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LETTER TO THE EDITOR

Critical properties of directed self-avoiding walks

S Redner and I Majid

Center for Polymer Studies† and Department of Physics, Boston University, Boston, MA, USA 02215

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Abstract. The generating functions and mean displacements of various two-dimensional directed self-avoiding walk models are calculated exactly by a simple transfer-matrix approach. Asymptotically, we find $\langle R_{\|N}\rangle \sim \langle R_N^2\rangle^{1/2} \sim N$, and $\langle R_{\perp N}^2\rangle^{1/2} \sim N^{1/2}$, where N is the number of steps in the walk, and $\|$ and $\perp$ refer to projections of the displacement parallel and perpendicular to the preferred axis of the walk respectively. Some general properties of directed self-avoiding walks for arbitrary dimensions are discussed as well.

Consider a self-avoiding walk (SAW) which is restricted not to step in several particular directions. Such a *directed* SAW model was apparently first studied by Fisher and Sykes (1959) in order to derive rigorous bounds for the connective constant of isotropic SAWs. More recently, it has been realised that introducing a global bias in geometrical models leads to novel anisotropic critical behaviour (see e.g. Kinzel 1983, and references therein). Examples include directed percolation, directed lattice animals, and directed SAWs.

The latter model is the simplest of the three, but it has received little theoretical attention, perhaps because it is so simple. A directed SAW can be decomposed as a forward walk along the preferred direction, and as a random walk perpendicular to this direction. Accordingly, the mean longitudinal displacement, $\langle R_{\|N}\rangle$, should vary linearly with N, while the root-mean-square perpendicular displacement, $\langle R_{\perp N}^2\rangle^{1/2}$, should vary as $N^{1/2}$ (see e.g. Nadal *et al* 1982). Very recently, however, Chakrabarti and Manna (1983) have claimed on the basis of series expansions that the mean end-to-end distance of a directed SAW varies as N^{ν} with $\nu = 0.86$, in contrast to the anisotropic behaviour mentioned above. In this letter, we calculate the generating functions and mean displacements exactly for a number of two-dimensional directed SAW models, and verify that the anisotropic behaviour is correct. In addition, we derive some exact properties for directed SAWs valid for all spatial dimensions. While our results are relatively straightforward, they may be of some pedagogical value, as well as serving to correct the result of Chakrabarti and Manna (1983).

We begin by considering directed SAWs on the square lattice. A two-choice model may be defined in which only steps upward or to the right are allowed with associated fugacities x and y respectively (figure 1(a)). This model is trivial as each step makes a projection of $+1/\sqrt{2}$ on the (1,1) diagonal, and a projection of $\pm 1/\sqrt{2}$ (randomly) perpendicular to the diagonal. Therefore $\langle R_{\|N}\rangle$ equals $N/\sqrt{2}$, while $\langle R_{\perp N}^2\rangle^{1/2}$ equals

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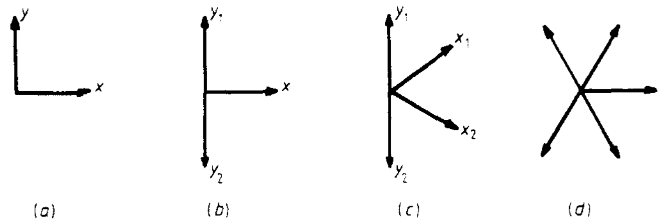

Figure 1. Directed SAW models. The fugacities associated with each step direction are also indicated. (*a*) Two-choice square lattice, (*b*) three-choice square lattice, (*c*) four-choice triangular lattice, (*d*) five-choice triangular lattice.

$(N/2)^{1/2}$. Since vertical and horizontal steps occur independently, the generating function is

$$G(x, y) = \sum_{n=0}^{\infty} (x+y)^n = 1/[1-(x+y)] \tag{1}$$

and for $x = y$, the critical point occurs at $x_c = \frac{1}{2}$.

The three-choice model, studied by Chakrabarti and Manna (1983), is considerably more interesting. This is a SAW restricted to step upward, to the right, or downward with associated fugacities y_1, x and y_2 respectively. In terms of the transfer matrix

$$T = \begin{pmatrix} x & x & x \\ y_1 & y_1 & 0 \\ y_2 & 0 & y_2 \end{pmatrix} \tag{2}$$

it may be readily verified that all configurations of three-choice directed SAWs of N steps are generated by

$$(1, 1, 1)T^N \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{3}$$

Therefore the generating function is

$$G(x, y_1, y_2) = (1, 1, 1)(1 + T + T^2 + \ldots) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = (1, 1, 1)(1-T)^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= (1 - y_1y_2)/[(1-y_1)(1-y_2) - x(1-y_1y_2)]. \tag{4}$$

This may be written alternatively as

$$G(x, y_1, y_2) = \frac{[(1-y_1y_2)/(1-y_1)(1-y_2)]}{1 - x[(1-y_1y_2)/(1-y_1)(1-y_2)]} \equiv G_{1D}/(1 - xG_{1D}). \tag{5}$$

Here G_{1D} denotes the generating function for SAWs in one dimension with step fugacities y_1 and y_2. This simple form shows that

$$\partial^n G(x, y_1, y_2)/\partial x^n|_{x=0} = (G_{1D})^{n+1}. \tag{6}$$

That is, the theorem of Liu and Stanley (1973) for anisotropic systems holds to all orders for directed systems. One may therefore write the generating function for directed SAWs in d dimensions (with one axis directed) in terms of the generating function for isotropic SAWs in $(d-1)$ dimensions in the same form as (5). Additionally, (4) is easily generalised to give the following generating function for directed SAWs in d dimensions with $d-1$ axes directed:

$$G_d(x, y_1, y_2) = (1-y_1y_2)/[(1-y_1)(1-y_2) - x(d-1)(1-y_1y_2)]. \tag{7}$$

Some of the above results were obtained originally by Fisher and Sykes (1959).

When $x = y_1 = y_2$, the generating function, equation (4), reduces to

$$G(x) = (1+x)/(1-2x-x^2) \equiv \sum_{N=0} a_N x^N \tag{8}$$

with a simple pole at $x_c = \sqrt{2}-1$. The coefficients a_N, the number of N-step directed SAWs, may be found by performing the contour integral

$$\frac{1}{2\pi i}\oint \frac{(1+x)\,dx}{(1-2x-x^2)x^{n+1}} \tag{9}$$

where the contour encloses only the pole at the origin. Evaluating the residues yields

$$a_N = [(\sqrt{2}+1)^{N+1} + (-1)^{N+1}(\sqrt{2}-1)^{N+1}]/2, \tag{10}$$

a result which may also be obtained directly from the recursion relation $a_N = 2a_{N-1} + a_{N-2}$.

The generating function for the total number of horizontal bonds in the ensemble of all N-step directed SAWs, $N_h(x, y_1, y_2)$, equals $x\,\partial G(x, y_1, y_2)/\partial x$. This gives

$$N_h(x, y_1, y_2) = x(1-y_1y_2)^2/[(1-y_1)(1-y_2) - x(1-y_1y_2)]^2. \tag{11}$$

For $x = y_1 = y_2$, the Nth term in the series representation, $N_h(x) \equiv \Sigma_N b_N x^N$, is found to be

$$b_N = \{(\sqrt{2}+1)^{N+1}[N\sqrt{2} - (\sqrt{2}-1)] + (-1)^{N+1}(\sqrt{2}-1)^{N+1}[N\sqrt{2} - (\sqrt{2}+1)]\}/4\sqrt{2}. \tag{12}$$

Here b_N is the number of horizontal steps in the ensemble of all N-step directed SAWs. Therefore the ratio b_N/a_N equals $\langle R_{\parallel N}\rangle$. As $N \to \infty$, we find

$$\langle R_{\parallel N}\rangle \sim (N-1)/2 + \tfrac{1}{2}\sqrt{2} \tag{13}$$

which agrees with enumeration results to five decimal places by order seven.

To find the mean-square displacement, we calculate the following generating function for the number of upward, downward and horizontal steps and their powers,

$$R^2(x, y_1, y_2) \equiv N_h^2(x, y_1, y_2) + (N_u(x, y_1, y_2) - N_d(x, y_1, y_2))^2, \tag{14}$$

by taking the requisite derivatives of $G(x, y_1, y_2)$. For $x = y_1 = y_2$, we obtain

$$R^2(x) = (3x - x^2 - 6x^3 - 2x^4 - x^5 - x^6)/(1-x)(1-2x-x^2)^3. \tag{15}$$

Upon extracting the Nth term in the series representation and dividing each term by a_N, we find the following asymptotic form for the mean-square displacement:

$$\langle R_N^2\rangle \sim (N+1)^2/4 + (N+1)(7\sqrt{2}-4)/8 + 5\sqrt{2}/4. \tag{16}$$

Therefore as $N \to \infty$, both $\langle R_{\parallel N}\rangle$ and $\langle R_N^2\rangle^{1/2}$ scale linearly with N. This disagrees with the results of Chakrabarti and Manna because they apparently did not perform systematic extrapolations. By order 14, a linear extrapolation of successive slopes of the logarithm of the mean end-to-end displacement, $\langle R_N \rangle$, against $\log N$ suggests a value $\nu \gtrsim 0.95$, and extrapolation based on 23 terms clearly converges to $\nu = 1$.

On the triangular lattice, the two- and three-choice models are essentially the same as the corresponding square lattice models. A four-choice model may be defined with fugacities x_1, x_2, y_1, y_2 for the four possible step directions (figure 1(*c*)). Defining the following transfer matrix,

$$\begin{pmatrix} x_1 & x_1 & x_1 & x_1 \\ x_2 & x_2 & x_2 & x_2 \\ y_1 & y_1 & y_1 & 0 \\ y_2 & y_2 & 0 & y_2 \end{pmatrix}, \tag{17}$$

and following (4), we obtain the generating function

$$G(x_1, x_2, y_1, y_2) = (1 - y_1y_2)/[(1-y_1)(1-y_2) - (x_1 + x_2)(1 - y_1y_2)] \tag{18}$$

which, for $x_1 = x_2 = y_1 = y_2$, has a simple pole at $x_c = (\sqrt{17} - 3)/4$. From (18), we find for the number of N-step directed SAWs

$$a_N = [(1+\sqrt{17})(3+\sqrt{17})^{N+1} - (1-\sqrt{17})(3-\sqrt{17})^{N+1}]/2^{N+3}\sqrt{17} \tag{19}$$

and for the total number of steps to the right in all N-step walks

$$b_N = \{(\sqrt{17}+3)^{N+1}[24 + 17N - \sqrt{17}(8-9N)] - (-1)^{N+1}(\sqrt{17}-3)^{N+1}[24+17N+\sqrt{17}(8-9N)]\}/2^{N+3}17\sqrt{17}. \tag{20}$$

The ratio b_N/a_N now equals $2\langle R_{\parallel N}\rangle/\sqrt{3}$ on the triangular lattice. As $N \to \infty$, we find

$$\langle R_{\parallel N}\rangle \sim [N(1+\sqrt{17})/2\sqrt{17} - (10+2\sqrt{17})/17]\sqrt{3}/2. \tag{21}$$

That is, $\langle R_{\parallel N}\rangle$ scales linearly with N for large N.

The five-choice model is potentially interesting because some vestiges of the excluded-volume interaction remain due to the possibility of forming closed loops. A T-matrix solution is therefore not possible, and we have therefore calculated series for $\langle R_{\parallel N}\rangle$ and $\langle R^2_{\perp N}\rangle^{1/2}$. Extrapolation of the data shows rather convincingly that $\nu_\perp = \frac{1}{2}$, and a value $\nu_\parallel \simeq 1$. We believe that with more series terms, we would ultimately find $\nu_\parallel = 1$.

In conclusion, directed SAWs have anisotropic behaviour with $\langle R_{\parallel N}\rangle \sim \langle R^2_N\rangle^{1/2} \sim N$, and $\langle R^2_{\perp N}\rangle^{1/2} \sim N^{1/2}$.

Note added in proof. After this work was completed we learned that A Szpilka has also derived our results for three-choice directed SAWs on the square lattice and that J L Cardy has obtained $\nu_\parallel = 1$ and $\nu_\perp = \frac{1}{2}$ from a field-theoretic approach.

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