as the essential parameter for describing the disorder): for small disorder ($m > 2$), the system is strong and its strength increases as a power of its size, whereas for large disorder ($m < 2$), the system has a finite strength which does not increase as its size increases. However, the exponents which are computed, are not universal (they depend on m).

The bubble model (Kahng *et al* 1987) constitutes another very simple way of building a system by association of links in series and parallel and determining the relative role of these types of associations in the overall strength of the system. Thus consider L bundles, each containing W links associated in parallel, which are associated in series. Each bundle can therefore be thought of as an effective bond. We are interested in the problem of a stress S pulling at the two extremities of the total system. This stress will be felt by each bundle since they are associated in series and the stress propagates in a scalar fashion from one side to the other. As soon as one bundle fails, the system is deconnected and global rupture has occurred. Our problem is thus to determine the scaling of the *weakest* bundle of W bonds out of L bundles. The following regimes, which are derived below, are found:

(I) For $L < \exp(\alpha W)$ where $0 < \alpha < \infty$ is a constant, the system strength takes the asymptotic limit obtained for a single bundle of W links associated in parallel: $S = W\theta$ for large W. Since the strength S/W is finite in the thermodynamic limit, we term this regime of behaviour as 'strong'.

(II) For $L \sim \exp(\alpha W)$, $S = W\theta\beta(\alpha)$ is linear in W but with a multiplicative factor $0 \leq \beta(\alpha) \leq 1$ which is monotonically dependent upon α. This defines a 'transition' regime where the strength of the system is still finite, but can be vanishingly small as $\alpha \to \infty$. When the system becomes even more tenous, i.e. $L \gg \exp(\alpha W)$, then the system is 'weak', as the strength vanishes in the thermodynamic limit. This 'weak' behaviour can be further classified according to the rate at which the strength vanishes with the system size, as detailed below.

(III) For $L \sim \exp(\alpha W(\log W)^y)$ with $y > 0$, $S \sim W \exp[-(\alpha/m)(\log W)^y]$.

(IV) For $L \sim \exp(\alpha W^y)$, $S \sim W \exp[-(\alpha/m) W^{y-1}]$.

One possible application of these ideas may concern the strength of cables. According to our results, when a cable becomes too long it becomes highly susceptible to failure. This is the well known size effect in rupture (Smith and Phoenix 1981, Phoenix and Smith 1989). When this point is reached, more strands in parallel would need to be added to ensure that the cable is not near its failure threshold. In fact, the bubble model suggests precise scaling laws for the association of long strands of fibres which would correspond to a given rated failure threshold.

Let us now outline the different regimes of strength versus size relations that can occur in the bubble model. As a preliminary, consider a single bundle of fibres. We denote by $X_1, X_2, \ldots, X_W$ the strength of the individual links of a given bundle and suppose that they are independent identically distributed random variables with the cumulative probability distribution $P(X_j < x) = F(x)$. In the following, we will consider the Weibull distribution defined by (3), which is usually taken to fit experimental results on material strengths. Furthermore, assume that the total load S is distributed equally on the individual links and let us consider the 'democratic' transfer mechanism: when a thread fails, the stress on the failed link is supposed to be transferred 'democratically' to the other links. This problem can be solved exactly using the theory of extreme order statistics (Galambos 1978) whose results can be stated as a theorem.

Theorem (Galambos 1978). Let $F(x)$ be absolutely continuous with finite second moment. Assume that $x(1-F(x))$ has a unique maximum at $x = x_0 > 0$ and let $\theta = x_0(1-F(x_0))$. If, in a neighbourhood of x_0, $F(x)$ has a positive continuous second derivative, then

$$\lim_{W\to\infty} P(S_W < W\theta + x\sqrt{W}) = (2\pi)^{-1/2}\int_{-\infty}^{x} \exp(-t^2/2)\,\mathrm{d}t. \tag{1}$$

It is remarkable that the asymptotic properties of the global failure threshold of the bundle can be reduced to the so-called central limit problem. Expression (1) implies that the probability of the global failure threshold S_W being equal to S is

$$P(S_W = S) \sim (2\pi W)^{-1/2} \exp[-(S - W\theta)^2/2W]. \tag{2}$$

The density distribution of global failure threshold is normally distributed around the maximum $S = W\theta$ with a dispersion scaling as $\sqrt{W}$. To illustrate this point, let us

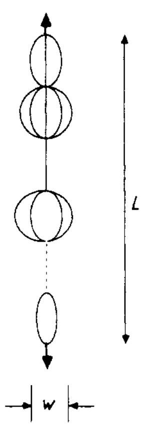

Figure 1. Schematic of the bubble model. In the mechanical case, the elongational stress is applied at the ends of the chain, while in the electrical problem the current flows between these two ends.

consider the usual Weibull distribution (Jayatilaka 1979) for failure threshold of each individual link

$$F(x) = 1 - \exp[-(x/\lambda)^m]. \tag{3}$$

Then,

$$x_0/\lambda = (1/m)^{1/m} \tag{4}$$

$$\theta/\lambda = (1/m)^{1/m} \exp(-1/m) \tag{5}$$

which are weakly dependent upon the order m of the Weibull distribution. For $m = 2$, one finds $\theta/\lambda = 0.429$. This value should be contrasted with the average strength $\langle x \rangle/\lambda = 0.886$ which is obtained from (3) for the same value $m = 2$.

Now, we consider the bubble model, i.e. a chain of bundles. First, for the case $L \ll \exp(\alpha W)$, we can compute from (2) the probability that all rupture thresholds S_W for the L bundles of W bonds are larger than some value $S_{\min}$: $P_L(S_{\min}) = (1 - \int_0^{S_{\min}} P(S)\, \mathrm{d}S)^L$. The strength $S_{\min}$ of the weakest bundle is thus determined by

$$L \int_0^{S_{\min}} P(S)\, \mathrm{d}S \sim 1 \tag{6}$$

which gives, for the Gaussian distribution (2),

$$S_{\min} = W\theta - \sqrt{2}\,(W \log L)^{1/2} + \mathrm{O}(\log(\log L)). \tag{7}$$

Note the existence of a correction $\mathrm{O}(\log(\log L))$ in the RHS of (7) which arises naturally in the asymptotic expansion of the error function implicit in (6). From (7), as long as $L \ll \exp(\alpha W)$ for all positive α, the correction $\sqrt{2}\,(W \log L)^{1/2}$ to the average strength $W\theta$ is very small. In this case, the global strength of the 'bubble' system, which is the strength of its weakest bundle, scales as

$$S_{W,L} = W\theta. \tag{8}$$

Thus the system is 'strong', as the global strength approaches a finite value in the thermodynamic limit.

Let us now turn to the case $L = \exp(\alpha W)$. Equation (7) indicates that, in this case, the first correction term is of order W, so that the strength of the system may vanish in the thermodynamic limit. Therefore let us pose the ansatz $S_{\min} = W\theta\beta$ with $0 \leq \beta \leq 1$. The probability that the strength of a bundle of W bonds takes this value is obtained from (2) and scales as

$$P_W(S_{\min}) \sim \exp[-(1-\beta)^2 \theta^2 W/2]. \tag{9}$$

Thus, $P_W(S_{\min})$ is exponentially small in W. This is the reason why L must be exponentially large in W in order to compensate this small factor. From the condition (6) with $L \sim \exp(\alpha W)$, we obtain

$$\beta(\alpha) = 1 - (2\alpha)^{1/2} x_0/\theta. \tag{10}$$

One can recover this result (10) from the previous analysis by substituting the expression $L \sim \exp(\alpha W)$ in (7). We obtain $S_{\min} = W\theta\beta$ with $\beta = 1 - h\alpha^{1/2} x_0/\theta$. Comparing with (10), this fixes the value of the coefficient h to be equal to $\sqrt{2}$ (cf equation (7)).

Surely this behaviour is valid for sufficiently small α such that $S_{\min}$ does not explore too far the tail of the Gaussian distribution (2). For large values of α, the previous type of arguments using the Gaussian asymptotic expression for $P_W(S_{\min})$ cannot be used anymore since very large fluctuations occur which sample the extreme tail of $P_W(S)$. This tail of course deviates significantly from a Gaussian law.

In order to tackle this large-fluctuation regime, we propose the approximation according to which the strength of the weakest bundle is controlled by the occurrence of the order of W bonds of strength less than $x(\alpha)$. In other words, in the bundle ensemble, we are looking at the existence of a bundle whose W bonds have all their strength less than or equal to $x(\alpha)$. The corresponding probability for finding such a bundle is $P_W(S_{\min}) = [F(x(\alpha))]^W \simeq \exp\{-W\log[1/F(x(\alpha))]\}$ for small $x(\alpha)$ where the relation between $S_{\min}$ and $x(\alpha)$ is to be determined below. The minimum value of $x(\alpha)$ for which such an event occurs among L bundles is determined from condition (6) which yields

$$F(x) = \mathrm{e}^{-\alpha} \tag{11}$$

which determines x as a function of α. Note that this approximation cannot be used in the previous 'small-fluctuation' regime since its validity relies on the smallness of $x(\alpha)$. The strength $S_{\min}$ of the bundle having W bonds with strength distributed according to the cumulative distribution $\underline{F}(x) = F(x)/F(x(\alpha))$ for $x \leqslant x(\alpha)$ and $\underline{F}(x) = 1$ for $x > x(\alpha)$ is given by the theorem stated above, which yields

$$S_{\min} = Wx(\alpha)[1 - F(x(\alpha))] = W\,\mathrm{e}^{-\alpha/m}(1 - \mathrm{e}^{-\alpha}). \tag{12}$$

In sum,

$$S_{W,L} \approx \begin{cases} W\theta[1 - (2\alpha)^{1/2} x_0/\theta] & \text{for small } \alpha \quad (13) \\ W\,\mathrm{e}^{-\alpha/m} & \text{for large } \alpha. \quad (14) \end{cases}$$

For intermediate values of α, we can use the method of asymptotic matching (Bender and Orzsag 1978) between the two solutions (13) and (14). The results are represented in figure 2. This shows that $S_{W,L}$ still scales as W but with a numerical factor which goes to zero as $\alpha = (\log L)/W \to \infty$.

Now we investigate the strength of the bubble model in the situation where the length L grows faster than exponentially in the width W. There are two natural cases to consider: one in which (roughly speaking) $L \sim W^{\alpha W}$, and the other in which $L \sim \exp(\alpha W^y)$ with $y > 1$. In the former case, we write the length-width relation as $L = \exp(\alpha W(\log W)^y)$ with $y > 0$. Using condition (6), the weakest bundle out of $L \sim \exp[\alpha W(\log W)^y]$ bundles has a strength $S_{\min}$ such that the probability that any bundle has a strength $S_{\min}$ scales as

$$P_W(S_{\min}) \sim \exp[-\alpha W(\log W)^y]. \tag{15}$$

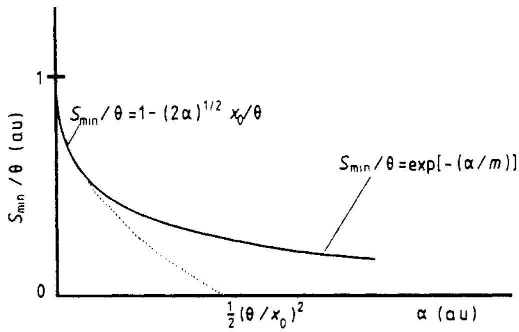

Figure 2. Variation of the rupture threshold in the bubble model as a function of α, where α is defined by $L \sim \exp(\alpha W)$, which gives the scaling of the total chain length L with respect to the number W of links in each bundle.

The problem is thus to determine the type of extremely rare events in an ensemble of bundles such that the corresponding probability is given by (15). We now argue that, for the Weibull distribution, this corresponds to

$$S_{\min} \sim W \exp[-(\alpha/m)(\log W)^y]. \tag{16}$$

Note in particular, that for $y=1$, $S=W^{\zeta}$ with an exponent $-\infty \leq \zeta(\alpha)=1-\alpha/m \leq 1$ which is a monotonic function of α.

Assuming, as in the previous discussion, that the strength of the weakest bundle will be controlled by the occurrence of the order of W bonds of strength less than some $x=x(\alpha, y)$, the corresponding probability is

$$P_W(S_{\min}) = [F(x)]^W = \exp(-mW \log x/\lambda) \tag{17}$$

valid for the Weibull distribution (3). Identifying (17) with (15) yields $x(\alpha, y)/\lambda \sim \exp[-(\alpha/m)(\log W)^y]$ for large W. Since $S_{\min}$ is of the form

$$S_{\min} = Wx(1-F(x)) \tag{18}$$

this yields the result announced in (16). In the case $y=1$, $S_{\min}=W^{\zeta}$ with $\zeta = 1-\alpha/m$. Therefore, for $\alpha \leq m$, $\zeta \geq 0$ and the system is strong since $S_{W,L}$ increases as W increases. However, if $\alpha > m$, $\zeta < 0$ and the system is weak since $S_{W,L}$ decreases to zero as W increases. This last regime is reminiscent of the behaviour of a chain of bonds associated in series (Galambos 1978).

For the case $L=\exp(\alpha W^y)$ with $y \geq 1$, applying step by step the previous line of reasoning, it is easy to show that the strength of the L bundles, and therefore the total strength of the system, is of the form

$$S_{\min} = S_{W,L} \sim W \exp[-(\alpha/m) W^{y-1}] \tag{19}$$

for the Weibull distribution. Note that we recover the previous results for $y=1$.

We have studied the asymptotic scaling of the rupture strength threshold $S(W)$ as a function of the system width W, of a quasi-one-dimensional structure, the 'bubble' model in which the system length $L(W)$ is a function of the system width W. The model exhibits a clear transition between a 'strong' behaviour for $L \leq \exp(\alpha W)$ characterised by a strength per bond S/W going to a constant in the thermodynamic limit and a 'weak' behaviour for $L \gg \exp(\alpha W)$ for which S/W goes to zero asymptotically. Figure 3 gives a summary of the different regimes found for the rupture threshold.

We have considered the case of a constant applied external current (respectively stress) in the electrical (respectively mechanical) version of the problem. The case of a constant applied voltage (respectively elongation) can be obtained straightforwardly from the previous discussion. As bonds break down, the external current flow is reduced according to the evolving resistance of the system. Now, it is easy to show that for almost all bundles, there subsists a finite fraction of the links which have survived. Indeed, the probability that a finite number of links holds the whole current is so vanishingly small that this probability does not control the behaviour of the resistance, which is an average over L resistances! From the current intensity threshold as a function of L, which has been discussed in this letter, and the average value of the electrical resistance R which is well behaved and changes only by a finite multiplicative factor, one obtains the voltage failure threshold by the simple formula $V=RI$.

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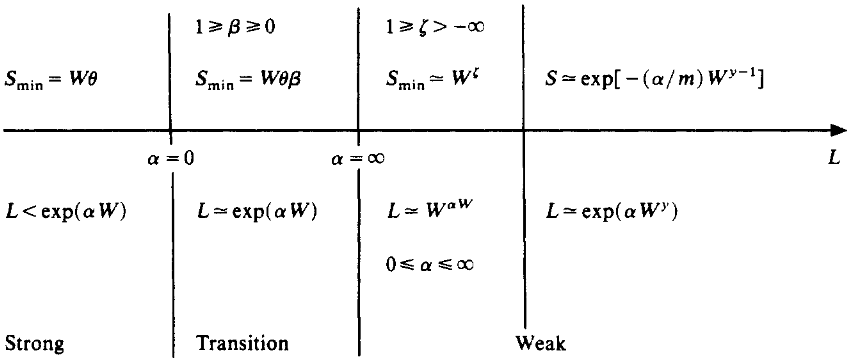

Figure 3. Summary of the different regimes found for the rupture threshold depending upon the scaling of the total chain length L with respect to the number W of links in each bundle. Note the transition from the 'strong' regime for $L < \exp(\alpha W)$ where $S_{min} = \theta W$ to the 'weak' regime for $L \gg \exp(\alpha W)$ where S_{min}/W goes to zero asymptotically for large W.

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