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LETTER TO THE EDITOR

Logarithmic voltage anomalies in random resistor networks

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Abstract. We investigate the behaviour of the maximum voltage drop across the bonds in a random resistor network above the percolation threshold. On the basis of numerical simulations on randomly diluted $L \times L$ square lattice networks, we find that the average value of the maximum voltage, $\langle V_{max}(p; L)\rangle$, exhibits a peak as a function of the bond concentration, p, which is located above the percolation threshold. This peak value appears to grow logarithmically with L, while the location of the peak appears to approach the percolation threshold very slowly as L increases. To help understand these results, we introduce the 'bubble' model, a quasi-one-dimensional structure in which the system length varies exponentially in the system width. This model is exactly soluble and a rather good description of percolation in systems of greater than one dimension can be obtained. Moreover, within the bubble model the peak value of the maximum voltage increases as $\ln L$, while the peak location approaches the percolation threshold as $\ln(\ln L)/\ln L$, in good agreement with our numerical results.

Very recently, the behaviour of the random resistor network has been the focus of renewed attention [1-5]. Part of the reason for this revival of interest stems from the fact that the distribution of voltage drops across the bonds in the network exhibits novel 'multifractal' aspects. This multifractality arises because there is no unique voltage (in a scaling sense) which controls the behaviour of all the moments of the voltage distribution. Consequently, many well established scaling ideas needed to be reformulated [1-6] to take account of this multiplicity of voltage scales inherent in such a distribution.

Another reason for considering the voltage distribution is that it underlies a variety of fundamental physical processes on linear networks. For example, the second moment gives the conductance of a resistor network, the fourth moment gives the magnitude of resistance noise [2], while the negative voltage moments are relevant for understanding transport and hydrodynamic dispersion processes in flow through random tube networks [7]. Another very important application arises from the investigation of the *maximum* voltage drop in the network [8-10], as this governs the failure of a random fuse network, and this, in turn, has been invoked as a model for the breaking of disordered media. Recently, analytical arguments [9, 10] and numerical simulations [9] have been presented which suggest that the configurational average of this maximal voltage drop, $\langle V_{max}(p; L)\rangle$, increases as $\ln L$, under conditions of fixed potential gradient, where L is the linear dimension of the system. This result appears to hold for any value of the bond concentration p greater than the percolation threshold p_c.

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A striking implication of this basic result is that the breaking strength of a random network of fuses decreases as $1/\ln L$.

In this letter, we investigate the concentration and size dependence of $\langle V_{max}(p; L)\rangle$ in greater detail. In addition to verifying the logarithmic behaviour found previously, we also find that for fixed L, $\langle V_{max}(p; L)\rangle$ is a non-monotonic function of p, exhibiting a peak at a value $p_{peak}(L)$, which is above the percolation threshold. As $L \to \infty$, $p_{peak}(L)$ does appear to approach the percolation threshold, although only very slowly. Physically, this peak originates from two competing effects as $p \to p_c$ from above. On the one hand, the total current flow in the network is reduced (under conditions of fixed external potential) and this clearly reduces the current flow in each bond of the network. On the other hand, as p decreases, local 'bottlenecks' in the network are formed which funnel current on progressively larger length scales. This gives rise to a strong local enhancement of current flow. The competition between these two effects is the mechanism that leads to a peak in $\langle V_{max}(p; L)\rangle$ for fixed L, which is above the percolation threshold.

This non-monotonicity may also have interesting experimental ramifications for various types of breakdown processes, as a finite-sized random resistor network should be most susceptible to 'burning out' at $p_{peak}(L)$, which is above the percolation threshold. Such an effect has, in fact, been observed (but not reported previously) in a study of the conductivity of a random resistor-diode network [11]. In the finite-size system studied, a burning point occured at $p \simeq 0.767$, compared to a threshold value of $p_c \simeq 0.578$. At this burning point, solder connections for circuit elements within the bottleneck melted and these solder joints began to smoke. This behaviour was reproducible both by randomly depleting the network to pass through the burning point, and then by rejoining connections in the reverse order that they were severed. Although the directionality constraint imposed by the diodes certainly enhances the bottleneck effect, similar breakdown behaviour can be anticipated to occur in suitably designed random resistor networks.

To provide a simple physical picture and an analytical account for these results, we introduce the 'bubble' model, a simple but exactly soluble quasi-one-dimensional structure which exhibits features of the percolation transition for systems in greater than one dimension. On this model, it is possible to investigate the behaviour of $\langle V_{max}(p; L)\rangle$, as well as a variety of related percolation properties, analytically. We find that for a fixed value of p, $\langle V_{max}(p; L)\rangle$ indeed increases as $\ln L$. More interestingly, for a fixed value of L, $\langle V_{max}(p; L)\rangle$ has a peak as a function of p, whose position approaches the percolation threshold as $\ln(\ln L)/\ln L$. While we cannot test the correctness of a double logarithm numerically, these predictions do provide a good account for our simulation results on random resistor networks on the square lattice.

Let us begin by describing our numerical results. We studied randomly diluted $L \times L$ square-lattice resistor networks, in which the opposite edges of the network were connected to bus bars, with one bus bar at potential $V = 0$ and the other at $V = L$. The current flow problem was solved numerically by using the Fourier-accelerated conjugate gradient scheme, introduced by one of the authors [3]. We considered networks of linear dimension up to $L = 128$, for a range of p values between $p_c(=\frac{1}{2})$ and 1. For each value of p, we obtained $\langle V_{max}(p; L)\rangle$ by averaging over 1000 spanning realisations for $L = 8$, 16 and 32, while for $L = 64$ and $L = 128$, averages were performed over 500 and 200 spanning realisations, respectively. Since the p dependence is smooth, we could estimate the location of the peak, $p_{peak}(L)$, and its magnitude, $V_{peak}(L)$, fairly accurately and the results are shown in figures 1 and 2. Evidently, $p_{peak}(L)$ is slowly

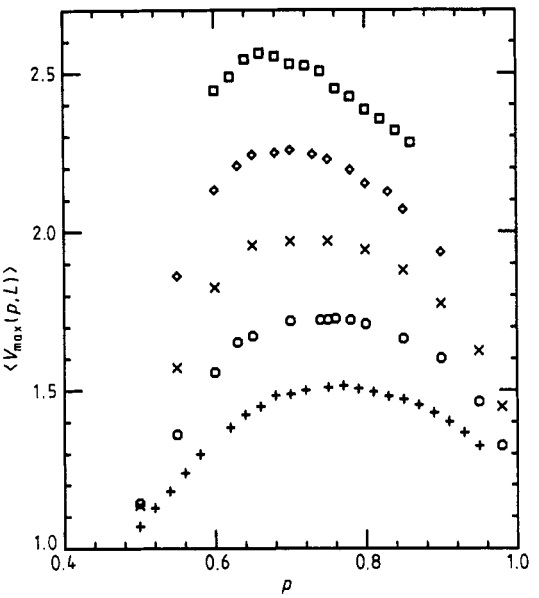

Figure 1. Numerical data for $\langle V_{max}(p; L)\rangle$ as a function p for various values of L. The data for $L=8$ (+), $L=16$ (○), $L=32$ (×), $L=64$ (◇), $L=128$ (□) are shown. Error bars are within the data points.

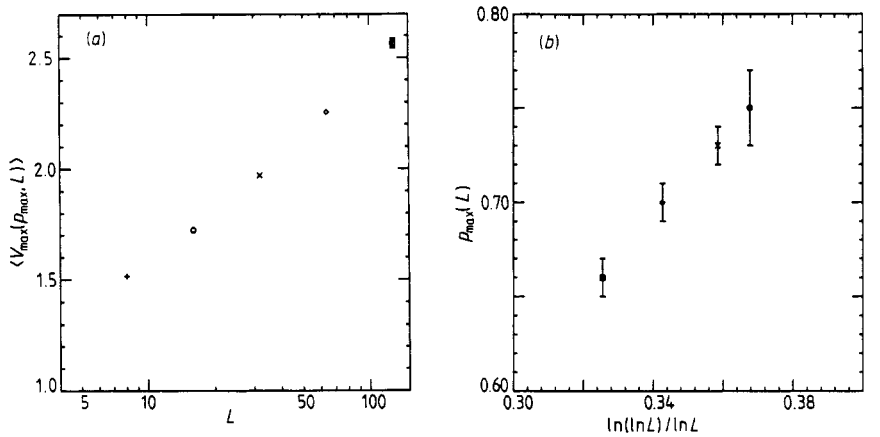

Figure 2. (*a*) Estimated values of $V_{peak}(L)$ as a function of L. Error bars are within the data points for $L<128$. (*b*) estimated values of p_{peak} (L) plotted as a function of $\ln(\ln L)/\ln L$. Error bars reflect the uncertainty in determining the location of $p_{peak}(L)$. The data symbols are the same as in figure 1.

approaching p_c as $L\to\infty$, while $V_{peak}(L)$ is increasing with L at a rate which is consistent with logarithmic growth, if the curvature in the data for small values of L is interpreted as a finite-size effect.

We now show how these peculiar results may be accounted for in terms of the 'bubble' model. To define this model, consider a percolation problem on a rectangular lattice of length L and width w in which the vertical bonds are either present with probability p, or absent with probability $1-p$, while the horizontal bonds are always

present. The resulting structure is effectively one dimensional as shown in figure 3. (This type of geometrical constraint has been invoked previously in order to develop exact renormalisation group recursion relations for percolation [12] and self-avoiding walks [13] in $d = 1 + \varepsilon$ dimensions.) In treating current flow on the bubble model, each occcupied vertical bond will be assigned a unit resistance and all the horizontal bonds will be considered as having zero resistance.

The nature of the percolation transition in the bubble model can be understood by considering the probability that a path spans the system. This probability is simply given by

$$p' = [1-(1-p)^w]^L \tag{1}$$

and as L, $w \to \infty$, this may be written approximately as

$$p' \sim \exp[-L\exp(-pw)]. \tag{2}$$

This spanning probability sharply changes in value from 0 to 1 at a value $p_c \sim \ln L/w$. Thus by choosing $L \sim e^w$, one generates a model which has a fixed point, i.e. a percolation transition, at a value of p that lies strictly within the interval $(0, 1)$ as $L \to \infty$. This non-trivial location of the transition arises because of the balancing influence of series and parallel paths in the asymptotic limit. Each bubble can be thought of as an effective bond which is present with probability $p_B = 1-(1-p)^w$, a quantity which is exponentially close to unity as $w \to \infty$. Consequently, the spanning probability, $p' = p_B^L$, can be freely varied between 0 and 1, by tuning the manner in which L scales with w. Owing to the one-dimensional nature of the model, all percolation properties can be obtained exactly. However, we will present these details elsewhere and concentrate instead on understanding the nature of the voltage anomalies through the study of the bubble model.

For the ensuing discussion, it is convenient to choose $L = 2^w$, as this gives rise to a percolation threshold at $p_c = \frac{1}{2}$ as $L \to \infty$. We will apply the results of this version of the bubble model in order to account for various features observed in our simulations of the square-lattice random resistor network. Parenthetically, it is noteworthy that percolation on the bubble model shares some features of percolation in a wedge [14] whose width grows logarithmically with the distance from the apex of the wedge. This

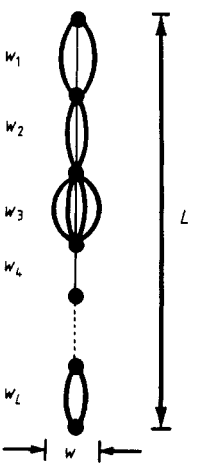

Figure 3. Schematic picture of the bubble model.

logarithmic relation between the width and length is the minimal growth law needed in order to have a value of p_c strictly less than 1.

The behaviour of $\langle V_{\max}(p; L)\rangle$ as a function of p and L in the bubble model is governed by two factors: the average current flow and the width of the expected narrowest bottleneck. For obtaining the average current flow, note that the average resistance of the bubble model is simply equal to

$$p'R = \sum_{\text{config}}{}' p_{\text{config}}\left(\frac{1}{w_1}+\frac{1}{w_2}+\ldots+\frac{1}{w_L}\right). \tag{3}$$

Here w_i is the number of occupied bonds in the ith bubble, the prime on the summation denotes that it is taken only over spanning configurations ($w_i > 0$ for all i), while $p_{\text{config}} = p_1 p_2 \ldots p_L$ is the product of configurational probabilities for each bubble. Due to this product form, we may write

$$\begin{aligned} R &= \sum_{\text{config}}{}' \frac{1}{p'}\left(\frac{p_1}{w_1}+\frac{p_2}{w_2}+\ldots+\frac{p_L}{w_L}\right) \\ &= \frac{L}{p'}\langle w_i^{-1}\rangle' \end{aligned} \tag{4}$$

where the average $\langle w_i^{-1}\rangle'$ again excludes the possibility that $w_i = 0$. As $w \to \infty$, it is clear that, as long as the probability of finding a bubble width w_i vanishes faster than linearly in w_i as $w_i \to 0$, then $\langle w_i^{-1}\rangle' \sim \langle w_i\rangle^{-1}$ and the latter quantity equals $1/pw$. Thus we obtain

$$R \simeq \frac{1}{p'}\frac{L}{pw} \tag{5}$$

or, conversely, the average conductance of the system is

$$G \simeq p'\frac{pw}{L}. \tag{6}$$

The conductance initially rises very sharply from 0 within a narrow transition region near p_c, where p' is rapidly increasing. However, once this rapid increase in p' has passed, G subsequently increases linearly with p.

Now consider the nature of bottlenecks in the bubble model. For this purpose, we appeal to Coniglio's theorem [15], which states that $\partial p'/\partial p$ equals the average number of singly connected bonds in the system. From (1), we obtain

$$\begin{aligned} \frac{\partial p'}{\partial p} &= Lw(1-q^w)^{L-1}q^{w-1} \\ &= \frac{wLq^w}{(1-q^w)}\frac{p'}{q} \end{aligned} \tag{7a}$$

where $q = 1-p$. Therefore at the fixed point (located at $\frac{1}{2}$, when $L = 2^w$) we find

$$\frac{\partial p'}{\partial p} = w + \mathcal{O}(e^{-w}) \sim \ln L. \tag{7b}$$

Thus at p_c, the bottlenecking effect is quite severe as there are $w \sim \ln L$ singly connected bonds, on average. However, the effect of this extreme bottlenecking at the percolation threshold is more than compensated for by the vanishing of the total current flow at this point.

To locate the burning point, we therefore require the behaviour of the bottlenecks above the percolation threshold. A particularly relevant quantity to consider is the concentration dependence of the probability that the width of the narrowest bottleneck in the system is equal to unity, $P_1(p, L)$. Questions of this nature have been studied previously through numerical simulations of percolation on the square lattice [16]. From Coniglio's theorem, it is clear that $P_1(p; L)$ is essentially equal to unity for $p \leq p_c$, but that $P_1(p; L)$ will decrease some finite distance above p_c, where the width of the narrowest bottleneck begins to grow, corresponding to the disappearance of singly connected bonds (figure 4). When this occurs, the total current flow in the network will split among the several bonds that comprise this narrowest bottleneck and the maximum single bond current flow, or equivalently, $\langle V_{max}(p; L)\rangle$, will decrease.

By elementary arguments

$$P_1(p; L) = 1 - \left(1 - \frac{wpq^{w-1}}{1-q^w}\right)^L$$

$$\sim 1 - \exp\left(-Lw\frac{p}{qp'^{1/L}}\exp(-wp)\right). \tag{8}$$

For $p < p_c$, $p' = 0$ leading to $P_1(p; L) = 1 - \mathcal{O}(\exp(-\exp(L))$, while for $p = p_c$, one finds $P_1(p; L) \simeq 1 - e^{-w} \sim 1 - L^{-1}$. On the other hand, $P_1(p; L) \to 0$, when the argument of the outer exponential in (8) is of the order of unity and this turns out to occur at a value of

$$\hat{p} \sim p_c + \mathcal{O}(\ln w/w) \sim p_c + \mathcal{O}\left(\frac{\ln(\ln L)}{\ln L}\right).$$

For larger values of p, $P_1(p; L)$ is effectively equal to 0, corresponding to the width of the narrowest bottleneck being larger than 1. We therefore anticipate that $p_{peak}(L)$ is essentially equal to $\hat{p}$. Although the location of $\hat{p}$ given by the bubble model should not be expected to be quantitatively correct, as the scaling behaviour of $\partial p'/\partial p$ in the bubble model is logarithmic in L (7*b*), while it is a power law in L for lattice systems, the numerical data are in surprisingly good agreement with the bubble model prediction (figure 2(*b*)).

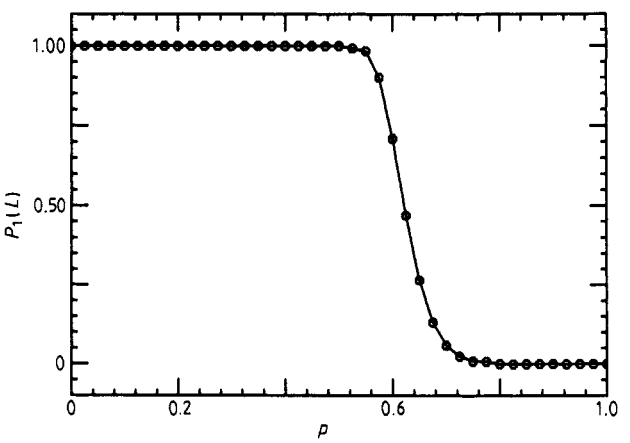

Figure 4. The dependence of $P_1(p, L)$, the probability that the narrowest bottleneck has width unity, against p for the bubble model with $w = 12$ and $L = 2^{12}$.

We now estimate the magnitude of the peak in $\langle V_{max}(p; L)\rangle$. Since the peak occurs at the largest value of p where a singly connected bond still exists, it is still true that G, $V_{max}(p; L)$, and I, where I is the total current flowing through the network, must coincide [1]. Then using (6), it immediately follows that

$$V_{peak}(L) \sim p_{peak}(L) w \sim p_c \ln L \tag{9a}$$

and this accords well with our data of figure 2(*a*).

It is also worth mentioning that this result can be readily adapted to treat electrical breakdown properties in the case where each bond is a fuse which will break down if the voltage across it is greater than some specified value, say 1. Then the breaking point is simply equal to the inverse of $V_{max}(p; L)$. The case where singly connected bonds still exist is particularly simple, as $V_{max}(p; L)$ coincides with the conductance of the system. In general, however, the behaviour of $V_{max}(p; L)$ is dominated by a linear dependence on w for all values of $p > p_c$. Therefore the breaking potential of the system, V_b, will scale as

$$\frac{V_b}{L} \sim \frac{1}{\ln L}. \tag{9b}$$

This result was obtained previously through a Lifshitz-type argument by Duxbury *et al* [9].

In conclusion, we have studied the behaviour of the average value of the maximum voltage drop, $\langle V_{max}(p; L)\rangle$, across the bonds in a random resistor network above the percolation threshold. On the basis of numerical simulations, $\langle V_{max}(p; L)\rangle$ exhibits a peak as a function of the bond concentration, whose value appears to increase logarithmically with the system size L. The location of the peak is above the percolation threshold for finite-size systems but appears to approach the percolation threshold at a rate which is consistent with the behaviour $\ln(\ln L)/\ln L$, as L increases. These unusual features can be accounted for within the bubble model. Although this model should not be taken literally as a description of percolation on lattice systems, it has the advantage of great simplicity, while still providing a very useful intuitive account of various geometrical properties of random systems above the percolation threshold. Furthermore, the bubble model predictions for the behaviour of $\langle V_{max}(p; L)\rangle$ are borne out rather nicely by numerical simulations of the random resistor network on the square lattice.

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