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LETTER TO THE EDITOR

Size and shape of directed lattice animals

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Abstract. We use the position-space renormalisation group and exact enumeration methods to study configurational properties of directed lattice animals. These are clusters made up of *directed* bonds in which the 'tail' of a new bond must be added to the 'head' of already existing bonds. Furthermore, the directed bonds have a net orientational order with respect to some preferred anisotropy axis. In the limit that the number of bonds $N \to \infty$, directed animals become extremely anisotropic and two independent correlation lengths, one parallel and one perpendicular to the preferred axis, are required to describe their shape. Mean-field theory suggests that these lengths diverge with exponents of $\nu_\parallel = \frac{1}{2}$ and $\nu_\perp = \frac{1}{4}$ respectively. Below seven dimensions mean-field theory breaks down, and we analyse our enumeration data to estimate the location of the critical point, the exponents $\nu_\parallel$ and $\nu_\perp$, and the exponent θ characterising the singularity of the directed animal generating function. In two dimensions, we estimate that $\nu_\parallel = 0.8$, $\nu_\perp = 0.5$ and $\theta = 0.5$, and our estimates interpolate smoothly with dimension to their respective mean-field limits.

In two dimensions, we also apply a one-parameter position-space renormalisation group using small cells which gives reasonable estimates for the location of the critical point and $\nu_\parallel$. In addition, we formulate a two-parameter renormalisation which allows us to study animals with a variable fraction of directed bonds, and thereby describe the crossover between directed and isotropic animals. We find that any anisotropy ensures that the critical behaviour belongs to the universality class of directed animals.

1. Introduction

Very recently there has been a considerable amount of interest in studying directed percolation (see e.g. Obukhov 1980, Kinzel and Yeomans 1981, Reynolds 1981, Redner 1981a, and references therein). This is a modification of the usual isotropic bond percolation problem in which a directionality constraint plays a fundamental role. In directed percolation, each occupied bond joining nearest-neighbour sites allows connectivity or information to 'flow' in only one direction along the bond. In this sense, a bond in directed percolation plays the role of a diode in an electrical network. Furthermore, the directed bonds are defined to have an overall orientation with respect to some anisotropy axis (see figure 1). Due to this global bias, the critical properties are considerably different from those of isotropic percolation. One of the more striking features is the existence of an anisotropy in which two independent diverging correlation lengths, one parallel and one transverse to the bias, are required to describe cluster shapes near the percolation threshold (Dhar and Barma 1981, Kinzel and Yeomans 1981, Klein and Kinzel 1981, Redner 1981a, Essam and De'Bell 1981).

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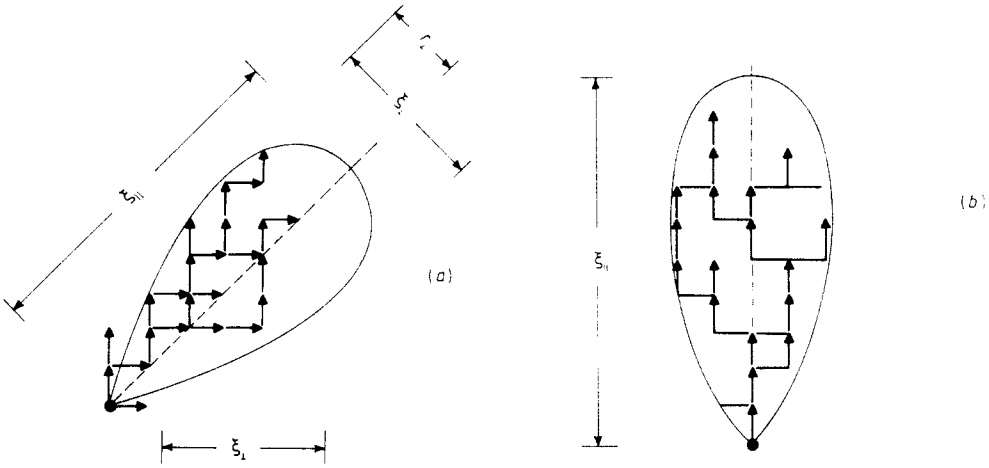

Figure 1. (*a*) A typical directed lattice animal on the square lattice. With respect to the origin, there is an anisotropy axis along which the cluster length, $\xi_{\parallel}$, may be defined. The cluster width, $\xi_{\perp}$, is also shown along with the perpendicular projection, $r_{\perp}$, that is calculated in our exact enumeration. (*b*) A typical partially directed animal on the square lattice.

A variation of percolation that has been of great utility in describing various random cluster phenomena is the lattice animal problem (Domb 1976, see also Stauffer 1979 and references therein). This is a lattice model in which each cluster of b bonds carries a weight of x^b, where x is a fugacity for each bond. This may be thought of as a special limit of bond percolation in which the statistical weight for the perimeter of any cluster is set equal to unity. The properties of lattice animals have been studied extensively both for their intrinsic interest and for their relationship to percolation.

In this letter, we shall study directed lattice animals in which the classical lattice animal problem is modified by the orientational constraint of directed percolation. We find that directed animals become extremely anisotropic in shape as the number of bonds N in the animal becomes infinite. Our goal is to study this anisotropy quantitatively, and to calculate exponents that characterise various geometrical properties of directed animals.

2. Isotropic and directed animals

For isotropic lattice animals, there are two basic properties that are generally studied. The first is the generating function defined by

$$\Gamma(x) = \sum_{b=0} a_b x^b \tag{1}$$

where a_b is the number of distinct animals containing b bonds per lattice site. It exhibits a power-law singularity of the form

$$\Gamma(x) \sim (x - x_c)^{\theta} \tag{2}$$

as $x \to x_c^-$. The exponent θ appears to be universal, depending only on the lattice dimensionality d (see e.g. Gaunt 1980 and references therein). It is equal to unity in $d = 2$ (Parisi and Sourlas 1981), and it interpolates smoothly with dimension to the

mean-field value of $\frac{5}{2}$ (Fisher and Essam 1961, Gaunt and Ruskin 1978) valid for $d \geqslant 8$ (Lubensky and Issacson 1979, de Gennes 1979).

A second, more physical, property is the dependence of the mean animal radius ξ on N. This dependence is written as

$$\xi \sim N^{\nu}, \tag{3}$$

with ν a universal exponent that is approximately equal to 0.6 in $d = 2$ (Isaacson and Lubensky 1980, de Gennes 1980 and references therein), and interpolates smoothly to the value of $\frac{1}{4}$ in the mean-field limit (see e.g. Lubensky and Isaacson 1979, de Gennes 1979).

For directed animals, equation (3) must be modified because two different correlation lengths exist (figure 1). Parallel to the anisotropy axis, a longitudinal radius diverges as

$$\xi_{\parallel} \sim N^{\nu_{\parallel}}, \tag{4a}$$

while perpendicular to the anisotropy, there is a transverse cluster width which diverges at a different rate

$$\xi_{\perp} \sim N^{\nu_{\perp}}. \tag{4b}$$

Our primary aim in this paper is to use both exact enumeration methods and the position-space renormalisation group (PSRG) to calculate the exponents θ, $\nu_{\parallel}$ and $\nu_{\perp}$ for directed animals in dimensions below the upper critical dimension, and to understand the crossover between isotropic and directed animals.

For purposes of calculation, we consider two directed animal models on d-dimensional hypercubic lattices. The first is *fully directed* animals in which directed bonds are constrained to point in the $+x, +y, +z, \ldots$ directions (figure 1(a)). Starting from an arbitrary origin, a fully directed animal 'grows' only into the first 2^d-tant of the lattice, corresponding to an external bias along the diagonal $(1, 1, 1, \ldots, 1)$. We also consider *partially directed* animals, in which directed bonds can occur along only one cartesian axis (Obukhov 1980, Redner 1981a), while isotropic two-way bonds occur along the remaining $(d-1)$ axes (figure 1(b)). In this model, the external bias is parallel to a cartesian axis. In some sense, the primary difference between the two models is the direction of the external bias. Thus one interesting question is whether the critical properties of directed animals depend on the direction of this axis.

Before embarking on detailed calculations, we investigate the behaviour in the mean-field limit. This is helpful in order to determine whether anisotropy in the animal shape is expected. In addition, it is then possible to find the upper critical dimension d_c and thereby ascertain the interesting range of dimensions where non-classical exponents should occur. To this end, we re-examine the mean-field theory of Lubensky and Issacson (1979) for isotropic animals, suitably generalised to account for directed bonds.

In the mean-field expansion of the free energy, spatial fluctuations are accounted for by including even powers of gradients in the order parameter. Due to the bias inherent in directed problems, we expect that odd powers of gradient will now appear (Obukhov 1980, Redner 1981a). To lowest order, this leads to a term of the form $\boldsymbol{a} \cdot \boldsymbol{k}$ multiplying the square of the order parameter. Here $\boldsymbol{k}$ is the wavevector and $\boldsymbol{a}$ is a vector which defines the orientational axis of the directed bonds. Thus, parallel to $\boldsymbol{a}$, large-distance correlations are governed by a linear term in $\boldsymbol{k}$, while in the transverse direction, the

correlations are governed by a quadratic term in $\boldsymbol{k}$. This leads to a parallel correlation length, $\xi_{\parallel}$, that should diverge at twice the rate of $\xi_{\perp}$, the correlation length perpendicular to $\boldsymbol{a}$. Furthermore, since the terms involving odd powers of $\boldsymbol{k}$ vanish in directions transverse to $\boldsymbol{a}$, the asymptotic behaviour of $\xi_{\perp}$ should be the same as the correlation length in isotropic animals. That is $\nu_{\perp}=\frac{1}{4}$, and as a result, $\nu_{\parallel}=\frac{1}{2}$. A typical directed animal is therefore shaped like a prolate spheroid of revolution with a major axis $\xi_{\parallel}$ and $d-1$ minor axes $\xi_{\perp}$ (figure 1). This leads naturally to a generalised Toulouse (1974) condition for d_c of the form $(d_c-1)\nu_{\perp}+\nu_{\parallel}=2\beta+\gamma$, compared with $d_c\nu=2\beta+\gamma$ for an isotropic system (Redner 1981a). Since the right-hand side of the latter equation must be 2 in order to obtain $d_c=8$ for isotropic animals, we find $d_c=7$ for directed animals. Thus we shall study the range of dimensions between two and eight for our series calculations.

3. Exact enumeration

We have primarily concentrated on fully directed animals because it is possible to calculate longer series. We have enumerated fully directed animals for $d=2$ through 8, and partially directed animals for $d=2$ and 3. In addition to calculating the number of animals, we have studied the cluster shape by measuring $\xi_{\parallel}$ and $\xi_{\perp}$ as a function of N. For fully directed animals, we measure $\xi_{\parallel}$ as defined in figure 1 directly, while in the perpendicular direction we calculate the maximum value of the projection of the animal onto a particular $(d-1)$-dimensional hyperplane that includes the anisotropy axis. The transverse cluster width, $\xi_{\perp}(N)$, is directly proportional to this projected length. For partially directed animals, we performed similar calculations, although it is somewhat less convenient to find the cluster width because azimuthal symmetry about the anisotropy axis is no longer exact. The data from this enumeration are shown in appendix 1.

We first consider the analysis for the generating function. For this purpose we first performed a Padé analysis of the logarithmic derivative of this generating function series to get a qualitative picture. Because this function is only weakly singular, it is useful to analyse the first and second derivatives of the series as well. In higher dimensions, the shortness of the series and the relatively slow convergence due to the possible presence of confluent singularities make the Padé analysis very inaccurate. Consequently, we have also used an alternative method following Gaunt *et al* (1976). We form the estimates $\theta_b=b(1-\lambda_b/\lambda)$, where $\lambda_b=a_b/a_{b-1}$ is the ratio of successive generating function coefficients, and λ is an estimate for the $b\to\infty$ limit of the λ_b. For λ, we use estimates based on linearly extrapolating the λ_b against $1/b$, that is $b\lambda_b-(b-1)\lambda_{b-1}$. It is then possible to extrapolate the resulting sequence for θ_b by a variety of methods (see e.g. Gaunt and Guttmann 1974), and thereby obtain estimates for θ which appear more reliable than the results of the Padé analysis. Our estimates for the critical fugacity x_c and the exponent θ are given in table 1.

Qualitatively, we find that θ interpolates smoothly from a value of 0.5 in two dimensions to a value of approximately 1.5 in seven and eight dimensions. Furthermore in two and three dimensions the estimates for θ in fully and partially directed animals are quite close. This supports a universality of directed animals with respect to the direction of the anisotropy axis.

In higher dimensions, our exponent estimates are somewhat imprecise, although the closeness of our estimates for θ in seven and eight dimensions is consistent with $d_c=7$.

Table 1. Series estimates for the critical parameters of directed animals. The error bars represent subjective estimates of the uncertainties in extrapolating the series.

Fully directed d	x_c	θ	$\nu_\parallel$	$\nu_\perp$
2	0.2849 ± 0.0001	0.53 ± 0.01	0.800 ± 0.001	0.50 ± 0.003
3	0.1586 ± 0.0004	0.94 ± 0.02	0.700 ± 0.005	0.45 ± 0.005
4	0.1100 ± 0.0005	1.20 ± 0.05	0.64 ± 0.01	0.40 ± 0.02
5	0.084 ± 0.001	1.35 ± 0.15	0.62 ± 0.02	0.39 ± 0.04
6	0.068 ± 0.001	1.40 ± 0.15	0.61 ± 0.03	0.35 ± 0.05
7	0.058 ± 0.001	1.43 ± 0.15	0.60 ± 0.05	0.35 ± 0.08
8	0.050 ± 0.001	1.44 ± 0.15	0.60 ± 0.05	0.35 ± 0.08
Partially directed				
2	0.2252 ± 0.0001	0.497 ± 0.003	0.792 ± 0.003	0.52 ± 0.01
3	0.108 ± 0.001	0.93 ± 0.02	0.64 ± 0.05	0.49 ± 0.03

To obtain the correct mean-field limit for θ, we consider direct animals on the Cayley tree. In this case, the only difference from isotropic animals is that only one site of a directed animal can serve as the origin. This leads to a reduction in the number of distinct directed animals of b bonds by a factor of b compared with the isotropic case, and a corresponding reduction of θ by 1. For isotropic animals, it is known that $\theta = \frac{5}{2}$ (Fisher and Essam 1961, Gaunt and Ruskin 1978), and consequently for directed animals we expect $\theta = \frac{3}{2}$. Our data for θ are in rough agreement with the predicted results of mean-field theory.

We now turn to the analysis of the correlation lengths in order to obtain estimates for $\nu_\parallel$ and $\nu_\perp$. To accomplish this, we first examine the dependence of $\xi_\parallel(N)$ and $\xi_\perp(N)$ against N on a double logarithmic scale. These data should be asymptotically linear, with limiting slopes $\nu_\parallel$ and $\nu_\perp$ respectively. We estimate these limiting values by first considering the sequence of slopes calculated from the straight lines that join each pair of successive data points. Then this sequence is extrapolated by means of Neville tables to yield the values shown in table 1. For low-dimensional systems, we have series of sufficient length to extrapolate with a good degree of confidence. The only exception is the $\xi_\parallel$ series for z-directed animals in three dimensions. The Neville tables are not well converged and our estimate for $\nu_\parallel$ seems anomalously low. In higher dimensions, however, there is substantial downward curvature in the data of successive slopes when plotted against $1/N$, and a visual extrapolation appears as reliable as the Neville tables. In six and higher dimensions, the series appear to be too short to display fully the curvature mentioned above. As a result, our estimates for $\nu_\parallel$ and $\nu_\perp$ will probably be too high for these dimensions.

The general trend of the data is that $\nu_\perp$ decreases from approximately 0.5 in two dimensions to a value near 0.35 in seven and eight dimensions. This is in qualitative agreement with the predictions of mean-field theory. In addition, $\nu_\parallel$ is approximately equal to 0.8 in two dimensions and appears to decrease to about 0.6 in seven dimensions. While the estimated values of $\nu_\parallel$ and $\nu_\perp$ in seven and eight dimensions are not in good agreement with mean-field prediction, the ratio of $\nu_\parallel/\nu_\perp$ is fairly close to the predicted value of 2. This suggests that there is a systematic effect which is cancelled in taking the ratio, thereby producing a relatively more accurate number. In sum, we see that the anisotropy is relevant as $N \to \infty$; both $\xi_\parallel$ and $\xi_\perp$ diverge, but at different rates so that $\xi_\parallel/\xi_\perp \to \infty$.

4. Position-space renormalisation group

For directed animals it is possible to formulate a rather simple cell PSRG approach in a manner similar to that employed for the percolation, self-avoiding walk and isotropic lattice animal problems (Reynolds *et al* 1977, de Queiroz and Chaves 1980, Family 1980, Redner and Reynolds 1981). The basic idea is to divide the lattice into finite cells of linear dimension l and rescale to cells of linear dimension 1. Within each cell we consider the generating function for all animals that start at an origin (the lower left-hand corner of the cell) and traverse the cell in a particular direction. These configurations map to an occupied directed bond on the rescaled level. We define the recursion relation by requiring that the generating function for traversing directed animals remain invariant on the original and rescaled levels.

This leads to a recursion relation of the form

$$x' = R(x) \tag{5}$$

where x' is simply the renormalised generating function of the 1×1 cell. We have calculated this recursion relation for the 2×2, 3×3 and 4×4 cells (appendix 2). These recursion relations each have three fixed points. Stable ones occur at $x^* = 0$, and there is also an unstable fixed point at an intermediate value of x^* which describes directed animals at criticality. The location of the fixed point provides an estimate for x_c, and the eigenvalue of the transformation is generally used to find a correlation length exponent. For directed animals there are two correlation length exponents, and the connection between the eigenvalue and $\nu_\parallel$ and $\nu_\perp$ is still not well understood. One can make a plausible argument that this type of PSRG approach is sensitive to $\nu_\parallel$ (Redner 1981a). However, to resolve the point definitively, a PSRG approach that unambiguously differentiates $\nu_\parallel$ and $\nu_\perp$ is required.

From our cell-to-bond transformations, we can also construct implicit cell-to-cell transformations in which a cell of linear dimension l is rescaled into a cell of linear dimension l' (Reynolds *et al* 1978, 1980, Family 1980, Redner and Reynolds 1981). This type of transformation has been useful because, with minimal computational labour, it gives accuracy comparable to the cell-to-bond transformation with $l \simeq 40$. The results for x^* and $\nu_\parallel$ with both types of transformations are given in table 2. It is somewhat surprising that any reasonable extrapolation procedure for x^* and $\nu_\parallel$ based on finite-size scaling (see e.g. Reynolds *et al* 1978, 1980 and references therein) gives estimates that are quite different than those found by series. One possible explanation for this discrepancy is the ambiguity in relating the eigenvalue to the

Table 2. Results for x^* and $\nu_\parallel$ upon rescaling a cell of linear dimension l to a cell of linear dimension l'.

l \ l'		1	2	3
2	x^*	0.3306		
	$\nu_\parallel$	0.6392		
3	x^*	0.3157	0.3048	
	$\nu_\parallel$	0.6463	0.6547	
4	x^*	0.3080	0.3003	0.2966
	$\nu_\parallel$	0.6515	0.6599	0.6656

exponent discussed above. In addition, there is a further problem in that isotropic finite-size scaling is not valid for directed animals or directed percolation. It is necessary to scale in an anisotropic way to reflect the anisotropy of the problem (Redner 1981b).

While a one-parameter approach is limited quantitatively, we can obtain very useful qualitative information by introducing a second parameter into the PSRG, a fugacity p for an isotropic bond. This permits us to study the crossover between directed and isotropic animals and obtain the general two-parameter phase diagram. To accomplish this, we examine all lattice animal configurations that start at the lower-left corner and span the cell. A given spanning animal may be made up of n_+ directed bonds and n isotropic bonds and is therefore weighted by $x^{n_+}p^n$. Then the statistical weight for all animals that span from the lower-left hand corner to the opposite edge is used to determine x' as before. However, if the spanning animal contains a path that allows one to traverse the cell in either direction, we renormalise to an isotropic bond.

These considerations lead to coupled recursion relations for x' and p' which can then be solved to yield the global phase diagram shown in figure 2. An important feature of this diagram is that there is a critical line in which the renormalisation flow is from the isotropic to the directed fixed point. This indicates that a lattice animal with any amount of anisotropy, no matter how weak, will have the same critical behaviour as directed animals.

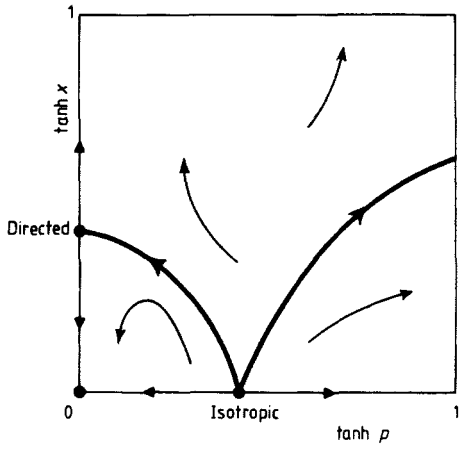

Figure 2. Schematic flow diagram for the two-parameter renormalisation defined by the coupled recursion relations described in the text. Fixed points are shown as heavy dots and the arrows give the direction of the flow under renormalisation. Notice the presence of a critical line along which the renormalisation flow is from the isotropic to the directed animal fixed point.

In summary, we have studied the properties of directed lattice animals, a random cluster model where the constituent bonds are directed and possess a preferred orientation. Directed animals are extremely anisotropic in shape in the asymptotic limit, being described by two diverging correlation lengths, one parallel ($\xi_\parallel$) and one transverse ($\xi_\perp$) to the anisotropy. For spatial dimensions greater than or equal to seven, mean-field theory appears to be valid and the two lengths diverge with exponents $\nu_\parallel = \frac{1}{2}$ and $\nu_\perp = \frac{1}{4}$ respectively. Moreover, the exponent θ characterising the singularity of the

generating function is equal to $\frac{3}{2}$ in mean-field theory, and this can be found by considering the Cayley tree limit. We have used series methods to estimate these exponents for directed animals between two and eight dimensions. Our data are fairly accurate in $d=2$, and in higher dimensions they are in rough agreement with the predictions of the mean-field limit. We have also used a small-cell position-space renormalisation group to study directed animals in two dimensions. A two-parameter approach gives a global phase diagram encompassing the crossover between isotropic and directed animals. We find that an animal with a non-zero fraction of directed bonds has the asymptotic behaviour of fully directed animals.

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Appendix 1. Exact enumeration data for fully and partially directed animals.

Number of fully directed animals

b \ d	2	3	4	5	6
1	2	3	4	5	6
2	5	12	22	35	51
3	14	55	140	285	506
4	42	273	969	2 530	5 481
5	130	1 419	7 060	23 701	62 742
6	412	7 617	53 304	230 480	746 293
7	1 326	41 838	413 028	2 303 445	9 130 254
8	4 318	233 826	3 264 214	23 507 305	114 133 905
9	14 188	1 324 596	26 201 392	243 890 210	
10	46 950	7 585 266	212 967 186		
11	156 258	43 822 671			
12	522 523	255 049 350			
13	1 754 254				
14	5 909 419				
15	19 964 450				
16	67 618 388				
17	229 526 054				

b \ d	7	8
1	7	8
2	70	92
3	819	1 240
4	10 472	18 278
5	141 631	285 160
6	1 991 661	4 627 708
7	28 821 066	77 308 184

Number of partially directed animals

b \ d	2	3
1	2	3
2	6	15
3	21	93
4	79	643
5	309	4 739
6	1 239	36 408
7	5 053	287 911
8	20 861	2 325 398
9	86 920	19 085 788
10	364 770	158 629 279
11	1 539 608	
12	6 528 865	
13	27 794 578	
14	118 717 437	
15	508 504 266	

Fully directed animals—Longitudinal cluster radius

b \ d	2	3	4	5	6	7	8
1	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2	1.800	1.750	-.727	1.714	1.705	1.700	1.696
3	2.571	2.472	2.428	2.403	2.387	2.376	2.368
4	3.238	3.087	3.022	2.986	2.962	2.947	2.935
5	3.892	3.674	3.583	3.532	3.500	3.479	3.462
6	4.512	4.223	4.105	4.041	4.000	3.973	3.953
7	5.111	4.745	4.598	4.519	4.469	4.436	4.412
8	5.693	5.243	5.066	4.972	4.913		
9	6.260	5.723	5.514	5.404			
10	6.813	6.187	5.944				
11	7.356	6.636					
12	7.888	7.073					
13	8.411						
14	8.926						
15	9.434						
16	9.935*e*						
17	10.430						

Transverse cluster radius

b \ d	2	3	4	5	6	7	8
1	0.500	0.333	0.250	0.200	0.167	0.143	0.125
2	0.800	0.583	0.454	0.371	0.313	0.271	0.239
3	1.071	0.781	0.621	0.515	0.440	0.385	0.341
4	1.285	0.948	0.762	0.639	0.551	0.485	0.433
5	1.484	1.095	0.886	0.749	0.650	0.575	0.516
6	1.665	1.228	0.997	0.847	0.739	0.657	0.591
7	1.831	1.348	1.098	0.936	0.820	0.732	0.661
8	1.987	1.460	1.190	1.018	0.894		
9	2.134	1.564	1.276	1.093			
10	2.273	1.662	1.357				
11	2.406	1.754					
12	2.532	1.842					
13	2.654						
14	2.771						
15	2.884						
16	2.993						
17	3.099						

Partially directed animals

	Longitudinal cluster radius		Transverse cluster radius	
b	$d=2$	$d=3$	$d=2$	$d=3$
1	0.500	0.333	0.500	0.333
2	1.000	0.667	1.000	0.800
3	1.428	0.956	1.428	0.967
4	1.810	1.219	1.810	1.237
5	2.190	1.459	2.158	1.482
6	2.548	1.681	2.481	1.706
7	2.887	1.889	2.782	1.913
8	3.216	2.086	3.065	2.106
9	3.535	2.272	3.333	2.288
10	3.847	2.451	3.588	2.459
11	4.151		3.831	
12	4.449		4.065	
13	4.742		4.289	
14	5.030		4.506	
15	5.314		4.715	

Appendix 2: Recursion relations for directed animals.

2×2 *cell*

$$x^2+4x^3+5x^4+4x^5+x^6$$

3×3 *cell*

$$x^3+6x^4+26x^5+67x^6+135x^7+219x^8+287x^9$$
$$+302x^{10}+244x^{11}+140x^{12}+52x^{13}+11x^{14}+x^{15}$$

4×4 *cell*

$$x^4+8x^5+43x^6+191x^7+646x^8+1839x^9+4600x^{10}$$
$$+10330x^{11}+21052x^{12}+39019x^{13}+65635x^{14}+99746x^{15}$$
$$+136049x^{16}+65090x^{17}+176143x^{18}+162720x^{19}+127714x^{20}$$
$$+83335x^{21}+44130x^{22}+18458x^{23}+5904x^{24}$$
$$+1384x^{25}+223x^{26}+22x^{27}+x^{28}$$

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