

Directed and diode percolation

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We study the novel percolation phenomena that occur in random-lattice networks consisting of resistor-like and diode-like bonds. Resistor bonds connect or "transmit information" in either direction along their length, while diodes connect in one direction only. We first treat the special case of directed bond percolation, in which the diodes are aligned along a preferred axis. Mean-field theory shows that clusters become extremely anisotropic near the percolation transition and that their shapes are characterized by two correlation lengths, one parallel and one transverse to the preferred axis. These lengths diverge with exponents $\nu_{\parallel}=1$ and $\nu_{\perp}=1/2$, respectively, from which we can show that the upper critical dimension for this system must be five. We also treat a more general random network on the square lattice containing resistors and diodes of arbitrary orientation. Duality arguments are applied to obtain exact results for the location of phase transitions in this system. We then use a position-space renormalization-group approach to map out the phase diagram and calculate critical exponents. This system has an isotropic percolating phase, and phases which percolate in only one direction. Novel types of transitions occur between these phases, in which the diode orientation plays a fundamental role. These percolating phases meet with the nonpercolating phase along a line of multicritical points, where concentration and orientational fluctuations are simultaneously critical.

I. INTRODUCTION

The percolation problem has been intensively studied, partly because it is an extremely simple model system exhibiting the full range of features in critical phenomena, and also because of the many realizations of percolation phenomena in random systems.^{1,2} Although these investigations have proved to be quite fruitful, there remain many interesting generalizations of percolation which are relatively unexplored. Many such generalizations are contained implicitly in the pioneering work of Broadbent and Hammersley.³ In their original model (Fig. 1), nearest-neighbor sites on a regular lattice could be joined by two directed bonds, each connecting or "transmitting information" in opposite directions. In this sense, the directed bonds act as diodes, in contrast to the bonds of usual isotropic percolation which act as resistors.

The primary purpose of this article is to investigate percolation phenomena in random networks containing both resistors and diodes. When the diodes have a preferred orientation, we shall see that the system is in the universality class of fully directed bond percolation, a model which has been the focus of recent investigation.⁴⁻¹⁷ Near the percolation threshold of this model, clusters become extremely anisotropic. Two independent diverging correlation lengths, one parallel, and one transverse to the special axis, are required to describe the phase transition. The influence of a preferred direction on a phase transition is a new feature not generally discussed in the theory of

critical phenomena. There do exist, however, several potentially important examples of such directed phenomena, e.g., irreversible chemical reactions,¹⁸ Markov processes with branching, absorption, and recombination,^{19,20} gelation in a flowing solvent, and transport in random systems with an external bias.^{21,22} Thus further study of the properties of directed percolation may provide interesting predictions that may be amenable to experimental verification.

Further interest in directed percolation stems from the existence of an exact mapping between directed percolation and Reggeon field theory²³ which models the creation, propagation, and destruction of a cascade of elementary particles. In turn, Reggeon field theory can be related to branching Markov processes.^{19,20} Thus many apparently diverse phenomena can be related within the context of directed percolation.

In addition to studying directed percolation, we are interested in understanding the percolation properties in a more general network, where the diode orientation can be continuously varied. We shall see that there are two parameters, the bond concentration and the bond orientation, which may be adjusted independently to drive the system to a percolation threshold. The range of phenomena which occurs in this system is considerably richer than that in the usual isotropic percolation problem. There exist transitions between nonpercolating, isotropically percolating, and unidirectionally percolating phases.⁹⁻¹¹ Moreover, these transitions meet along a line of mul-

critical points where concentration and orientational fluctuations play an equally important role. Our goal is to understand, in greater detail, the general physical features of this interesting system. We will give a more complete account of some preliminary work on this model that was published earlier.¹⁰

The organization of the remainder of this article is as follows: In Sec. II, we first define the fully directed percolation model. We also recount recent work on directed percolation along with some examples of potential physical applications of this model. We then define our more general model, which we call the random resistor-diode network, and briefly outline its interesting features. In Sec. III, we describe a Landau-Ginzburg free energy expansion of directed percolation that gives an intuitive picture of cluster shapes near the percolation threshold. From this, we construct a simple argument that shows that the upper critical dimension is five, a result first obtained by field-theoretic methods.^{7,23}

In Sec. IV, we study the general random resistor-diode network. First, we present duality arguments to obtain exact results for the location of some of the percolation thresholds and exponents of the system. Then a position-space renormalization group (PSRG) approach is given to map out the phase diagram and elucidate the critical behavior of this model. In Sec. V, we give some conclusions and summarize our work. The Appendix contains the recursion relation for the random resistor-diode network.

II. MODELS

A. Fully directed bond percolation

For concreteness, we will define directed percolation on a d -dimensional hypercubic lattice. A site may connect to its nearest neighbors, which are separated by one lattice spacing in the positive direction along a given cartesian axis, with a probability p_+ [see Fig. 1(b)]. Thus only sites confined to the first 2^d -tant of the lattice—the “light cone”—may be reached by a connected path from the origin. Due to this biased way of forming connected paths, there exists a special “symmetry-breaking” direction, $(1, 1, \dots, 1)$, which may be regarded as a “time” axis.^{19,20,23}

For a small concentration of directed bonds, we have only finite clusters. At the critical concentration, an infinite cluster forms which percolates in the direction of the time axis. Above the threshold, percolation is confined to a cone-shaped region of finite opening angle about the time axis. This angle goes to zero⁶ as the transition is approached from above, and as $p_+ \rightarrow 1$, the percolating region fills the light cone.

Early low-density series work⁴ indicated that the critical behavior of directed and isotropic percolation

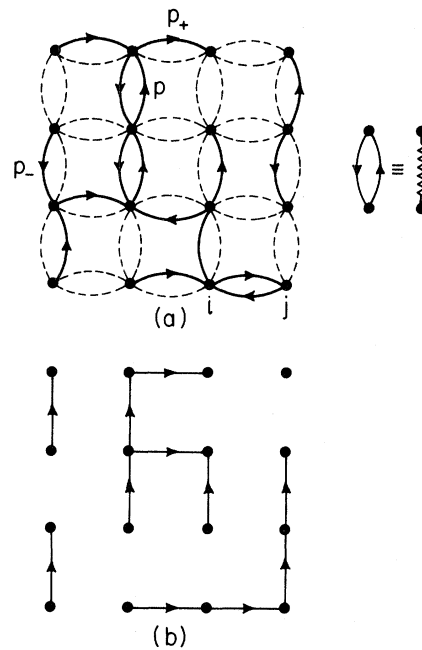


FIG. 1. (a) The random resistor-diode network as a realization of the general Broadbent-Hammersley percolation model. Each pair of nearest-neighbor sites may be joined by up to two directed bonds, one defining a connected path from i to j , and the other defining a path in the opposite direction. When both oppositely oriented directed bonds are occupied, we have a resistor, while a single directed bond acts as a diode. (b) The special case of fully directed bond percolation. Each pair of nearest-neighbor sites may be joined by a directed bond of positive orientation with respect to any given cartesian axis.

were different. More recently, Obukhov⁷ studied directed percolation with field theory methods. He showed that the system becomes extremely anisotropic near the percolation threshold, leading to a lowering of the critical dimension, from six in the isotropic case, to five. Because of the anisotropy, the decay of connectivity is characterized by two lengths, one parallel and one transverse to the time axis. In two dimensions, a variety of numerical methods have been applied to estimate the critical behavior of the anisotropy.^{5,6,8,16} In addition, recent PSRG approaches⁹⁻¹¹ give an intuitive way of seeing that directed percolation is in a different universality class than isotropic percolation. Moreover, these treatments can be extended in a relatively simple fashion, to provide information about random networks in which the orientation of the diodes may be random.

B. Random resistor-diode network

In this model, each bond may be vacant with probability q or occupied by one of three bond elements

[see Fig. 1(a)]. A resistor may occur with a probability p ; a diode oriented along the positive direction of a given cartesian axis may occur with probability p_+ ; or a diode oriented in the *opposite* direction may occur with probability p_- . Clearly $q = 1 - p - p_+ - p_-$. On the square lattice, the time axis is parallel to

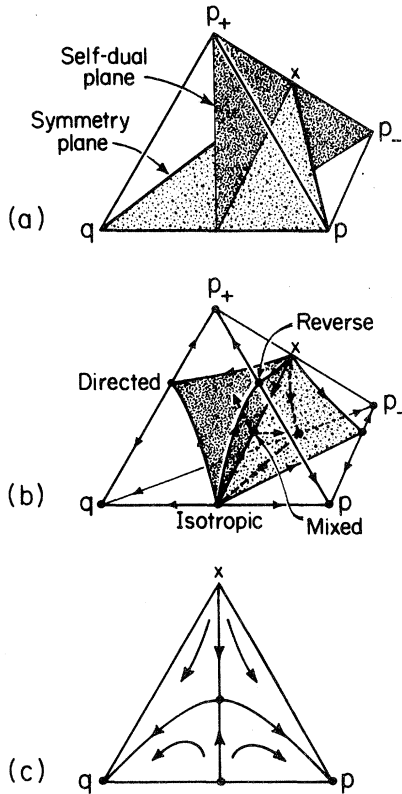


FIG. 2. (a) The composition tetrahedron of the random resistor-diode network. For any point inside the tetrahedron, the perpendicular distance to any face gives the relative concentration of the species labeled at the opposite apex. The self-dual ($p = q$), and the symmetry ($p_+ = p_-$) planes are shown shaded. Their intersection defines a line where the concentration of directed bonds per lattice edge is unity: (b) Phase diagram of the random resistor-diode network. Fixed points are shown by heavy dots, and the arrows indicate the direction of flow under renormalization. At the mixed fixed point, the two perpendicular axes define the “ p ” and “ p_{\pm} ” directions. The point marked x represents a lattice in which each edge is occupied by a diode, but there is no orientational order of the diodes (“randomized Manhattan”). Shown shaded are the surfaces of second-order transitions which divide the diagram into the positive diode, negative diode, resistor, and vacancy phases. These surfaces meet at a common multicritical line defined by the intersection of the self-dual and symmetry planes. (c) The renormalization flow within the $p_+ = p_-$ symmetry plane. Notice that the top vertex of the triangle is not a fixed point. The flow is confined to lie within the symmetry plane.

(1, 1) if $p_+ > p_-$, or parallel to $(-1, -1)$ if $p_+ < p_-$. The case $p_+ = p_-$ is particularly interesting. The system is isotropic, but connectivity may be mediated through either the resistors or the randomly oriented diodes. The limit $p_+ = p_- = 0$, $p \neq 0$, corresponds to conventional isotropic percolation, while the limits $p_- = p = 0$, $p_+ \neq 0$, and $p_+ = p = 0$, $p_- \neq 0$, correspond to fully directed percolation.

The phase diagram of this system can be visualized within the composition tetrahedron. This is defined by the intersection of the three-dimensional subspace $p_+ + p_- + p + q = 1$ with the half spaces p_+ , p_- , p , $q \leq 1$ within the composition space defined by $p_+ + p_- + p + q = 1$ [see Fig. 2(a)]. Within this phase diagram, we can describe the crossover between directed and isotropic percolation as the relative concentration of the bond elements varies. We find four phases in the system. There are the usual nonpercolating and isotropic percolating phases. In addition, there are two phases which percolate in only one direction, either along the (1,1) axis or in the opposite direction.

Novel types of transitions arise because of the possibility of forming an infinite cluster with either an isotropic or unidirectional connectivity. Furthermore, on the square lattice, there exist additional symmetries in the phase diagram, due to the self-dual nature of the lattice. As a result, there exists a special line where the self-dual plane, defined by $p = q$ (cf. Sec. IV), and the $p_+ = p_-$ symmetry plane intersect. We shall see that this is a line of multicritical points where the four phases meet and concentration and orientational fluctuations are both critical.

III. MEAN-FIELD THEORY FOR DIRECTED PERCOLATION

A mean-field theory for isotropic percolation can be derived through a Landau-Ginzburg-Wilson expansion for the free energy of the corresponding Q -state Potts model.²⁴ In the $Q \rightarrow 1$ limit, we obtain a free energy

$$F(P) = [(p - p_c) + bk^2 + \dots]P^2 + cP^3 + \dots \quad (3.1a)$$

Here p is the bond probability of p_c is its critical value, k is the wave vector, b and c are constants, and P is the percolation probability. This latter quantity is the order parameter in percolation, defined as the probability that a randomly chosen site belongs to the infinite cluster. By minimizing (3.1a) with respect to P , we find the well-known results the P vanishes as $p \rightarrow p_c^+$ as $(p - p_c)^\beta$, with $\beta = 1$, and that the mean cluster size (the percolation analog of the susceptibility) diverges as $(p - p_c)^{-\gamma}$ with $\gamma = 1$. Moreover, this wave-vector dependent susceptibility analog is a Lorentzian with a characteristic width that goes to

zero at the transition. From this, we deduce that there exists a characteristic length scale ξ diverging as $(p - p_c)^{-\nu}$ with $\nu = 1/2$.

In directed percolation, the up-down symmetry along the time axis is broken and we wish to understand how this is manifested in the free energy. In the free energy, there exist gradient terms to account for the presence of spatial fluctuations in the system. For percolation, the clusters represent these fluctuations. Because of the anisotropic structure of the clusters in directed percolation, both even and odd terms in the gradient should now occur. In the first approximation, we keep the linear and quadratic terms, leading to a free energy in momentum space

$$F(P) = [(\bar{p} - \bar{p}_c) + \vec{a} \cdot \vec{k} + bk^2 + \dots] P^2 + cP^3 + \dots, \quad (3.1b)$$

where \vec{a} is parallel to the time direction. In this expression, \bar{p} is again the bond probability, but it may now be some linear combination of p_+ and p .

Such a modified free energy is consistent with the anisotropic propagator in the field-theoretic treatment of directed percolation,⁷ or in Reggeon field theory.²³ The free energy (3.1b) still yields the exponents $\beta = \gamma = 1$. However, *two* characteristic length scales are evident. Parallel to \vec{a} , the linear term in k dominates as $k \rightarrow 0$, and we obtain a longitudinal correlation length, $\xi_{||}$, which diverges as $(\bar{p} - \bar{p}_c)^{-\nu_{||}}$, with $\nu_{||} = 1$. However, perpendicular to \vec{a} , we obtain a transverse correlation length, ξ_{\perp} , diverging as $(\bar{p} - \bar{p}_c)^{-\nu_{\perp}}$, with $\nu_{\perp} = 1/2$. Thus as \bar{p}_c is approached, there exists a strong anisotropy in the cluster shapes (see Fig. 3).

From this picture, we can obtain the upper critical dimension d_c , above which mean-field theory is quantitatively correct, in a simple fashion. Generally, to find d_c , we require that the relative fluctuations of the order parameter in a suitably-chosen volume are small.^{25,26} For an isotropic system, the fluctuations are important over a linear dimension of the order of ξ . Thus the appropriate volume is a sphere of

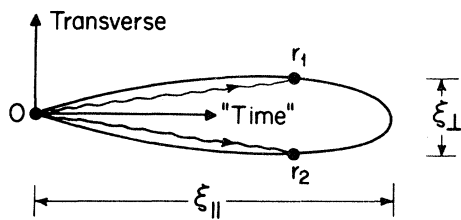


FIG. 3. Schematic picture of a typical cluster, defined with respect to a fixed origin, near the threshold of directed percolation. Notice that the sites r_1 and r_2 on opposite sides of the cluster are related only through their common connection to the origin.

radius ξ , and we are led to the condition $d_c \nu = 2\beta + \gamma$, or $d_c = 6$.²⁷ However for directed percolation, the fluctuations are important over a dimension $\xi_{||}$ in the time direction, but only over a dimension ξ_{\perp} transverse to the time. Thus we are led to consider fluctuations in a prolate ellipsoid of revolution about the time direction, with a major axis $\xi_{||}$ and with $d - 1$ minor axes ξ_{\perp} .²⁸ As a result, the Ginzburg criterion generalizes to

$$(d_c - 1)\nu_{\perp} + \nu_{||} = 2\beta + \gamma. \quad (3.2)$$

Employing the mean-field exponents calculated above, we find $d_c = 5$, a result first obtained by field-theoretic methods.^{7,23} We see that the critical dimension has been reduced compared to isotropic percolation because of the effective longer-range interaction along the time axis. This situation is analogous to the reduction of the critical dimension in systems with long-range interactions, such a dipolar-coupled ferromagnets.

IV. RANDOM RESISTOR-DIODE NETWORK ON THE SQUARE LATTICE

A. Exact duality relations

Because the square lattice is self-dual, one can determine exactly that the critical concentration is $1/2$ for isotropic bond percolation.²⁹ We shall now extend these duality arguments to the full random resistor-diode network, yielding more general exact results for the structure of the phase diagram.

The dual transformation constructs a one-to-one correspondence between the configurations on a lattice \mathcal{L} , and configurations on a closely related dual lattice \mathcal{L}^D (see Fig. 4). For the random resistor-diode network, we define the dual transformation by the following rules: if an edge on \mathcal{L} is occupied by a resistor, then on \mathcal{L}^D the edge crossing the resistor is defined to be vacant. Similarly, if an edge is vacant on \mathcal{L} then on \mathcal{L}^D the edge crossing the vacancy is occupied by a resistor. These rules give the conventional dual transformation between resistor-vacancy configurations on \mathcal{L} and \mathcal{L}^D . In addition, if an edge on \mathcal{L} is occupied by a diode, then on \mathcal{L}^D we define the corresponding dual edge to be occupied by a diode rotated by $\pi/2$ clockwise with respect to the original diode. Such a correspondence also rotates the time axis by the same amount so that the directionality of the diodes with respect to the time axis is preserved under duality. Under these rules, the bond probabilities are related by $p^D = q$, $q^D = p$ and $p_{\pm}^D = p_{\pm}$, where the superscript D refers to the probability on the dual lattice.

Under duality, consider how various bond configurations transform. We can classify four distinct

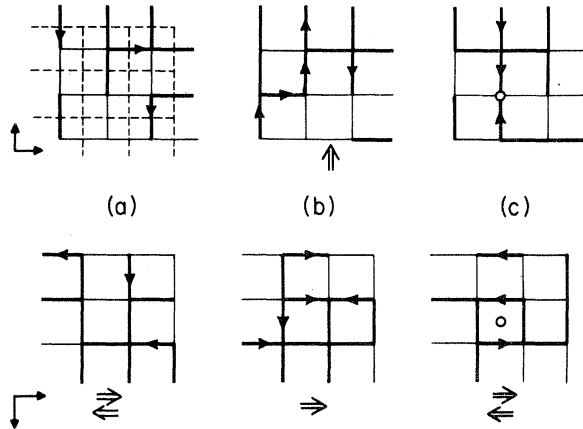


FIG. 4. Relationship between bond configurations on the original lattice (top row) and on the dual (bottom row). A heavy line represents a resistor, a heavy line with an arrow represents a diode and a light line represents a vacant bond. The coordinate axes show the orientation of the positive diodes on \mathcal{L} and \mathcal{L}^D . In addition, the arrows exterior to the lattices indicate the direction(s) of the percolating paths present. We define spanning (either forward or backward) along the vertical in \mathcal{L} . On \mathcal{L}^D , this axis becomes the horizontal. In (a), a finite configuration is shown and a superimposed dual lattice is shown dashed. Below, the spanning configuration obtained after the dual transformation is shown. In (b), we show a unidirectional forward spanning configuration that transforms into another such configuration under duality. In (c), we sketch a configuration containing “mismatched” diodes (open circle), and its dual counterpart which spans in both directions.

types of configurations with different transformation properties. First, consider configurations consisting of only finite clusters [Fig. 4(a)]. A percolating cluster of vacancies must occur in order to isolate the clusters of occupied bonds. Under duality, we obtain a percolating cluster of resistors. This result, together with the correspondence between p and p^D and the self-duality of the square lattice, lead to the well-known result²⁹ that $p_c = 1/2$ for isotropic percolation.

Next, consider a bond configuration that percolates in only one direction. Under the dual transformation, the percolating path maps into a “barrier” that allows paths to pass through it in only one direction [Fig. 4(b)]. As a result, a unidirectional spanning path transforms into a unidirectional path which spans in the same sense with respect to the time axis. Finally, consider configurations such as the one shown in Fig. 4(c). We could percolate in any direction, if the directionality constraints imposed by the diodes are ignored. Under the dual mapping, this cluster transforms into an impenetrable barrier of vacancies, except at the locations of the diodes which are “mismatched” [see Fig. 4(c)]. At these locations, a “gap” occurs in the barrier which allows a connected

path to pass in either direction. Thus we transform to a configuration that percolates in both directions.

In summary, under duality the positive diode phase transforms into itself and the negative diode phase also transforms into itself. The resistor and vacancy phases transform into each other. Because the lattice is self-dual, we thus conclude that the phase diagram must be symmetric about the $p = q$ plane. We will see in the next section, that due to the additional symmetry with respect to the interchange of p_+ with p_- , there exists a critical line defined by the interaction of the self-dual ($p = q$) and the symmetry ($p_+ = p_-$) planes. Along this line, there is an isotropic percolation threshold given by the condition $p_c = 1/2 - p_{\pm}$. This generalizes the result $p_c = 1/2$ of isotropic percolation, to the random resistor-diode network.

B. Position-space renormalization-group treatment

Our PSRG approach is based on rescaling a $b \times b$ bond cell to a 1×1 cell first used by Reynolds *et al.*^{30–32} for isotropic percolation (see Fig. 5). In their procedure, a “connectivity” weight function was used in which “percolating” cells map to an occupied bond upon rescaling. Our treatment is based on generalizing this connectivity weight function to account for cells which percolate in only one or in both directions. In the larger cell, if a configuration

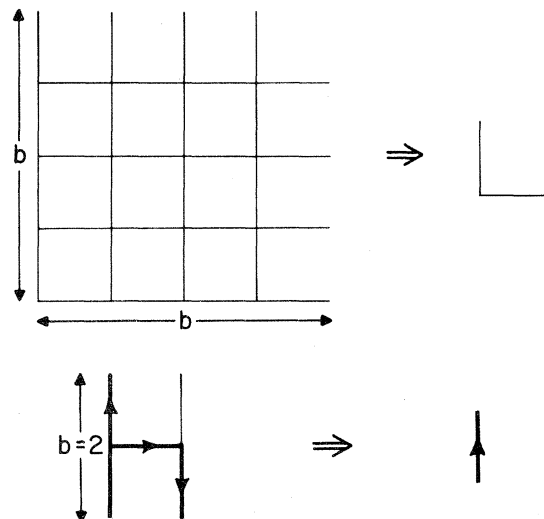


FIG. 5. A $b \times b$ cell on the square lattice which maps into the 1×1 cell shown to the right. A regular array of these cells cover the lattice. Shown below is the 5-bond cell required for calculating the probability of getting across the cell vertically when the rescaling factor is 2. On this cell, we sketch a typical configuration and its rescaled counterpart.

of resistors and diodes traverses the cell in both directions, then after rescaling, the configuration maps to a resistor. Similarly, if a configuration traverses the cell in only one direction, then on the rescaled level the cell maps to a diode whose orientation is determined by the direction of traversing. Finally, a nontraversing configuration maps to a vacancy upon renormalization.

In the simplest approximation, we treat the 2×2 cell. We calculate the probabilities of traversing the cell either from the top to the bottom or in the opposite direction. In this case, three of the eight cell bonds do not contribute to these probabilities; hence we need to examine 4^5 distinct configurations on the remaining 5-bond cell. The probability that we traverse the cell in both directions then gives p' , the probability of a rescaled resistor, while the probability of traversing in only one direction gives p_{\pm}' the probability of a rescaled diode. These recursion relations are displayed in the Appendix.

From these relations, we obtain the phase diagram shown in Fig. 2(b) in the composition space defined by p_+ , p_- , p , and q . The diagram divides into four distinct phases. There exists a "vacancy" phase in which only finite clusters occur. There also exists a "resistor-like" phase in which an infinite cluster forms that percolates isotropically via both resistors and diodes. There are also two "diode-like" phases with an infinite cluster that percolates in only one direction.

In the phase diagram, the q and p_{\pm} phases, and the p and p_{\pm} phases are separated by surfaces of second-order phase transitions. By duality, these surfaces are symmetrically located with respect to the self-dual ($p = q$) plane. In addition, the obvious symmetry under the interchange of p_+ and p_- means that pairs of surfaces on opposite sides of the $p_+ = p_-$ plane are also symmetrically positioned. All four phases meet along a common line, which is the intersection of the self-dual plane with the $p_+ = p_-$ symmetry plane. This is a line of higher-order critical points where the four phases become simultaneously critical.

In order to determine the critical behavior, we examine the linearized recursion relations in the vicinity of the fixed points. There are ten fixed points, but four of them are trivial corresponding to a lattice completely occupied by only one type of bond species. Of the remaining nontrivial fixed points, there is the isotropic percolation threshold located at $p^* = q^* = 1/2$, $p_{\pm}^* = p_{\mp}^* = 0$. At this point, the linearized transformation matrix $T_{\alpha\beta} \equiv \partial\alpha'/\partial\beta$, ($\alpha, \beta = p, p_+,$ or p_-) is equal to

$$\begin{pmatrix} 1.625 & 0 & 0 \\ 0 & 1.5625 & 0.0625 \\ 0 & 0.0625 & 1.5625 \end{pmatrix}. \quad (4.1)$$

Thus there are three relevant eigenvalues, 1.625

(doubly degenerate) and 1.5, with associated eigenvectors (1,0,0), (0,1,1) and (0,1,-1), respectively. The double degeneracy of the larger eigenvalue implies that when we approach the isotropic fixed point along any path within the symmetry plane, the correlation length has the same singular behavior. Our 2×2 approximation yields a correlation length exponent $\nu = \ln 2 / \ln 1.625 \cong 1.428$. This value has been obtained previously in the PSRG treatment of Reynolds *et al.*³⁰ and is accurate to within 7% of the best estimates for ν .³³ Thus we might anticipate similar accuracy at the other fixed points.

The remaining eigenvector is perpendicular to the symmetry plane and therefore points outside the composition tetrahedron. Thus it is not clear how to define an exponent associated with this eigenvector. Because all three eigenvectors are relevant, the isotropic fixed point is a fourth-order critical point.

There also exist two directed fixed points which signal the onset of an anisotropic infinite cluster percolating predominantly along the time axis. These fixed points occur at $p_{\pm}^* = 0.5550$, $q^* = 1 - p_{\pm}^*$ and $p^* = 0$. Our value for p_{\pm}^* is a rough approximation to the best numerical estimates^{5,6,8} of 0.6447 for the threshold of directed percolation. At the p_+ directed fixed point, the linearized transformation matrix is

$$\begin{pmatrix} 0 & 0 & 0 \\ 1.628 & 1.567 & 0.061 \\ 0 & 0 & 0 \end{pmatrix}. \quad (4.2)$$

In this case, there is only one relevant eigenvalue conjugate to the eigenvector (0,1,0). In the plane perpendicular to this direction, we have two irrelevant eigenvectors which may be chosen arbitrarily due to the double degeneracy of the eigenvalue zero. Since these eigenvalues are zero, the renormalization flow is asymptotically tangent to this perpendicular plane. The directed fixed point is a domain of attraction for all points on the critical surface separating the vacancy and diode phases. This indicates that a random network containing resistors and diodes with a preferred orientation, no matter how weak, is in the universality class of fully directed percolation.

From the relevant eigenvalue, we obtain a longitudinal correlation length exponent $\nu_{\parallel} = 1.543$, compared to the numerical estimates, $\nu_{\parallel} = 1.74$.^{8,16,23} It should be stressed that our PSRG approach based on a connectivity rule should be sensitive *only* to the longitudinal correlation length, ξ_{\parallel} . This is defined in terms of connected paths, while the transverse length, which is proportional to the mean cluster width, must be defined differently. It is not possible to construct a connected path joining opposite sides of the cluster. The connection between the two sides may be defined by their being joined to a common site at the origin of the cluster (see Fig. 3). Thus the properties of ξ_{\perp} are considerably more subtle,⁸ ap-

parently rescaling as ξ_{\parallel} times an angle^{14,15} which defines the width of the infinite cluster. It would be very interesting to understand the properties of ξ_{\perp} within the PSRG framework.

Because of the dual symmetry of the problem, there also exist two more fixed points located at $p_{\pm}^* = 0.5550$, $p^* = 1 - p_{\pm}^*$, and $q = 0$. At these fixed points, there is a transition from a diode phase for $p_{\pm} > p_{\pm}^*$, in which there is connectivity along the diode "polarization" to the resistor phase for $p_{\pm} < p_{\pm}^*$, in which isotropic connectivity occurs. Thus as $p_{\pm} \rightarrow p_{\pm}^*$ from above, connected paths oriented opposite to the diode polarization begin to span the lattice. We term this situation "reverse" percolation. By duality, the matrix $T_{\alpha\beta}$ at the reverse fixed points is related to $T_{\alpha\beta}$ at the directed fixed points. For example, at the p_+ reverse fixed point $T_{\alpha\beta}$ is

$$\begin{pmatrix} 1.628 & 0.061 & 1.567 \\ -1.628 & -0.061 & -1.567 \\ 0 & 0 & 0 \end{pmatrix}. \quad (4.3)$$

This leads to an identical renormalization flow structure, and the same exponents as those found at the directed fixed points.

One of the more intriguing aspects of this system is the existence of a "mixed" fixed point located at $p^* = q^* = 0.2543$, $p_+^* = p_-^* \equiv p_{\pm}^* = 0.2457$, very close to the center of the composition tetrahedron. It describes a transition in which an isotropic infinite cluster forms with percolating paths containing both resistors and diodes. Even though the infinite cluster is isotropic at the threshold, there are two interesting and distinct critical behaviors that depend on the direction of approach to the fixed point.

To gain further insight into this behavior, we examine the linearized transformation matrix at the fixed point

$$\begin{pmatrix} 1.625 & 0.5373 & 0.5373 \\ 0 & 1.0252 & -0.4748 \\ 0 & -0.4748 & 1.0252 \end{pmatrix}. \quad (4.4)$$

We find two relevant eigenvalues 1.625 and 1.5 with eigenvectors (1,0,0) and (0,1,-1), respectively. Notice that these two eigenvalues and their corresponding eigenvectors are identical to those at the isotropic fixed point! This feature is quite unexpected, and it would be worthwhile to understand whether this equality is merely an artifact of the approximations inherent in the small-cell PSRG approach, or whether it reflects an additional symmetry in the system.

From (4.4), we see that one type of critical behavior occurs when we fix $p_+ = p_- = p_{\pm}^*$ to maintain isotropy, and we exchange resistors with vacancies to approach the transition. This corresponds to a path in the symmetry plane which is parallel to the $p - q$ base of the composition tetrahedron—the " p " direction (see Fig. 2). For $p < p^*$, only finite clusters ex-

ist (which may contain both resistors and diodes), but for $p > p^*$, an isotropic percolating phase occurs. As we approach p^* , the correlation length diverges with the *identical* exponent to that found at the isotropic fixed point, in the 2×2 approximation.

On the other hand, we can also approach the transition by fixing $p = q = p^*$ and varying the diode orientation. This corresponds to a path perpendicular to the symmetry plane—the " p_{\pm} " direction. To understand the nature of the transition, suppose that we start with $p_- > p_{\pm}^* > p_+$, so that the diode orientation is along the (-1,-1) axis. Thus an infinite connected path propagates along this axis. As p_- decreases and p_+ increases, longer connected paths begin to form, which propagate primarily in the (1,1) direction. Finally, as $p_+, p_- \rightarrow 0$, the diode polarization goes to zero, and we have the onset of percolation along (1,1). The divergence of the correlation length along this axis is governed by the eigenvalue 1.5, with an associated exponent of 1.710. On the other hand, for $p_+ > p_-$, connected paths in the (-1,-1) direction are now finite in length. As p_+ increases these paths shrink at a rate governed by the same exponent given above.

The irrelevant eigenvalue is 0.5504, and the corresponding eigenvector is (0,1,1), parallel to the line defined by the intersection of the self-dual and symmetry planes. Thus the mixed fixed point controls the entire critical behavior along this line where the four phases in the system meet. We have multicritical behavior where concentration fluctuations—corresponding to deviations from criticality in the p direction, and orientational fluctuations—corresponding to deviations in the p_{\pm} direction—are simultaneously critical.

Finally, it is amusing to note that we can characterize this multicritical line in a simple fashion. If we view a resistor as a directed bond pair (see Fig. 1), then all along this line the concentration of directed bonds per lattice edge is unity. This perspective indicates that there may be additional symmetry in the model which may provide a clue concerning the apparent universality between two evidently distinct fixed points on the multicritical line.

V. SUMMARY AND DISCUSSION

We have studied percolation phenomena in a random network consisting of resistor-like and diode-like bonds. We first treated a special case, in which diodes of only one orientation can occur, by mean-field theory. The existence of a preferred direction in the model can be accounted for by a very simple modification of the standard Landau-Ginzburg-Wilson free energy for isotropic percolation. From this, we find that clusters become extremely anisotropic near the percolation threshold and are character-

ized by two diverging correlation lengths, one parallel and one transverse to the preferred or time axis. The longitudinal length represents a mean length of connected paths, and this appears amenable to study through a PSRG approach based on a connectivity rule. However, the transverse length represents a mean cluster width, which we are unable to treat within our PSRG framework. Further analytic work on this interesting problem would be quite useful. Additionally, it would be interesting to make closer connections between directed percolation and various physical systems. Along these lines, we are currently studying the conductivity properties of resistor-diode networks.

We then introduced a more general system, the random resistor-diode network, in which the lattice edges may be occupied by either resistors or diodes of arbitrary orientation. Because of the self-dual nature of the square lattice, exact results for the structure of the phase diagram could be derived. The percolation phenomena of this model are quite diverse because of the possibility of forming an infinite cluster that percolates either isotropically or in only one direction. Moreover there exists multicritical behavior where the directed and isotropic percolating phases become identical simultaneously. Our study thus far has been of a qualitative nature, and it would be quite interesting to use numerical simulations to test our theoretical predictions.

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APPENDIX

Recursion relations for the random resistor-diode network. The symbols p_+ , p_- , p , and q refer respectively to the probability that a lattice edge is occupied by a positive diode, negative diode, resistor, or vacancy. The prime refers to the renormalized bond probabilities.

$$\begin{aligned}
 p'_+ &= \{2p^3 p_+^2 + p^2(7p_+^3 + 7p_+^2 p_-) + p(5p_+^4 + 14p_+^3 p_- + 6p_+^2 p_-^2) + (p_+^5 + 5p_+^4 p_- + 4p_+^3 p_-^2)\} \\
 &\quad + q\{p^3(4p_+) + p^2(22p_+^2 + 14p_+ p_-) + p(20p_+^3 + 36p_+^2 p_- + 8p_+ p_-^2) + (5p_+^4 + 14p_+^3 p_- + 6p_+^2 p_-^2)\} \\
 &\quad + q^2\{p^2(17p_+ + p_-) + p(22p_+^2 + 14p_+ p_-) + (7p_+^3 + 7p_+^2 p_-)\} + q^3\{4pp_+ + 2p_+^2\} \\
 &\equiv f(p_+, p_-, p, q) \ , \\
 p' &= \{p^5 + p^4(5p_+ + 5p_-) + p^3(8p_+^2 + 20p_+ p_- + 8p_-^2) + p^2(3p_+^3 + 23p_+^2 p_- + 23p_+ p_-^2 + 3p_-^3) \\
 &\quad + p(6p_+^3 p_- + 16p_+^2 p_-^2 + 6p_+ p_-^3) + (3p_+^3 p_-^2 + 3p_+^2 p_-^3)\} \\
 &\quad + q\{5p^4 + p^3(16p_+ + 16p_-) + p^2(8p_+^2 + 32p_+ p_- + 8p_-^2) + p(8p_+^2 p_- + 8p_+ p_-^2) + (2p_+^2 p_-^2)\} \\
 &\quad + q^2\{8p^3 + p^2(6p_+ + 6p_-)\} + q^3\{2p^2\} \\
 &\equiv g(p_+, p_-, p, q) \ , \\
 p'_- &= f(p_-, p_+, p, q) \ , \\
 q' &= g(p_+, p_-, q, p) \ .
 \end{aligned}$$

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